An iterative method for fixed point problems and variational inequality problems*

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Abstract. *In this paper, we present an iterative method for fixed point problems and variational inequality problems. Our method is based on the so-called extragradient method and viscosity approximation method. Using this method, we can find the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for monotone mapping.*

Key words: *variational inequality, fixed point, monotone mapping, nonexpansive mapping*

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1. Introduction

Let C be a closed convex subset of a real Hilbert space H. A mapping A of C into H is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0,$$

for all $x, y \in C$. A is called α-inverse-strongly-monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2,$$

for all $x, y \in C$. It is clear that an α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous.

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It is well known that the variational inequality problem VI(A,C) is to find $x \in C$ such that

$$\text{VI(A,C):} \qquad \langle Ax, y - x \rangle \geq 0,$$

for all $y \in C$ (see [1-3]). The variational inequality has been extensively studied in the literature; see, e.g., [9-20] and the references therein. A mapping S of C into itself is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|,$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. Recall that a mapping $f : C \to C$ is called contractive if there exists a constant $\beta \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \beta \|x - y\|, \quad \forall x, y \in C.$$

For finding an element of $F(S) \cap VI(A, C)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive and a mapping A of C into H is α-inverse-strongly-monotone, Takahashi and Toyoda [4] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \tag{1}$$

for every $n = 0, 1, 2, \cdots$, where P_C is the metric projection of H onto C, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $F(S) \cap VI(A, C)$ is nonempty, then the sequence $\{x_n\}$ generated by (1) converges weakly to some $z \in F(S) \cap VI(A, C)$. Recently, Nadezhkina and Takahashi [12] introduced a so-called extragradient method motivated by the idea of Korpelevich [5] for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem. They obtained the following weak convergence theorem.

Theorem NT ([12]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(A, C) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by*

$$\begin{cases} x_0 = x \in H, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Ay_n) \ \forall n \geq 0, \end{cases} \tag{2}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ converge weakly to the same point $P_{F(S) \cap VI(A,C)}(x_0)$.

Recently, inspired by Nadezhkina and Takahashi's results, Zeng and Yao [13] introduced another iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem. They obtained the following strong convergence theorem.

Theorem ZY1 ([13]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping* *and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(A, C) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by*

$$\begin{cases} x_0 = x \in H, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) SP_C(x_n - \lambda_n Ay_n) \ \forall n \geq 0, \end{cases} \tag{3}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions:

(a) $\{\lambda_n k\} \subset (0, 1 - \delta)$ for some $\delta \in (0, 1)$;

(b) $\{\alpha_n\} \subset (0, 1), \sum_{n=0}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point

$$P_{F(S)\cap VI(A,C)}(x_0)$$

provided

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4}$$

Further, very recently, Zeng and Yao [14] introduced an extragradient-like approximation method basing on the extragradient method and viscosity approximation method and obtained the following very interesting result.

Theorem ZY2 ([14]). *Let C be a nonempty closed convex subset of a real Hilbert space. Let $f : C \to C$ be a contractive mapping, $A : C \to H$ be a monotone, L-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(A, C) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequence generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = (1 - \gamma_n)x_n + P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n f(y_n) + \beta_n SP_C(x_n - \lambda_n Ay_n) \ \forall n \geq 0, \end{cases} \tag{5}$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leq 1$ for all $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequences $\{x_n\}, \{y_n\}$ converge strongly to $q = P_{F(S)\cap VI(A,C)} f(q)$ if and only if

$$\{Ax_n\} \text{ is bounded and } \liminf_{n\to\infty} \langle Ax_n, y - x_n \rangle \geq 0, \forall y \in C. \tag{6}$$

Remark 1. *(1) The authors think that all of the above results are very interesting and important.*

(2) We note that the iterative scheme (2) in Theorem NT *has only weak convergence. The iterative scheme (3) in* Theorem ZY1 *has strong convergence but imposed the assumption (4) on the sequence $\{x_n\}$. The iterative scheme (5) has strong convergence but also imposed the assumption (6) on the sequence $\{x_n\}$.*

This natural bring us the following question?

Question 2. *Could we construct an iterative algorithm to approximate the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for monotone mapping without any assumptions on the sequence* $\{x_n\}$*?*

In this paper, motivated by the iterative schemes (2), (3) and (5), we introduced an new iterative method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for monotone mapping. We obtain a strong convergence theorem under the some mild conditions.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a closed convex subset of H. It is well known that, for any $x \in H$, there exists unique $y_0 \in C$ such that

$$\|x - y_0\| = \inf\{\|x - y\| : y \in C\}.$$

We denote y_0 by $P_C x$, where P_C is called the metric projection of H onto C. The metric projection P_C of H onto C has the following basic properties:

(i) $\|P_C x - P_C y\| \leq \|x - y\|$ for all $x, y \in H$,

(ii) $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$ for every $x, y \in H$,

(iii) $\langle x - P_C x, y - P_C x \rangle \leq 0$ for all $x \in H, y \in C$,

(iv) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$ for all $x \in H, y \in C$.

Such properties of P_C will be crucial in the proof of our main results. Let A be a monotone mapping of C into H. In the context of the variational inequality problem, it is easy to see from (iv) that

$$x^* \in VI(A, C) \Leftrightarrow x^* = P_C(x^* - \lambda A x^*), \quad \forall \lambda > 0.$$

A set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$; i.e.,

$$N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (see [6]).

Now, we introduce several lemmas for our main results in this paper. The first lemma is an immediate consequence of an inner product. The second lemma is well known demiclosedness principle.

Lemma 2.1. *In a real Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.2. *Assume that S is a nonexpansive self-mapping of a nonempty closed convex subset of C of a real Hilbert space H. If $F(S) \neq \emptyset$, then I_S is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y, it follows that $(I - S)x = y$. Here I is the identity operator of H.*

Lemma 2.3. ([7]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.4. ([8]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \delta_n,$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \sigma_n = \infty$*;*

(2) $\limsup_{n\to\infty} \delta_n/\sigma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we will state and prove our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone, L-Lipschitz continuous mapping of C into H, S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A, C) \neq \emptyset$ and $f : C \to C$ be a contraction. For given $x_0 \in C$ arbitrary, let the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be generated by*

$$\begin{cases} z_n = P_C(x_n - \lambda_n A x_n), \\ y_n = (1 - \gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A z_n), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S[\beta_n f(x_n) + (1 - \beta_n)y_n], \end{cases} \tag{7}$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\lim_{n\to\infty} \lambda_n = 0$, and $\{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the conditions:

(i) $\lim_{n\to\infty} \beta_n = 0, \sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$*;*

(iii) $\lim_{n\to\infty}(\gamma_{n+1}-\gamma_n)=0$.

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *converge strongly to the same point* $q = P_{F(S)\cap VI(A,C)}f(q)$.

Proof. We divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Let $x^* \in F(S)\cap VI(A,C)$, then $x^* = P_C(x^* - \lambda_n Ax^*)$. Put $t_n = P_C(x_n - \lambda_n Az_n)$. Substituting x by $x_n - \lambda_n Az_n$ and y by x^* in (iv), we have

$$\begin{aligned}
\|t_n - x^*\|^2 &\le \|x_n - \lambda_n Az_n - x^*\|^2 - \|x_n - \lambda_n Az_n - t_n\|^2 \\
&= \|x_n - x^*\|^2 - 2\lambda_n\langle Az_n, x_n - x^*\rangle + \lambda_n^2\|Az_n\|^2 \\
&\quad -\|x_n - t_n\|^2 + 2\lambda_n\langle Az_n, x_n - t_n\rangle - \lambda_n^2\|Az_n\|^2 \\
&= \|x_n - x^*\|^2 + 2\lambda_n\langle Az_n, x^* - t_n\rangle - \|x_n - t_n\|^2 \\
&= \|x_n - x^*\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle Az_n - Ax^*, x^* - z_n\rangle \\
&\quad +2\lambda_n\langle Ax^*, x^* - z_n\rangle + 2\lambda_n\langle Az_n, z_n - t_n\rangle.
\end{aligned} \tag{8}$$

Using the fact that A is monotonic and x^* is a solution of the variational inequality problem VI(A,C), we have

$$\langle Az_n - Ax^*, x^* - z_n\rangle \le 0 \quad \text{and} \quad \langle Ax^*, x^* - z_n\rangle \le 0. \tag{9}$$

It follows from (8) and (9) that

$$\begin{aligned}
\|t_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle Az_n, z_n - t_n\rangle \\
&= \|x_n - x^*\|^2 - \|(x_n - z_n) + (z_n - t_n)\|^2 \\
&\quad +2\lambda_n\langle Az_n, z_n - t_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - z_n\|^2 - 2\langle x_n - z_n, z_n - t_n\rangle \\
&\quad -\|z_n - t_n\|^2 + 2\lambda_n\langle Az_n, z_n - t_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - z_n\|^2 - \|z_n - t_n\|^2 \\
&\quad +2\langle x_n - \lambda_n Az_n - z_n, t_n - z_n\rangle.
\end{aligned} \tag{10}$$

Substituting x by $x_n - \lambda_n Ax_n$ and y by t_n in (iii), we have

$$\langle x_n - \lambda_n Ax_n - z_n, t_n - z_n\rangle \le 0. \tag{11}$$

It follows that

$$\begin{aligned}
\langle x_n - \lambda_n Az_n - z_n, t_n - z_n\rangle &= \langle x_n - \lambda_n Ax_n - z_n, t_n - z_n\rangle \\
&\quad +\langle \lambda_n Ax_n - \lambda_n Az_n, t_n - z_n\rangle \\
&\le \langle \lambda_n Ax_n - \lambda_n Az_n, t_n - z_n\rangle \\
&\le \lambda_n L\|x_n - z_n\|\|t_n - z_n\|.
\end{aligned} \tag{12}$$

By (10) and (12), we obtain

$$\begin{aligned}\|t_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - z_n\|^2 - \|z_n - t_n\|^2 \\ &\quad +2\lambda_n L\|x_n - z_n\|\|t_n - z_n\| \\ &\le \|x_n - x^*\|^2 - \|x_n - z_n\|^2 - \|z_n - t_n\|^2 \\ &\quad +\lambda_n L(\|x_n - z_n\|^2 + \|z_n - t_n\|^2) \\ &\le \|x_n - x^*\|^2 + (\lambda_n L - 1)\|x_n - z_n\|^2 \\ &\quad +(\lambda_n L - 1)\|z_n - t_n\|^2.\end{aligned} \tag{13}$$

Since $\lambda_n \to 0$ as $n \to \infty$, there exists a positive integer N_0 such that $\lambda_n L - 1 \le -\frac{1}{2}$ when $n \ge N_0$. It follows from (13) that

$$\|t_n - x^*\| \le \|x_n - x^*\|.$$

From (7), we have

$$\begin{aligned}\|y_n - x^*\| &= \|(1-\gamma_n)(x_n - x^*) + \gamma_n(t_n - x^*)\| \\ &\le (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\ &= \|x_n - x^*\|.\end{aligned}$$

Write $W_n = \beta_n f(x_n) + (1-\beta_n)y_n$, then we have

$$\begin{aligned}\|W_n - x^*\| &= \|\beta_n(f(x_n) - x^*) + (1-\beta_n)(y_n - x^*)\| \\ &\le \beta_n\|f(x_n) - x^*\| + (1-\beta_n)\|y_n - x^*\| \\ &\le \beta_n\|f(x_n) - f(x^*)\| + \beta_n\|f(x^*) - x^*\| + (1-\beta_n)\|x_n - x^*\| \\ &\le \beta\beta_n\|x_n - x^*\| + \beta_n\|f(x^*) - x^*\| + (1-\beta_n)\|x_n - x^*\| \\ &= \beta_n\|f(x^*) - x^*\| + [1 - (1-\beta)\beta_n]\|x_n - x^*\|.\end{aligned}$$

From (7), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(SW_n - Sx^*)\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|W_n - x^*\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\beta_n\|f(x^*) - x^*\| \\ &\quad d + \alpha_n[1 - (1-\beta)\beta_n]\|x_n - x^*\| \\ &= [1 - (1-\beta)\alpha_n\beta_n]\|x_n - x^*\| \\ &\quad +(1-\beta)\alpha_n\beta_n\frac{1}{1-\beta}\|f(x^*) - x^*\| \\ &\le \max\{\|x_0 - x^*\|, \frac{1}{1-\beta}\|f(x^*) - x^*\|\}.\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Hence $\{t_n\}$, $\{St_n\}$, $\{Ax_n\}$ and $\{Az_n\}$ are also bounded.

Step 2. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

For all $x, y \in C$, we get

$$\begin{aligned}\|(I - \lambda_n A)x - (I - \lambda_n A)y\| &\le \|x - y\| + \lambda_n\|Ax - Ay\| \\ &\le (1 + L\lambda_n)\|x - y\|.\end{aligned} \tag{14}$$

By (7) and (14), we have

$$\begin{aligned}\|t_{n+1}-t_n\| &= \|P_C(x_{n+1}-\lambda_{n+1}Az_{n+1})-P_C(x_n-\lambda_nAz_n)\|\\ &\le \|(x_{n+1}-\lambda_{n+1}Az_{n+1})-(x_n-\lambda_nAz_n)\|\\ &= \|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_{n+1}Ax_n)\\ &\quad +\lambda_{n+1}(Ax_{n+1}-Az_{n+1}-Ax_n)+\lambda_nAz_n\|\\ &\le \|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_{n+1}Ax_n)\| \qquad (15)\\ &\quad +\lambda_{n+1}(\|Ax_{n+1}\|+\|Az_{n+1}\|+\|Ax_n\|)+\lambda_n\|Az_n\|\\ &\le (1+\lambda_{n+1}L)\|x_{n+1}-x_n\|+\lambda_{n+1}(\|Ax_{n+1}\|\\ &\quad +\|Az_{n+1}\|+\|Ax_n\|)+\lambda_n\|Az_n\|.\end{aligned}$$

Put $u_n = SW_n$, then $x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n$. We can obtain

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|SW_{n+1}-SW_n\| \le \|W_{n+1}-W_n\|\\ &= \|\beta_{n+1}f(x_{n+1})+(1-\beta_{n+1})y_{n+1}-\beta_nf(x_n)-(1-\beta_n)y_n\| \qquad (16)\\ &\le \beta_{n+1}(\|f(x_{n+1})\|+\|y_{n+1}\|)+\beta_n(\|f(x_n)\|+\|y_n\|)\\ &\quad +\|y_{n+1}-y_n\|\end{aligned}$$

We note that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|(1-\gamma_{n+1})x_{n+1}+\gamma_{n+1}t_{n+1}-(1-\gamma_n)x_n-\gamma_nt_n\|\\ &= \|(1-\gamma_{n+1})(x_{n+1}-x_n)+(\gamma_n-\gamma_{n+1})x_n\\ &\quad +\gamma_{n+1}(t_{n+1}-t_n)+(\gamma_{n+1}-\gamma_n)t_n\| \qquad (17)\\ &\le (1-\gamma_{n+1})\|x_{n+1}-x_n\|+|\gamma_{n+1}-\gamma_n|\|x_n\|\\ &\quad +\gamma_{n+1}\|t_{n+1}-t_n\|+|\gamma_{n+1}-\gamma_n|\|t_n\|\end{aligned}$$

Combining (15) and (17), we have

$$\begin{aligned}\|y_{n+1}-y_n\| \le\ & \|x_{n+1}-x_n\|+|\gamma_{n+1}-\gamma_n|(\|x_n\|+\|t_n\|)+\lambda_n\|Az_n\|\\ &+\lambda_{n+1}(L\|x_{n+1}-x_n\|+\|Ax_{n+1}\|+\|Az_{n+1}\|+\|Ax_n\|),\end{aligned}$$

this together with (16) implies that

$$\begin{aligned}&\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|\\ &\le \beta_{n+1}(\|f(x_{n+1})\|+\|y_{n+1}\|)+\beta_n(\|f(x_n)\|+\|y_n\|)\\ &\qquad +|\gamma_{n+1}-\gamma_n|(\|x_n\|+\|t_n\|)+\lambda_n\|Az_n\|\\ &\qquad +\lambda_{n+1}(L\|x_{n+1}-x_n\|+\|Ax_{n+1}\|+\|Az_{n+1}\|+\|Ax_n\|).\end{aligned}$$

It follows that

$$\limsup_{n\to\infty}(\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|) \le 0.$$

Hence by *Lemma 2.3*, we obtain $\|u_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\alpha_n\|u_n-x_n\| = 0.$$

Noting that (15), we also have $\|t_{n+1}-t_n\| \to 0$ as $n \to \infty$.

Step 3. $\lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} \|St_n - t_n\| = 0$.

Indeed, we observe that

$$\begin{aligned}\|Sx_n - x_n\| &\le \|Sx_n - SW_n\| + \|SW_n - x_n\| \\ &\le \|x_n - W_n\| + \|u_n - x_n\| \\ &\le \beta_n\|f(x_n) - x_n\| + (1-\beta_n)\|y_n - x_n\| + \|u_n - x_n\|.\end{aligned}$$

Noting that $x_n = P_C(x_n)$, then we have

$$\|t_n - x_n\| = \|P_C(x_n - \lambda_n A z_n) - P_C x_n\| \le \lambda_n \|Az_n\| \to 0,$$

and hence

$$\|y_n - x_n\| = \|\gamma_n(t_n - x_n)\| \le \gamma_n\|t_n - x_n\| \to 0.$$

Thus from the last three inequalities we conclude that

$$\|Sx_n - x_n\| \to 0.$$

At the same time, we have

$$\begin{aligned}\|St_n - t_n\| &\le \|St_n - Sx_n\| + \|Sx_n - x_n\| + \|x_n - t_n\| \\ &\le 2\|x_n - t_n\| + \|Sx_n - x_n\| \to 0.\end{aligned}$$

Step 4. $\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle \le 0$.

Indeed, we pick a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ so that

$$\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle = \lim_{i\to\infty} \langle f(q) - q, x_{n_i} - q\rangle. \tag{18}$$

Without loss of generality, let $x_{n_i} \rightharpoonup \hat{x} \in C$. Then (18) reduces to

$$\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle = \langle f(q) - q, \hat{x} - q\rangle.$$

In order to show $\langle f(q) - q, \hat{x} - q\rangle \le 0$, it suffices to show that $\hat{x} \in F(S) \cap VI(A, C)$. Note that by *Lemma 2.2* and *Step 3*, we have $\hat{x} \in F(S)$. Now we claim that $\hat{x} \in VI(A, C)$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$. Let $(v, w) \in G(T)$. Then we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. Therefore we have $\langle v - y, w - Av\rangle \ge 0$ for all $y \in C$. In particular, taking $y = x_{n_i}$, we get

$$\begin{aligned}\langle v - \hat{x}, w\rangle &= \liminf_{i\to\infty} \langle v - x_{n_i}, w\rangle \\ &\ge \liminf_{i\to\infty} \langle v - x_{n_i}, Av\rangle \\ &= \liminf_{i\infty} [\langle v - x_{n_i}, Av - Ax_{n_i}\rangle + \langle v - x_{n_i}, Ax_{n_i}\rangle] \\ &\ge \liminf_{i\to\infty} \langle v - x_{n_i}, Ax_{n_i}\rangle \\ &\ge \liminf_{i\to\infty} \langle v - x_n, Ax_n\rangle \\ &\ge 0\end{aligned}$$

and so $\langle v - \hat{x}, w \rangle \geq 0$. Since T is maximal monotone, we have $\hat{x} \in T^{-1}0$ and hence $\hat{x} \in VI(A,C)$. This shows that $\hat{x} \in F(S) \cap VI(A,C)$. Therefore, we derive $\langle f(q) - q, \hat{x} - q \rangle \leq 0$. Then we obtain $\limsup_{n\to\infty} \langle f(q) - q, x_n - q \rangle \leq 0$. At the same time, it is easily seen that $\|W_n - x_n\| \to 0$, it follows that

$$\limsup_{n\to\infty} \langle f(q) - q, W_n - q \rangle \leq 0. \tag{19}$$

Step 5. $\lim_{n\to\infty} \|x_n - q\| = 0$ where $q = P_{F(S)\cap VI(A,C)} f(q)$.

First we observe that

$$\begin{aligned}
\|f(x_n) - f(q)\| \|W_n - q\| &\leq \beta \|x_n - q\| \|\beta_n (f(x_n) - q) + (1-\beta_n)(y_n - q)\| \\
&\leq \beta_n \beta \|x_n - q\| \|f(x_n) - q\| \\
&\quad + (1-\beta_n)\beta \|x_n - q\| \|y_n - q\| \\
&\leq \beta_n \beta \|x_n - q\| \|f(x_n) - q\| + (1-\beta_n)\beta \|x_n - q\|^2 \\
&\leq \beta_n \|x_n - q\| \|f(x_n) - q\| + \beta \|x_n - q\|^2.
\end{aligned} \tag{20}$$

From (7), (20) and *Lemma 2.1*, we get

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1-\alpha_n)(x_n - q) + \alpha_n (SW_n - q)\|^2 \\
&\leq (1-\alpha_n)\|x_n - q\|^2 + \alpha_n \|W_n - q\|^2 \\
&= (1-\alpha_n)\|x_n - q\|^2 + \alpha_n \|\beta_n (f(x_n) - q) + (1-\beta_n)(y_n - q)\|^2 \\
&\leq (1-\alpha_n)\|x_n - q\|^2 + \alpha_n [(1-\beta_n)^2 \|y_n - q\|^2 \\
&\quad + 2\beta_n \langle f(x_n) - q, W_n - q \rangle] \\
&\leq (1 - 2\alpha_n\beta_n)\|x_n - q\|^2 + 2\alpha_n\beta_n \langle f(x_n) - f(q), W_n - q \rangle \\
&\quad + 2\alpha_n\beta_n \langle f(q) - q, W_n - q \rangle + \alpha_n \beta_n^2 \|y_n - q\|^2 \\
&\leq (1 - 2\alpha_n\beta_n)\|x_n - q\|^2 + 2\alpha_n\beta_n \|f(x_n) - f(q)\| \|W_n - q\| \\
&\quad + 2\alpha_n\beta_n \langle f(q) - q, W_n - q \rangle + \alpha_n \beta_n^2 \|y_n - q\|^2 \\
&\leq [1 - 2(1-\beta)\alpha_n\beta_n]\|x_n - q\|^2 + 2\alpha_n\beta_n^2 \|x_n - q\| \|f(x_n) - q\| \\
&\quad + 2\alpha_n\beta_n \langle f(q) - q, W_n - q \rangle + \alpha_n \beta_n^2 \|y_n - q\|^2 \\
&= [1 - 2(1-\beta)\alpha_n\beta_n]\|x_n - q\|^2 \\
&\quad + 2(1-\beta)\alpha_n\beta_n \times \{ \frac{1}{1-\beta} \langle f(q) - q, W_n - q \rangle \\
&\quad + \frac{\beta_n}{1-\beta} \|x_n - q\| \|f(x_n) - q\| + \frac{\beta_n}{2(1-\beta)} \|y_n - q\|^2 \} \\
&= (1 - \sigma_n)\|x_n - q\|^2 + \delta_n,
\end{aligned} \tag{21}$$

where $\sigma_n = 2(1-\beta)\alpha_n\beta_n$ and $\delta_n = 2(1-\beta)\alpha_n\beta_n \times \{\frac{1}{1-\beta}\langle f(q) - q, W_n - q\rangle + \frac{\beta_n}{1-\beta}\|x_n - q\|\|f(x_n) - q\| + \frac{\beta_n}{2(1-\beta)}\|y_n - q\|^2\}$. It is easily seen that $\limsup_{n\to\infty} \delta_n/\sigma_n \leq 0$. There fore, according to *Lemma 2.4* we conclude from (21) that $\|x_n - q\| \to 0$. Further from $\|y_n - x_n\| \to 0$ and

$$\begin{aligned}
\|z_n - x_n\| &= \|P_C(x_n - \lambda_n A x_n) - P_C(x_n)\| \\
&\leq \lambda_n \|A x_n\| \to 0,
\end{aligned}$$

we get $\|y_n - q\| \to 0$ and $\|z_n - q\| \to 0$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone, L-Lipschitz continuous mapping of C into H such that $F(S) \neq \emptyset$ and $f : C \to C$ be a contraction. For given $x_0 \in C$ arbitrary, let the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be generated by*

$$\begin{cases} z_n = P_C(x_n - \lambda_n A x_n), \\ y_n = (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A z_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n f(x_n) + (1-\beta_n)y_n], \end{cases}$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\lim_{n\to\infty} \lambda_n = 0$, and $\{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the conditions:

(i) $\lim_{n\to\infty} \beta_n = 0, \sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\lim_{n\to\infty}(\gamma_{n+1} - \gamma_n) = 0$.

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to the same point $q = P_{F(S)}f(q)$.

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