A condition that a tangential quadrilateral is also a chordal one

Mirko Radić[*], Zoran Kaliman[†] and Vladimir Kadum[‡]

Abstract. *In this article we present a condition that a tangential quadrilateral is also a chordal one. The main result is given by* Theorem 1 *and* Theorem 2.

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1. Introduction

A polygon which is both tangential and chordal will be called a bicentric polygon. The following notation will be used.

If $A_1A_2A_3A_4$ is a considered bicentric quadrilateral, then its incircle is denoted by C_1, circumcircle by C_2, radius of C_1 by r, radius of C_2 by R, center of C_1 by I, center of C_2 by O, distance between I and O by d.

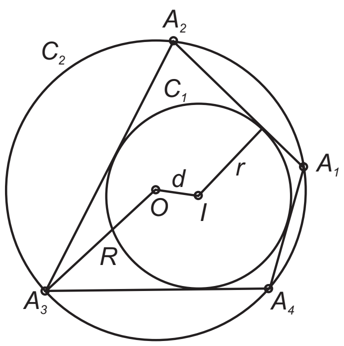

Figure 1.1

[*]Faculty of Philosophy, University of Rijeka, Omladinska 14, HR-51 000 Rijeka, Croatia, e-mail: `mradic@ffri.hr`

[†]Faculty of Philosophy, University of Rijeka, Omladinska 14, HR-51 000 Rijeka, Croatia, e-mail: `kaliman@ffri.hr`

[‡]University "Juraj Dobrila" of Pula, Preradovićeva 1, HR-52 100 Pula, Croatia, e-mail: `vladimir.kadum@pu.t-com.hr`

The first one who was concerned with bicentric quadrilaterals was a German mathematician Nicolaus Fuss (1755-1826), see [2]. He found that C_1 is the incircle and C_2 the circumcircle of a bicentric quadrilateral $A_1A_2A_3A_4$ iff

$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2). \tag{1.1}$$

The problem of findings relation (1.1) has ranged in [1] as one of 100 great problems of elementary mathematics.

A very remarkable theorem concerning bicentric polygons is given by a French mathematician Poncelet (1788-1867). This theorem is known as the Poncelet's closure theorem. For the case when conics are circles, one inside the other, this theorem can be stated as follows:

If there is a bicentric n-gon whose incircle is C_1 and circumcircle C_2, then there are infinitely many bicentric n-gons whose incircle is C_1 and circumcircle C_2. For every point P_1 on C_2 there are points $P_2, \ldots, P_n$ on C_2 such that $P_1 \ldots P_n$ are a bicentric n-gon whose incircle is C_1 and circumcircle C_2.

In the following (Section 3) bicentric quadrilaterals will also be considered, where instead of an incircle there is an excircle. As will be seen, there is a great analogy between those two kinds of bicentric quadrilaterals.

2. About one condition concerning bicentric quadrilaterals

First, let us briefly discuss the notations to be used.

If $A_1A_2A_3A_4$ is a given tangential quadrilateral, then by t_1, t_2, t_3, t_4 we denote its tangent lengths such that

$$t_i + t_{i+1} = |A_iA_{i+1}|, \quad i = 1, 2, 3, 4. \tag{2.1}$$

By β_1, β_2, β_3, β_4 we denote angles $\measuredangle IA_iA_{i+1}$, $i = 1, 2, 3, 4$, where I is the center of the incircle of $A_1A_2A_3A_4$. (See *Figure 2.1*)

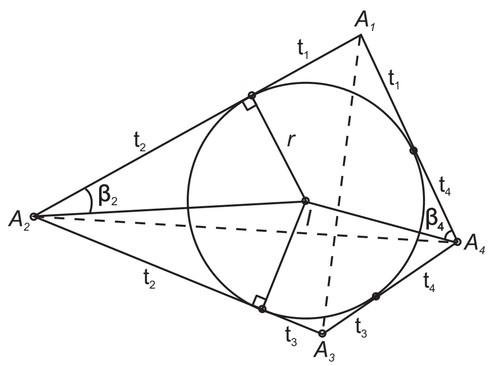

Figure 2.1.

The following theorem will be proved.

Theorem 1. *Let $A_1A_2A_3A_4$ be any given tangential quadrilateral, and let t_1, t_2, t_3, t_4 be its tangent lengths such that* (2.1) *holds. Then this quadrilateral is also a chordal one if and only if*

$$\frac{|A_1A_3|}{t_1+t_3} = \frac{|A_2A_4|}{t_2+t_4} = \sqrt{k}, \tag{2.2}$$

where

$$1 < k \leq 2. \tag{2.3}$$

Proof. First we suppose that (2.2) holds. From *Figure 2.1* we see that the equality $|A_1A_3|^2 = k(t_1+t_3)^2$ can be written as

$$|A_1A_2|^2 + |A_2A_3|^2 - 2|A_1A_2||A_2A_3|\cos 2\beta_2 = k(t_1+t_3)^2$$

or

$$(t_1+t_2)^2 + (t_2+t_3)^2 - 2(t_1+t_2)(t_2+t_3)\frac{t_2^2-r^2}{t_2^2+r^2} = k(t_1+t_3)^2, \tag{2.4}$$

since

$$\cos 2\beta_2 = \frac{1-\tan^2\beta_2}{1+\tan^2\beta_2}, \quad \tan\beta_2 = \frac{r}{t_2}.$$

The equality $|A_1A_3|^2 = k(t_1+t_3)^2$ can also be written as

$$(t_1+t_4)^2 + (t_4+t_3)^2 - 2(t_1+t_4)(t_4+t_3)\frac{t_4^2-r^2}{t_4^2+r^2} = k(t_1+t_3)^2, \tag{2.5}$$

where

$$2\beta_4 = \text{ measure of } \sphericalangle A_1A_4A_3, \quad \cos 2\beta_4 = (t_4^2-r^2)/(t_4^2+r^2).$$

In the same way can see that the equality $|A_2A_4|^2 = k(t_2+t_4)^2$ can be written in the following two ways:

$$(t_1+t_2)^2 + (t_1+t_4)^2 - 2(t_1+t_2)(t_1+t_4)\frac{t_1^2-r^2}{t_1^2+r^2} = k(t_2+t_4)^2, \tag{2.6}$$

$$(t_3+t_2)^2 + (t_3+t_4)^2 - 2(t_3+t_2)(t_3+t_4)\frac{t_3^2-r^2}{t_3^2+r^2} = k(t_2+t_4)^2. \tag{2.7}$$

Solving equation (2.4) for t_2 we get

$$\begin{aligned}(t_2)_1 = \Big[& -4r^2t_1 - 4r^2t_3 - \\ & \Big((4r^2t_1+4r^2t_3)^2 - 4(4r^2+t_1^2-kt_1^2-2t_1t_3-2kt_1t_3+t_3^2-kt_3^2) \\ & (r^2t_1^2-kr^2t_1^2+2r^2t_1t_3-2kr^2t_1t_3+r^2t_3^2-kr^2t_3^2)\Big)^{\frac{1}{2}}\Big]/ \\ & \big(2(4r^2+t_1^2-kt_1^2-2t_1t_3-2kt_1t_3+t_3^2-kt_3^2)\big),\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
(t_2)_2 = \Big[&- 4r^2t_1 - 4r^2t_3 + \\
&\Big((4r^2t_1 + 4r^2t_3)^2 - 4(4r^2 + t_1^2 - kt_1^2 - 2t_1t_3 - 2kt_1t_3 + t_3^2 - kt_3^2) \\
&(r^2t_1^2 - kr^2t_1^2 + 2r^2t_1t_3 - 2kr^2t_1t_3 + r^2t_3^2 - kr^2t_3^2)\Big)^{\frac{1}{2}}\Big]/ \\
&\big(2(4r^2 + t_1^2 - kt_1^2 - 2t_1t_3 - 2kt_1t_3 + t_3^2 - kt_3^2)\big). \qquad (2.9)
\end{aligned}
$$

It is easy to see that equation (2.4) in t_2 has the same solutions as equation (2.5) in t_4, that is

$$\{(t_2)_1, (t_2)_2\} = \{(t_4)_1, (t_4)_2\}.$$

Since equation (2.4) has t_2 as one solution, and equation (2.5) has t_4 as one solution, it follows that

$$\{(t_2)_1, (t_2)_2\} = \{(t_4)_1, (t_4)_2\} = \{t_2, t_4\}. \qquad (2.10)$$

Putting $t_2 = (t_2)_1$, $t_4 = (t_2)_2$ in (2.6) we get

$$\frac{(-1+k)r^2(t_1+t_3)^2}{-4r^2 + (-1+k)t_1^2 + 2(1+k)t_1t_3 + (-1+k)t_3^2} = t_1t_3. \qquad (2.11)$$

Solving this equation for t_3 yields

$$t_3 \in \left\{ \frac{r^2}{t_1}, \frac{-t_1 - 2\sqrt{k}\,t_1 - kt_1}{-1+k}, \frac{-t_1 + 2\sqrt{k}\,t_1 - kt_1}{-1+k} \right\}$$

Thus, the only positive t_3 is given by

$$t_3 = \frac{r^2}{t_1}. \qquad (2.12)$$

Now we find that from (2.8) and (2.9) there follows

$$(t_2)_1 \cdot (t_2)_2 = \frac{r^2(t_1+t_3)^2(1-k)}{(t_1-t_3)^2 + 4r^2 - k(t_1+t_3)^2},$$

which according to (2.10) and (2.12) can be written as

$$t_2t_4 = r^2. \qquad (2.13)$$

That also $t_1t_3 = r^2$, that is

$$t_1t_3 = t_2t_4 = r^2, \qquad (2.14)$$

follows from $(t_1+t_2+t_3+t_4)r^2 = t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2$ putting $t_4 = r^2/t_2$. Namely, we get $r^2(t_2+t_4) = t_1t_3(t_2+t_4)$, from which follows $t_1t_3 = r^2$.

We shall prove that these relations are sufficient for a tangential quadrilateral to be a chordal one. The proof is as follows.

Since

$$\cos 2\beta_2 = \frac{t_2^2 - r^2}{t_2^2 + r^2}, \quad \cos 2\beta_4 = \frac{t_4^2 - r^2}{t_4^2 + r^2}$$

using (2.14) we can write

$$\cos 2\beta_4 = \frac{(r^2/t_2)^2 - r^2}{(r^2/t_2)^2 + r^2} = -\frac{t_2^2 - r^2}{t_2^2 + r^2} = -\cos 2\beta_2.$$

In the same way we find that $\cos 2\beta_3 = -\cos 2\beta_1$.

Thus, from (2.2) it follows that the given tangential quadrilateral $A_1A_2A_3A_4$ is also a chordal one since $2\beta_1 + 2\beta_3 = 2\beta_2 + 2\beta_4 = \pi$. In this connection let us remark that it is not difficult to check that identically holds

$$r(t_1 + t_2 + t_3 + t_4) = \sqrt{(t_1 + t_2)(t_2 + t_3)(t_3 + t_4)(t_4 + t_1)}$$

for every positive numbers r, t_1, t_2, t_3, t_4 such that $t_1t_3 = t_2t_4 = r^2$.

Now we prove that relations (2.14) are necessarily for a tangential quadrilateral to be a chordal one. The proof is easy; namely, it is easy to see that

$$\cos 2\beta_2 = -\cos 2\beta 4$$

or

$$\frac{t_2^2 - r^2}{t_2^2 + r^2} = -\frac{t_4^2 - r^2}{t_4^2 + r^2}$$

is valid only if $t_2t_4 = r^2$.

In the same way it can be seen that $\cos 2\beta_1 = -\cos 2\beta_3$ only if $t_1t_3 = r^2$.

Here let us remark that the following holds. If $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are two bicentric quadrilaterals which have the same incircle and

$$\begin{aligned} t_i + t_{i+1} &= |A_iA_{i+1}|, \quad i = 1, 2, 3, 4 \\ u_i + u_{i+1} &= |B_iB_{i+1}|, \quad i = 1, 2, 3, 4 \\ t_1t_3 &= t_2t_4 = r^2, \\ u_1u_3 &= u_2u_4 = r^2, \end{aligned}$$

then these quadrilateral need not have the same circumcirle. It will be only if

$$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = u_1u_2 + u_2u_3 + u_3u_4 + u_4u_1 = 2(R^2 - d^2).$$

(See Theorem 3.2 in [3].)

In this connection may be interesting how radius R can be obtained and some other relations. In short about this.

Let C_1 and C_2 denote the incircle and the circumcircle of the considered bicentric quadrilateral $A_1A_2A_3A_4$, and let the other notation be as stated in the introduction.

The radius of C_2 can be obtained using well-known relations which hold for a bicentric quadrilateral:

$$R^2 = \frac{(a_1a_2 + a_3a_4)(a_1a_3 + a_2a_4)(a_1a_4 + a_2a_3)}{16J^2}, \quad J^2 = a_1a_2a_3a_4$$

where $a_1 = t_1 + t_2$, $a_2 = t_2 + t_3$, $a_3 = t_3 + t_4$, $a_4 = t_4 + t_1$, $J =$ area of $A_1A_2A_3A_4$. It can be found that

$$16R^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2 + \frac{a_1a_2a_3}{a_4} + \frac{a_2a_3a_4}{a_1} + \frac{a_3a_4a_1}{a_2} + \frac{a_4a_1a_2}{a_3} \tag{2.15}$$

or, using relations (2.14),

$$R^2 = \frac{[(r^2+t_1^2)(r^2+t_2^2)][(r^2+t_1^2)(r^2+t_2^2)+4r^2t_1t_2]}{16r^2t_1^2t_2^2}\,. \tag{2.16}$$

Now we shall prove that

$$k = \frac{2R^2}{R^2+d^2}. \tag{2.17}$$

For this purpose, in (2.8) and (2.9) we shall put $\frac{2R^2}{R^2+d^2}$ instead of k, and $\frac{r^2}{t_1}$ instead of t_3. It can be found that

$$(t_2)_1 = \frac{(R^2-d^2)t_1+\sqrt{D}}{r^2+t_1^2}, \tag{2.18}$$

$$(t_2)_2 = \frac{(R^2-d^2)t_1-\sqrt{D}}{r^2+t_1^2}, \tag{2.19}$$

where

$$D = (R^2-d^2)^2t_1^2 - r^2(r^2+t_1^2)^2. \tag{2.20}$$

It is easy to check that $(t_2)_1\cdot(t_2)_2 = r^2$ or, since (2.10) holds,

$$t_2t_4 = r^2. \tag{2.21}$$

Besides, we have to prove one lemma. In this lemma will be used values t_m and t_M given by

$$t_m = \sqrt{(R-d)^2-r^2}, \quad t_M = \sqrt{(R+d)^2-r^2}. \tag{2.22}$$

See *Figure 2.2.* As can be seen, t_m and t_M are the lengths of the least and the largest tangent that can be drawn from C_2 to C_1.

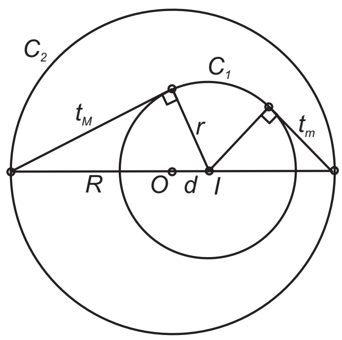

Figure 2.2

Lemma 1. *Let u_1 be any given value (tangent length) such that*

$$t_m \le u_1 \le t_M, \tag{2.23}$$

and let u_2, u_3, u_4 *be given by*

$$u_2 = \frac{(R^2-d^2)u_1 + \sqrt{D}}{r^2+u_1^2}, \tag{2.24}$$

$$u_3 = \frac{r^2}{u_1}, \tag{2.25}$$

$$u_4 = \frac{r^2}{u_2}, \tag{2.26}$$

where

$$D = (R^2-d^2)^2u_1^2 - r^2(r^2+u_1^2)^2. \tag{2.27}$$

Then the bicentric quadrilateral $B_1B_2B_3B_4$, *where* $|B_iB_{i+1}| = u_i + u_{i+1}$, $i = 1,2,3,4$, *has the same incircle and circumcircle as the considered quadrilateral* $A_1A_2A_3A_4$.

Proof. Since in the expression of u_2 appears the term $\sqrt{D}$, we have to prove that $D \geq 0$ for every u_1 such that $t_m \leq u_1 \leq t_M$. For this purpose, as can be readily seen, it is sufficient to prove that $D = 0$ for $u_1 = t_m$ and $u_1 = t_M$. The proof is as follows:

$$(R^2-d^2)^2t_m^2 - r^2(r^2+t_m^2)^2 = (R-d)^2[(R^2-d^2)^2 - 2r^2(R^2+d^2)] = 0,$$

because of (1.1)

$$(R^2-d^2)^2t_M^2 - r^2(r^2+t_M^2)^2 = (R-d)^2[(R^2-d^2)^2 - 2r^2(R^2+d^2)] = 0.$$

That C_1 is incircle of $B_1B_2B_3B_4$ it is clear from

$$\begin{aligned} r^2(u_1+u_2+u_3+u_4) &= u_1u_2u_3 + u_2u_3u_4 + u_3u_4u_1 + u_4u_1u_2 \\ &= r^2(u_2+u_3+u_4+u_1), \text{ since } u_1u_3 = u_2u_4 = r^2. \end{aligned}$$

To prove that C_2 is circumcircle of $B_1B_2B_3B_4$ we have to prove that

$$\frac{[(r^2+u_1^2)(r^2+u_2^2)][(r^2+u_1^2)(r^2+u_2^2)+4r^2u_1u_2]}{16r^2u_1^2u_2^2} = R^2. \tag{2.28}$$

First, using u_2 given by (2.24), we find that $(r^2+u_1^2)(r^2+u_2^2)$ in (2.28) can be written as $2(R^2-d^2)u_1u_2$.

Now, it is easy to see that

$$2(R^2-d^2)u_1u_2\left[2(R^2-d^2)u_1u_2 + 4r^2u_1u_2\right] = 16R^2r^2u_1^2u_2^2$$

is equivalent to Fuss' relation (1.1).

Thus, *Lemma 1* is proved. (Cf. with Theorem 3.3 in [3].) □

It remains to prove that k given by (2.17) is not only sufficient but also necessary for $A_1A_2A_3A_4$ to be a bicentric one. It will be proved using one of the relations

(2.4)-(2.7). So, starting from (2.4) we can write

$$
\begin{aligned}
&t_2^2[(t_1+t_3)^2-k(t_1+t_3)^2]+4r^2(t_1+t_3)t_1t_2+r^2(t_1+t_3)^2(1-k)=0,\\
&t_2^2(t_1+t_3)(1-k)+4r^2t_1t_2+r^2(t_1+t_3)(1-k)=0,\\
&1-k=\frac{-4r^2t_1t_2}{(r^2+t_1^2)(r^2+t_2^2)}\ ,\\
&1-k=\frac{-4r^2t_1t_2}{2(R^2-d^2)t_1\cdot\frac{(R^2-d^2)t_1+\sqrt{D}}{r^2+t_1^2}}=-\frac{2r^2}{R^2-d^2}
\end{aligned}
$$

since $t_2=(t_2)_1$ given by (2.18) .

Now, we have

$$1-\frac{2R^2}{R^2+d^2}=-\frac{2r^2}{R^2-d^2}\quad \text{or}\quad \frac{R^2-d^2}{R^2+d^2}=-\frac{2r^2}{R^2-d^2},$$

since Fuss' relation (1.1) holds.

At the end we prove the following assertion: If $A_1A_2A_3A_4$ is a bicentric quadrilateral, then $\frac{|A_1A_3|}{t_1+t_3}=\frac{|A_2A_4|}{t_2+t_4}=\sqrt{k}$.

Proof. Let denote by F relation obtained from (2.4) putting

$$t_2=\frac{(R^2-d^2)t_1+\sqrt{D}}{r^2+t_1^2},\quad t_3=\frac{r^2}{t_1},\quad t_4=\frac{r^2}{t_2},\quad k=\frac{2R^2}{R^2+d^2},$$

where

$$D=(R^2-d^2)^2t_1^2-r^2(r^2+t_1^2)^2.$$

Using computer algebra it is easy to show that

$$F\iff (R^2-d^2)^2-2r^2(R^2+d^2)=0,$$

which proves $|A_1A_3|=(t_1+t_3)\sqrt{k}$. In the same way can be proved that $|A_2A_4|=(t_2+t_4)\sqrt{k}$. □

This completes the proof of *Theorem 1.* □

Now some of its corollaries will be stated.

Corollary 1. *Let t_1, t_2, t_3, t_4 be any given lengths (in fact positive numbers) such that $t_1t_3=t_2t_4=r^2$, and let R^2 and d^2 be given by*

$$R^2=\frac{[(r^2+t_1^2)(r^2+t_2^2)][(r^2+t_1^2)(r^2+t_2^2)+4r^2t_1t_2]}{16r^2t_1^2t_2^2}\ ,\tag{2.29}$$

$$d^2=\frac{[(r^2+t_1^2)(r^2+t_2^2)][(r^2+t_1^2)(r^2+t_2^2)-4r^2t_1t_2]}{16r^2t_1^2t_2^2}\ .\tag{2.30}$$

Then holds Fuss' relation (1.1).

Proof. From (2.29) and (2.30) it follows

$$
\begin{aligned}
(R^2-d^2)^2&=\tfrac{[(r^2+t_1^2)(r^2+t_2^2)]^2}{4t_1^2t_2^2}\ , \qquad (2.31)\\
2r^2(R^2+d^2)&=\tfrac{[(r^2+t_1^2)(r^2+t_2^2)]^2}{4t_1^2t_2^2}\ .
\end{aligned}
$$

□

Corollary 2. *Under the condition of* Corollary 1 *it holds*

$$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = 2(R^2 - d^2).$$

Proof. Since (2.31) holds we can write

$$\begin{aligned} 2(R^2-d^2) &= \tfrac{(r^2+t_1^2)(r^2+t_2^2)}{t_1t_2} \\ &= (t_1+\tfrac{r^2}{t_1})(t_2+\tfrac{r^2}{t_2}) \\ &= (t_1+t_3)(t_2+t_4)(\text{ since } t_3=\tfrac{r^2}{t_1}, t_4=\tfrac{r^2}{t_2}) \\ &= t_1t_2+t_2t_3+t_3t_4+t_4t_1. \end{aligned} \tag{2.32}$$

□

Corollary 3. *If* $k = \frac{2R^2}{R^2+d^2}$ *and* (1.1) *hold, then every positive solution of the system with equations* (2.4)-(2.7) *can be expressed such that there holds*

$$t_m \le t_1 \le t_M,$$

$$t_2 = \frac{(R^2-d^2)t_1 + \sqrt{D}}{r^2+t_1^2}\ , \quad t_3 = \frac{r^2}{t_1}\ , \quad t_4 = \frac{r^2}{t_2}$$

where $D = (R^2-d^2)^2t_1^2 - r^2(r^2+t_1^2)^2$.

Corollary 4. *Let* $A_1A_2A_3A_4$ *be any given tangential quadrilateral and let* t_1, t_2, t_3, t_4 *be lengths of its tangents such that*

$$t_i + t_{i+1} = |A_iA_{i+1}|, \quad i = 1,2,3,4.$$

Then this quadrilateral is also a chordal one iff

$$t_1t_3 = r^2, \tag{2.33}$$

where r *is radius of the incircle of* $A_1A_2A_3A_4$.

Proof. From $(t_1+t_2+t_3+t_4)r^2 = t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2$ it follows

$$t_4 = \frac{t_1t_2t_3 - r^2(t_1+t_2+t_3)}{r^2 - t_1t_2 - t_2t_3 - t_3t_1}\ .$$

Putting $t_3 = \frac{r^2}{t_1}$ we get

$$t_4 = \frac{r^2(t_1^2+r^2)}{(t_1^2+r^2)t_2} = \frac{r^2}{t_2}\ .$$

Thus, (2.14) it holds and *Corollary 4* is proved. □

Corollary 5. *Instead of* (2.33) *in* Corollary 4 *it can be put* $t_2t_4 = r^2$.

Corollary 6. *Instead of* (2.33) *in* Corollary 4 *it can be put*

$$\frac{t_1}{t_1^2+r^2} = \frac{t_3}{t_3^2+r^2}\ . \tag{2.34}$$

Proof. From (2.34) it follows

$$t_1 t_3 (t_1 - t_3) = r^2 (t_1 - t_3).$$

□

Let us remark that $t_1 = t_3$ only if $d = 0$, and in this case it holds $t_1 = t_3 = r$, $t_1 t_3 = r^2$.

Corollary 7. *Instead of* (2.33) *in* Corollary 4 *can be put*

$$\frac{t_2}{t_2^2 + r^2} = \frac{t_4}{t_4^2 + r^2} \ . \tag{2.35}$$

Corollary 8. *Instead of* (2.33) *in* Corollary 4 *can be put*

$$\frac{t_1^2 - r^2}{t_1^2 + r^2} = \frac{r^2 - t_3^2}{r^2 + t_3^2} \ .$$

Corollary 9. *If* (2.33) *is fulfilled, then*

$$\prod_{i=1}^{4} \sin \alpha_i = \frac{2r^2}{R^2 + d^2} \ ,$$

where $\alpha_i =$ *measure of* $\sphericalangle A_{i-1} A_i A_{i+1}$ *(Of course,* $A_0 = A_4$*).*

Proof. As

$$\sin \alpha_i = \frac{2rt_i}{t_i^2 + r^2} = \frac{2rt_i}{t_i^2 + t_i t_{i+2}} = \frac{2r}{t_i + t_{i+2}} \ ,$$

we can write

$$\prod_{i=1}^{4} \sin \alpha_i = \frac{16r^4}{[(t_1 + t_3)(t_2 + t_4)]^2} = \frac{4r^4}{(R^2 - d^2)^2} = \frac{2r^2}{R^2 + d^2} \ ,$$

since $(t_1 + t_3)(t_2 + t_4) = t_1 t_2 + t_2 t_3 + t_3 t_4 + t_4 t_1 = 2(R^2 - d^2)$ and holds (1.1). □

Corollary 10. *It holds*

$$\sum_{i=1}^{4} \sin \alpha_i \sin \alpha_{i+1} = \frac{8r^2}{R^2 - d^2} \ .$$

Corollary 11. *It holds*

$$\sum_{i=1}^{4} \cos \alpha_i \cos \alpha_{i+1} = 0.$$

Proof. $\cos \alpha_i = \dfrac{t_i^2 - r^2}{t_i^2 + r^2}, \quad \cos \alpha_{i+2} = \dfrac{r^2 - t_i^2}{r^2 + t_i^2} \ .$ □

Corollary 12. *Let t_1, t_2, t_3 be any given lengths (in fact positive numbers). Then there are lengths t_4 and r such that*

$$(\sum_{i=1}^{4} t_i)r^2 = \sum_{i=1}^{4} t_i t_{i+1} t_{i+2}, \quad t_1 t_2 t_3 t_4 = r^4. \tag{2.36}$$

Proof. From

$$t_4 = \frac{t_1 t_2 t_3 - r^2(t_1 + t_2 + t_3)}{r^2 - t_1 t_2 - t_2 t_3 - t_3 t_1}, \quad t_4 = \frac{r^4}{t_1 t_2 t_3}$$

we get the following cubic equation for r^2

$$r^6 - r^4(t_1 t_2 + t_2 t_3 + t_3 t_1) + r^2(t_1 + t_2 + t_3)t_1 t_2 t_3 - t_1^2 t_2^2 t_3^2 = 0.$$

Its roots are given by

$$(r^2)_1 = t_1 t_2, \quad (r^2)_2 = t_2 t_3, \quad (r^2)_3 = t_3 t_1.$$

□

Corollary 13. *If* (2.36) *holds, then there are three possibilities:*

$$t_1 t_2 = t_3 t_4, \quad t_2 t_3 = t_4 t_1, \quad t_1 t_3 = t_2 t_4.$$

Proof. According to *Corollary 12*, it holds

$$(t_4)_1 = \frac{t_1 t_2}{t_3}, \quad (t_4)_2 = \frac{t_2 t_3}{t_1}, \quad (t_4)_3 = \frac{t_3 t_1}{t_2}.$$

□

In the third case we have a bicentric quadrilateral.

Corollary 14. *If the first part of* (2.36) *holds, then*

$$t_1 t_2 t_3 t_4 = r^4 \Longleftrightarrow \sum_{i=1}^{4} \frac{r}{t_i} = \sum_{i=1}^{4} \frac{t_i}{r} .$$

Corollary 15. *All of the bicentric quadrilaterals which have the same incircle and the same circumcircle have the same product of diagonals. In other words, if $A_1A_2A_3A_4$ is a bicentric quadrilateral, then*

$$|A_1A_3| \cdot |A_2A_4| = 2(R^2 + 2r^2 - d^2)$$

Proof. Since

$$|A_1A_3| \cdot |A_2A_4| = (t_1 + t_3)(t_2 + t_4)\frac{2R^2}{R^2 + d^2} = 2(R^2 - d^2)\frac{2R^2}{R^2 + d^2}$$

it is easy to show that

$$2(R^2 - d^2) \cdot \frac{2R^2}{R^2 + d^2} - 2(R^2 + 2r^2 - d^2) = 0 \Longleftrightarrow (R^2 - d^2)^2 - 2r^2(R^2 + d^2) = 0.$$

□

3. The case when a quadrilateral is a tangential one in relation to an excircle

Let $A_1A_2A_3A_4$ be a tangential quadrilateral such that there is a circle C_1 with the property that

$$|A_iA_{i+1}| = |t_i - t_{i+1}|, \quad i = 1, 2, 3, 4 \tag{3.1}$$

where t_i is the length of the tangent drawn from A_i to C_1 (see *Figure 3.1*).

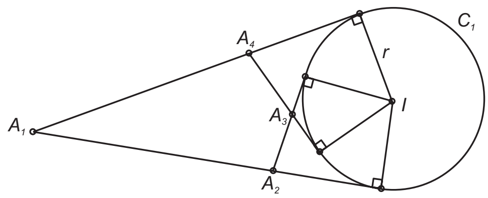

Figure 3.1.

Such a tangential quadrilateral, for convenience in the following expression, will be called ex-tangential quadrilateral. In the case when $A_1A_2A_3A_4$ is also a chordal one, then such a quadrilateral will be called ex-bicentric quadrilateral. The following notation will be used.

If $A_1A_2A_3A_4$ is a considered ex-bicentric quadrilateral, then its excircle is denoted by C_1, circumcircle by C_2, radius of C_1 by r, radius of C_2 by R, center of C_1 by I, center of C_2 by O, distance between I and O by d.

As it is well-known, the Fuss' relation (1.1) also holds for ex-bicentric quadrilaterals. In this connection let us remark that from (1.1) it follows

$$d^2 = R^2 + r^2 \pm \sqrt{4R^2r^2 + r^4},$$

and that for ex-bicentric quadrilaterals holds

$$d^2 = R^2 + r^2 + \sqrt{4R^2r^2 + r^4}, \tag{3.2}$$

whereas for bicentric quadrilateral considered in the preceding section holds

$$d^2 = R^2 + r^2 - \sqrt{4R^2r^2 + r^4}. \tag{3.3}$$

Also let us remark that circles C_1 and C_2 are not intersecting in the case of ex-bicentric quadrilateral since from (3.2) it follows

$$d^2 > R^2 + r^2 + 2Rr \quad \text{or} \quad d > R + r.$$

Now we can prove the following theorem.

Theorem 2. *Let $A_1A_2A_3A_4$ be any given ex-tangential quadrilateral and let t_1, t_2, t_3, t_4 be its tangent lengths such that*

$$|t_i - t_{i+1}| = |A_iA_{i+1}|, \quad i = 1, 2, 3, 4. \tag{3.4}$$

Then this quadrilateral is also a chordal one if and only if

$$\frac{|A_1A_3|}{t_1+t_3} = \frac{|A_2A_4|}{t_2+t_4} = \sqrt{k}, \tag{3.5}$$

where

$$0 < k < 1. \tag{3.6}$$

Proof. First we suppose that (3.5) holds. From *Figure 3.2* we see that the equality $|A_1A_3|^2 = k(t_1+t_3)^2$ can be written as

$$|A_1A_2|^2 + |A_2A_3|^2 - 2|A_1A_2||A_2A_3|\cos\alpha_2 = k(t_1+t_3)^2$$

or

$$(t_1-t_2)^2 + (t_2-t_3)^2 + 2(t_1-t_2)(t_2-t_3)\frac{t_2^2-r^2}{t_2^2+r^2} = k(t_1+t_3)^2, \tag{3.7}$$

since

$$\cos\alpha_2 = -\cos 2\beta_2 = -\frac{t_2^2-r^2}{t_2^2+r^2}\ .$$

The equality $|A_1A_3|^2 = k(t_1+t_3)^2$ can also be written as

$$(t_1-t_4)^2 + (t_4-t_3)^2 + 2(t_1-t_4)(t_4-t_3)\frac{t_4^2-r^2}{t_4^2+r^2} = k(t_1+t_3)^2. \tag{3.8}$$

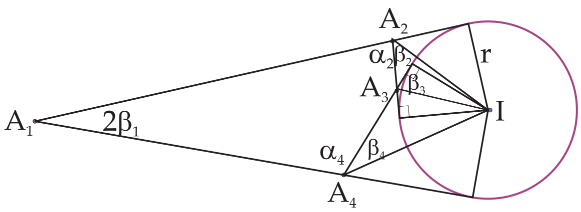

Figure 3.2.

In the same way can be seen that equality $|A_2A_4|^2 = k(t_2+t_4)^2$ can be written in the following two ways:

$$(t_1-t_2)^2 + (t_1-t_4)^2 - 2(t_1-t_2)(t_1-t_4)\tfrac{t_1^2-r^2}{t_1^2+r^2} = k(t_2+t_4)^2, \tag{3.9}$$

$$(t_3-t_2)^2 + (t_3-t_4)^2 - 2(t_3-t_2)(t_3-t_4)\tfrac{t_3^2-r^2}{t_3^2+r^2} = k(t_2+t_4)^2. \tag{3.10}$$

Solving equation(3.7) for t_2 we get

$$(t_2)_{1,2} = \frac{2r^2(t_1+t_3) \pm \sqrt{D}}{(t_1-t_3)^2 - k(t_1+t_3)^2 + 4r^2}\ , \tag{3.11}$$

where

$$D = 4r^4(t_1 + t_3)^2 - [(t_1 - t_3)^2 - k(t_1 + t_3)^2 + 4r^2][r^2(t_+ t_3)^2(1 - k)]. \tag{3.12}$$

It is easy to see that equation (3.7) in t_2 has the same solutions as equation (3.8) in t_4, that is

$$\{(t_2)_1, (t_2)_2\} = \{(t_4)_1, (t_4)_2\}.$$

Since equation (3.7) has t_2 as one solution, and equation (3.8) has t_4 as one solution, it follows that

$$\{(t_2)_1, (t_2)_2\} = \{(t_4)_1, (t_4)_2\} = \{t_2, t_4\}. \tag{3.13}$$

Putting $t_2 = (t_2)_1$, $t_4 = (t_2)_2$ in (3.9) we get

$$\frac{r^2(t_1 + t_3)^2(1 - k)}{(t_1 - t_3)^2 - k(t_1 + t_3)^2 + 4r^2} = t_1 t_3. \tag{3.14}$$

From (3.11) it follows

$$(t_2)_1(t_2)_2 = \frac{r^2(t_1 + t_3)^2(1 - k)}{(t_1 - t_3)^2 - k(t_1 + t_3)^2 + 4r^2}\ , \tag{3.15}$$

which according to (3.13) and (3.14) can be written as

$$t_1 t_3 = t_2 t_4. \tag{3.16}$$

Solving equation (3.14) for t_3 we get

$$(t_3)_1 = \frac{r^2}{t_1}, \quad (t_3)_2 = \frac{(1 + \sqrt{k})t_1}{1 - \sqrt{k}}, \quad (t_3)_3 = \frac{(1 - \sqrt{k})t_1}{1 + \sqrt{k}}\ . \tag{3.17}$$

First we consider the case when t_3 is given by

$$t_3 = \frac{r^2}{t_1}\ . \tag{3.18}$$

In this case, according to (3.16), it holds

$$t_1 t_3 = t_2 t_4 = r^2. \tag{3.19}$$

The proof that $A_1A_2A_3A_4$ is in this case also a chordal one is done in the same way as that in *Theorem 1.*

Let C_2 denote the circumcircle of $A_1A_2A_3A_4$ and let the other notation be stated as in the beginning of this section. The radius of C_2 is given by

$$R^2 = \frac{(ab + cd)(ac + bd)(ad + bc)}{16J^2}, \quad J^2 = abcd$$

where $a = t_1 - t_2$, $b = t_2 - t_3$, $c = t_4 - t_3$, $d = t_1 - t_4$. It can be found that

$$R^2 = \frac{[(r^2 + t_1)^2(r^2 + t_2^2)][(r^2 + t_1^2)(r^2 + t_2)^2 - 4r^2 t_1 t_2]}{16r^2 t_1^2 t_2^2}\ . \tag{3.20}$$

Now we shall prove that

$$k = \frac{2R^2}{R^2 + d^2} \,. \tag{3.21}$$

For this purpose in $(t_2)_1$ and $(t_2)_2$, given by (3.11), we shall put $\frac{2R^2}{R^2+d^2}$ instead of k, and $\frac{r^2}{t_1}$ instead of t_3. It can be found that

$$(t_2)_1 = \frac{(d^2 - R^2)t_1 + \sqrt{D}}{r^2 + t_1^2}, \quad (t_2)_2 = \frac{(d^2 - R^2)t_1 - \sqrt{D}}{r^2 + t_1^2} \tag{3.22}$$

where

$$D = (d^2 - R^2)^2 t_1^2 - r^2(r^2 + t_1^2)^2. \tag{3.23}$$

It is easy to check that $(t_2)_1 \cdot (t_2)_2 = r^2$ or, since (3.13) holds,

$$t_2 t_4 = r^2.$$

In the following lemma will be used lengths t_m and t_M given by

$$t_m = \sqrt{(d-R)^2 - r^2}, \quad t_M = \sqrt{(d+R)^2 - r^2}. \tag{3.24}$$

See *Figure 3.3*. It holds

$$t_m = |MN| = \sqrt{(d-R)^2 - r^2}, \quad t_M = |PQ| = \sqrt{(d+R)^2 - r^2}.$$

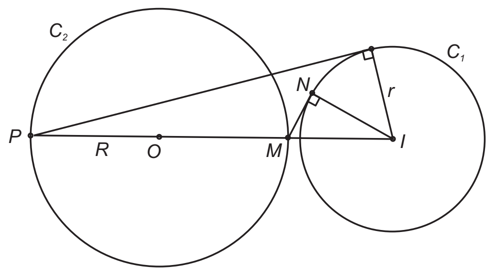

Figure 3.3.

Lemma 2. *Let u_1 be any given value (tangent length) such that*

$$t_m \le u_1 \le t_M, \tag{3.25}$$

and let u_2, u_3, u_4 be given by

$$u_2 = \frac{(d^2-R^2)u_1+\sqrt{D}}{r^2+u_1^2}\,, \tag{3.26}$$

$$u_3 = \frac{r^2}{u_1}\,, \tag{3.27}$$

$$u_4 = \frac{r^2}{u_2}\,, \tag{3.28}$$

where

$$D = (d^2 - R^2)^2 u_1^2 - r^2(r^2 + u_1^2)^2. \tag{3.29}$$

Then the ex-bicentric quadrilateral $B_1B_2B_3B_4$, where $|B_iB_{i+1}| = |u_i - u_{i+1}|$, $i = 1, 2, 3, 4$, has the same excircle and circumcircle as the considered quadrilateral $A_1A_2A_3A_4$.

Proof. Since in the expression of u_2 there appears the term $\sqrt{D}$, we have to prove that $D \geq 0$ for every u_1 such that $t_m \leq u_1 \leq t_M$. For this purpose, as can be readily seen, it is sufficient to prove that $D = 0$ for $u_1 = t_m$ and $u_1 = t_M$. The proof is as follows:

$$(d^2 - R^2)^2 t_m^2 - r^2(r^2 + t_m^2)^2 = (d - R)^2[(d^2 - R^2)^2 - 2r^2(d^2 + R^2)] = 0,$$
$$(d^2 - R^2)^2 t_M^2 - r^2(r^2 + t_M^2)^2 = (d - R)^2[(d^2 - R^2)^2 - 2r^2(d^2 + R^2)] = 0.$$

That C_1 is the excircle of $B_1B_2B_3B_4$ is clear from

$$r^2(u_1 - u_2 + u_3 - u_4) = -u_1u_2u_3 + u_2u_3u_4 - u_3u_4u_1 + u_4u_1u_2,$$

since $u_1u_3 = u_2u_3 = r^2$. (See relation (3.10) in [4].)

To prove that C_2 is the circumcircle of $B_1B_2B_3B_4$ we have to prove that

$$\frac{[(r^2 + u_1^2)(r^2 + u_2^2)][(r^2 + u_1^2)(r^2 + u_2^2) - 4r^2u_1u_2]}{16r^2u_1^2u_2^2} = R^2. \tag{3.30}$$

The proof is analogous to the proof of (2.28). The *Lemma 2* is proved. □

In connection with the relation (3.21) let us remark that in the case when (3.19) holds, then from each of the relations (3.7)-(3.10) follows k given by (3.21). So, starting from the relation (3.7), we can write:

$$(t_1^2 + r^2)(1 - k)t_2^2 - 4r^2t_1t_2 + r^2(t_1^2 + r^2)(1 - k) = 0,$$
$$1 - k = \tfrac{4r^2t_1t_2}{(r^2+t_1^2)(r^2+t_2^2)},$$

from which, since $t_2 = \frac{(d^2-R^2)t_1+\sqrt{D}}{r^2+t_1^2}$, we get

$$1 - k = \frac{2r^2}{d^2 - R^2}.$$

Putting $k = \frac{2R^2}{d^2+R^2}$ we have the equality

$$1 - \frac{2R^2}{d^2 + R^2} = \frac{2r^2}{d^2 - R^2},$$

since Fuss' relation $(d^2 - R^2)^2 = 2r^2(d^2 + R^2)$ holds.

Now we shall consider the other two solutions for t_3 given by (3.17), that is

$$(t_3)_2 = \frac{(1 + \sqrt{k})t_1}{1 - \sqrt{k}}, \quad (t_3)_3 = \frac{(1 + \sqrt{k})t_1}{1 - \sqrt{k}}\ .$$

Putting $(t_3)_2$ instead of t_3 in (3.11) we get

$$(t_2)_1 = \frac{(1 + \sqrt{k})t_1}{1 - \sqrt{k}}, \quad (t_2)_2 = t_1.$$

It is not difficult to see that from

$$\{t_1, t_2, t_3, t_4\} = \left\{t_1, \frac{(1+\sqrt{k})t_1}{1-\sqrt{k}}, \frac{(1+\sqrt{k})t_1}{1-\sqrt{k}}, t_1\right\}$$

follows that C_2 must be a point, that is, $t_1 = 0$.

In the same way can be seen that $(t_3)_3$ is possible only if $t_1 = 0$.

At the end we prove the following assertion: If $A_1A_2A_3A_4$ is an ex-bicentric quadrilateral, then

$$\frac{|A_1A_3|}{t_1+t_3} = \frac{|A_2A_4|}{t_2+t_4} = \sqrt{k}.$$

Proof. Let denote by F relation obtained from (3.8) putting

$$t_2 = \frac{(d^2-R^2)t_1+\sqrt{D}}{r^2+t_1^2}, \quad t_3 = \frac{r^2}{t_1}, \quad t_4 = \frac{r^2}{t_2}, \quad k = \frac{2R^2}{R^2+d^2},$$

where

$$D = (d^2-R^2)^2t_1^2 - r^2(r^2+t_1^2)^2.$$

Using computer algebra it is easy to show that

$$F \Longleftrightarrow (d^2-R^2)^2 - 2r^2(d^2+R^2),$$

which proves $|A_1A_3| = (t_1+t_3)\sqrt{k}$. In the same way can be proved that $|A_2A_4| = (t_2+t_4)\sqrt{k}$. □

This completes the proof of *Theorem 2.* □

Here are some of its corollaries.

Corollary 16. *Let $A_1A_2A_3A_4$ be an ex-bicentric quadrilateral and let $|A_iA_{i+1}| = |t_i - t_{i+1}|,\ i = 1,2,3,4$. Then*

$$R^2 = \frac{[(r^2+t_1^2)(r^2+t_2^2)][(r^2+t_1^2)(r^2+t_2^2) - 4r^2t_1t_2]}{16r^2t_1^2t_2^2}, \tag{3.31}$$

$$d^2 = \frac{[(r^2+t_1^2)(r^2+t_2^2)][(r^2+t_1^2)(r^2+t_2^2) + 4r^2t_1t_2]}{16r^2t_1^2t_2^2}. \tag{3.32}$$

The proof is analogous to the proof of Corollary 1.

Corollary 17. *It holds*

$$2(d^2-R^2) = \frac{(r^2+t_1^2)(r^2+t_2^2)}{t_1t_2} = t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1.$$

The proof is analogous to the proof of Corollary 2.

Corollary 18. *If $k = \frac{2R^2}{d^2+R^2}$ and* (1.1) *holds, then every positive solution of the system with equations* (3.7)-(3.10) *can be expressed such that following holds*

$$t_m \le t_1 \le t_M,$$

$$t_2 = \frac{(d^2-R^2)t_1+\sqrt{D}}{r^2+t_1^2}, \quad t_3 = \frac{r^2}{t_1}, \quad t_4 = \frac{r^2}{t_2}$$

where $D = (d^2 - R^2)^2 t_1^2 - r^2(r^2 + t_1^2)^2$.

Corollary 19. *Let* $A_1A_2A_3A_4$ *be any given ex-tangential quadrilateral and let* t_1, t_2, t_3, t_4 *be lengths of its tangents such that*

$$|t_i - t_{i+1}| = |A_iA_{i+1}|, \quad i = 1, 2, 3, 4.$$

Then this quadrilateral is also a chordal one iff

$$t_1t_3 = r^2, \tag{3.33}$$

where r *is the radius of the excircle of* $A_1A_2A_3A_4$.

Proof. From

$$(t_1 - t_2 + t_3 - t_4)r^2 = -t_1t_2t_3 + t_2t_3t_4 - t_3t_4t_1 + t_4t_1t_2$$

it follows

$$t_4 = \frac{t_1t_2t_3 + r^2(t_1 - t_2 + t_3)}{r^2 + t_1t_2 + t_2t_3 - t_3t_1}.$$

Putting $t_3 = \frac{r^2}{t_1}$ we get

$$t_4 = \frac{r^2(t_1 + r^2)}{t_2(t_1^2 + r^2)} = \frac{r^2}{t_2}.$$

□

Corollary 20. *Instead of the relation given by* (3.33) *each of the following five relations can be put:*

$$t_2t_4 = r^2,$$

$$\frac{t_1}{r^2 + t_1^2} = \frac{t_3}{r^2 + t_3^2}, \quad \frac{t_2}{r^2 + t_2^2} = \frac{t_4}{r^2 + t_4^2},$$

$$\frac{t_1^2 - r^2}{t_1^2 + r^2} = \frac{r^2 - t_3^2}{r^2 + t_3^2}, \quad \frac{t_2^2 - r^2}{t_2^2 + r^2} = \frac{r^2 - t_4^2}{r^2 + t_4^2}.$$

Corollary 21. *Let* (3.33) *be fulfilled. Then*

$$\sum_{i=1}^{4} \sin\alpha_i = \frac{2r^2}{d^2 + R^2},$$

where $\alpha_i =$ *measure of* $\sphericalangle A_{i-1}A_iA_{i+1}$, $A_0 = A_4$.

Proof. Analogous to the proof of *Corollary 9.* □

Corollary 22. *It holds*

$$\sum_{i=1}^{4} \sin\alpha_i \sin\alpha_{i+1} = \frac{8r^2}{d^2 - r^2}.$$

Corollary 23. *It holds*

$$\sum_{i=1}^{4} \cos\alpha_i \cos\alpha_{i+1} = 0.$$

Corollary 24. *Let t_1, t_2, t_3 be any given lengths (in fact positive numbers). Then there are lengths t_4 and r such that*

$$(t_1 - t_2 + t_3 - t_4)r^2 = -t_1t_2t_3 + t_2t_3t_4 - t_3t_4t_1 + t_4t_1t_2, \tag{3.34}$$

$$t_1t_2t_3t_4 = r^4 \tag{3.35}$$

Proof. Analogous to the proof of *Corollary 12*. Here we have the equation

$$r^6 + r^4(t_1t_2 + t_2t_3 - t_3t_1) - r^2(t_1 - t_2 + t_3)t_1t_2t_3 - t_1^2t_2^2t_3^2 = 0,$$

whose roots for r^2 are given by

$$(r^2)_1 = -t_1t_2, \quad (r^2)_2 = -t_2t_3, \quad (r^2)_3 = t_1t_3.$$

□

Corollary 25. *Let* (3.34) *and* (3.35) *be fulfilled. Then*

$$t_1t_2t_3t_4 = r^4 \Longleftrightarrow \sum_{i=1}^{4}(-1)^i\frac{r}{t_i} = \sum_{i=1}^{4}(-1)^i\frac{t_i}{r}.$$

Corollary 26. *All of ex-bicentric quadrilaterals which have the same excircle and the same circumcircle have the same product of diagonals. In other words, if $A_1A_2A_3A_4$ is an ex-bicentric quadrilateral, then*

$$|A_1A_3| \cdot |A_2A_4| = 2(d^2 - 2r^2 - R^2).$$

Proof. The proof is obtained in the same way as the proof of *Corollary 15*, namely it holds

$$2(d^2 - R^2) \cdot \frac{2R^2}{d^2 + R^2} - 2(d^2 - 2r^2 - R^2) = 0 \Longleftrightarrow (d^2 - R^2)^2 - 2r^2(d^2 + R^2) = 0.$$

□

In this connection let us remark that in [4, Theorem 3.2] it is proved that

$$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = 2(d^2 - R^2).$$

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