Quantum heaps, cops and heapy categories

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Abstract. *A heap is a structure with a ternary operation which is intuitively a group with a forgotten unit element. Quantum heaps are associative algebras with a ternary cooperation which are to Hopf algebras what heaps are to groups, and, in particular, the category of copointed quantum heaps is isomorphic to the category of Hopf algebras. There is an intermediate structure of a cop in a monoidal category which is in the case of vector spaces to a quantum heap about what a coalgebra is to a Hopf algebra. The representations of Hopf algebras make a rigid monoidal category. Similarly, the representations of quantum heaps make a kind of category with ternary products, which we call a heapy category.*

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1. Classical background: heaps

A reference for this section is [1].

1.1 *Before forgetting: group as heap.* Let G be a group. Then the ternary operation $t : G \times G \times G \to G$ given by

$$t(a, b, c) = ab^{-1}c, \tag{1}$$

satisfies the following relations:

$$\begin{array}{l} t(b, b, c) = c = t(c, b, b) \\ t(a, b, t(c, d, e)) = t(t(a, b, c), d, e) \end{array} \tag{2}$$

A heap (H, t) is a pair of nonempty set H and a ternary operation $t : H \times H \times H$ satisfying relation (2). A morphism $f : (H, t) \to (H', t')$ of heaps is a set map $f : H \to H'$ satisfying $t' \circ (f \times f \times f) = f \circ t$.

Thus every group has its canonical heap, what defines a faithful functor Heap : $\underline{\text{Groups}} \to \underline{\text{Heaps}}$.

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1.2 The automorphism group of a heap (H, t) denoted by $\mathrm{Aut}H$ is the subgroup of a symmetric group of H consisting of the maps of the form $t(\cdot, a, b) : H \to H$ where $a, b \in H$, and $\cdot$ is a place-holder. Composition (group operation) satisfies

$$t(\cdot, c, d) \cdot_{\mathrm{Aut}H} t(\cdot, a, b) = t(t(\cdot, c, d), a, b) = t(\cdot, c, t(d, a, b)).$$

The rightmost equality implies that the result of the composition is in $\mathrm{Aut}H$. The inverse of $t(\cdot, a, b)$ is $t(\cdot, b, a)$ by (2) and the unit is $t(\cdot, x, x)$ (independent of $x \in H$).

1.3 The following are equivalent

(i) bijections $t(\cdot, a, b)$ and $t(\cdot, a', b')$ are the same maps,

(ii) $t(a, a', b') = b$,

(iii) $t(b, b', a') = a$.

Proof. (ii) follows from (i) and $t(a, a, b) = b$.

(iii) follows from (ii) by applying $t(\cdot, b', a')$ on the right. Similarly, (ii) follows from (iii).

(i) follows from (ii) by the calculation:

$$t(x, a', b') = t(t(x, a, a), a', b') = t(x, a, t(a, a', b')) = t(x, a, b).$$

1.4 The reader should conclude noticing that the defining action of $\mathrm{Aut}H$ is transitive (by $t(a, a, b) = b$) and free (if $t(a, b, c) = a$, then by last $t(x, b, c) = x$ for each x, in particular $t(b, b, c) = b$ but also $t(b, b, c) = c$ by (2).

1.5 If we started with a group G, then we can recover it *up to an isomorphism* from the corresponding heap (as the automorphism group of the heap). Indeed, then $t(\cdot, e, a)$ is the multiplication by a. A byproduct of this construction is that we now know that all the other possible identities for the group-induced ternary operation (1) follow from (2).

1.6 Similarly, every heap is isomorphic (in the category of heaps, where morphisms are defined as usual for algebraic structures) to the heap of operation (1) on its automorphism group. However, the isomorphism is not natural but one needs to specify which element will be unity. In other words, we have a natural isomorphism (not only equivalence) of the category of groups with the category of **pointed heaps**, that is heaps with a nullary operation $\star$ and morphisms respecting also this operation.

The isomorphism in question is $H \ni a \mapsto t(\cdot, \star, a) \in \iota(H) = \mathrm{Heap}(\mathrm{Aut}H)$.

$$\begin{aligned} t(\cdot, \star, a)[t(\cdot, \star, b)]^{-1} t(\cdot, \star, c) &= t(\cdot, \star, a) t(\cdot, b, \star) t(\cdot, \star, c) \\ &= t(\cdot, \star, a) t(\cdot, b, c) \\ &= t(\cdot, \star, t(a, b, c)). \end{aligned}$$

For the morphism of heaps $f : (H, t) \to (H', t')$ we define $\iota(f)(t(\cdot, \star, a)) = t'(\cdot, \star, f(a))$ and ι becomes a covariant functor.

The identities for the ternary operation t play an important role in universal algebra (theory of Mal'cev algebras and Mal'cev categories).

2. Cops

2.1 Let $(\mathcal{C}, \otimes, \mathbf{1})$, or $\mathcal{C}$ for short, be a strict monoidal category with a unit object $\mathbf{1}$. A **cop** C in $(\mathcal{C}, \otimes, \mathbf{1})$ is a pair (C, τ), where C is an object in $\mathcal{C}$ and $\tau : C \to C \otimes C \otimes C$ a morphism in $\mathcal{C}$ satisfying the law

$$(\mathrm{id} \otimes \mathrm{id} \otimes \tau) \circ \tau = (\tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ \tau. \tag{3}$$

Let $(\mathcal{C}, \otimes, \mathbf{1}, \sigma)$ be a strict symmetric monoidal category and C a monoid (=algebra) object in that category, i.e. C is equipped with a product $\mu : C \otimes C \to C$ and a unit morphism $\eta : \mathbf{1} \to C$ satisfying standard axioms. Then an **opposite monoid** C_{op} is the same object C equipped with product $\sigma_{C,C} \circ \mu$ and with the same unit map η. A **symmetric cop monoid** C in a strict symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, \sigma)$ is a *monoid* object C with a *morphism of monoids* $\tau : C \to C \otimes C_{\mathrm{op}} \otimes C$ satisfying the law (3). Here the tensor product has a usual tensor product structure of a monoid in a strict symmetric monoidal category (for two monoids A and B one takes $(\mu \otimes \mu) \circ (\mathrm{id} \otimes \sigma_{B,A} \otimes \mathrm{id})$ as a product on $A \otimes B$).

2.2 A **counit** of a cop C in $\mathcal{C}$ is a morphism $\epsilon : C \to \mathbf{1}$ such that

$$(\mathrm{id} \otimes \epsilon \otimes \epsilon) \circ \tau = \mathrm{id} = (\epsilon \otimes \epsilon \otimes \mathrm{id}) \circ \tau, \tag{4}$$

where the identification morphism $\mathbf{1} \otimes \mathbf{1} \otimes C \equiv C \equiv C \otimes \mathbf{1} \otimes \mathbf{1}$ is used.

Our interest is in the cops in the (symmetric) monoidal category of vector spaces $\underline{Vec}$ or supervector spaces $\underline{SVec}$ over some fixed field.

2.3 A **coheap monoid** in a symmetric monoidal category is a symmetric cop monoid such that $(\mathrm{id} \otimes \mu) \circ \tau = (\mu \otimes \mathrm{id}) \circ \tau = \mathrm{id}$, where the identification $C \otimes \mathbf{1} = C = \mathbf{1} \otimes C$ is implicitly used.

2.3.a A **character** of a monoid (C, μ, η) in a strict monoidal category is a morphism $\epsilon : C \to \mathbf{1}$ such that $\epsilon \circ \eta = \mathrm{id}_{\mathbf{1}}$ and $(\epsilon \otimes \epsilon) = \epsilon \circ \mu$.

2.3.b A **character** of a symmetric cop monoid C is any character of (C, η, μ) in $\mathcal{C}$.

2.3.c *Proposition.* A **character** of a coheap monoid is automatically a counit of the underlying cop.

The proof is straightforward:

$$\begin{aligned}(\mathrm{id} \otimes \epsilon \otimes \epsilon)\tau &= (\mathrm{id} \otimes (\epsilon \circ \mu))\tau = (\mathrm{id} \otimes \epsilon)(\mathrm{id} \otimes \mu)\tau = (\mathrm{id} \otimes \epsilon)(\mathrm{id} \otimes \eta) = \mathrm{id},\\ (\epsilon \otimes \epsilon \otimes \mathrm{id})\tau &= ((\epsilon \circ \mu) \circ \mathrm{id})\tau = (\mathrm{id} \otimes \epsilon)(\mu \otimes \mathrm{id})\tau = (\epsilon \otimes \mathrm{id})(\eta \otimes \mathrm{id}) = \mathrm{id},\end{aligned}$$

where again obvious identifications are implicitly used, e.g. $\mathrm{id}_{\mathbf{1}} \otimes \mathrm{id} \cong \mathrm{id}$.

2.3.d A **copointed cop** is a pair (C, ϵ) of a cop C and a counit ϵ of C. A **copointed coheap monoid** is a coheap monoid with a *character* ϵ of C. Warning: a copointed cop which is also a coheap is not necessarily a copointed coheap, as the counit does not need to be a character of a coheap. Clearly, the above theory may be modified for nonstrict monoidal categories.

3. Quantum heaps

3.1 Heap is morally a group with a forgotten unit. Quantum heap is morally a Hopf algebra with a forgotten counit. We fix a ground field $\mathbf{k}$ throughout.

3.2 Quantum heap is an associative unital **k**-algebra (H, μ, η) together with a ternary algebra cooperation

$$\tau : H \to H \otimes H_{\rm op} \otimes H,$$

satisfying the following properties

$$\begin{aligned} (\mathrm{id} \otimes \mathrm{id} \otimes \tau)\tau &= (\tau \otimes \mathrm{id} \otimes \mathrm{id})\tau \\ (\mathrm{id} \otimes \mu)\tau &= \mathrm{id} \otimes 1_H \\ (\mu \otimes \mathrm{id})\tau &= 1_H \otimes \mathrm{id} \end{aligned} \tag{5}$$

Moreover, τ is required to be an algebra homomorphism from H into $H \otimes H_{\rm op} \otimes H$, where $H_{\rm op}$ has the opposite algebra structure and the tensor product has a usual algebra structure. In other words, it is a coheap in the symmetric monoidal category of vector spaces.

We use a heap analogue of the Sweedler notation:

$$\tau(h) = \sum h^{(1)} \otimes h^{(2)} \otimes h^{(3)},$$

and because of the first of the above identities, it is justified to extend it to any odd number ≥ 3 factors, e.g.

$$(\mathrm{id} \otimes \mathrm{id} \otimes \tau)\tau(h) = \sum h^{(1)} \otimes h^{(2)} \otimes h^{(3)} \otimes h^{(4)} \otimes h^{(5)}.$$

In this paper a heap-Sweedler notation has *upper* indices while the Sweedler notation for coalgebras will have *lower* indices. Heap-Sweedler indices extend to any odd number ≥ 3 of indices. The requirement that τ is an algebra homomorphism from H into $H \otimes H_{\rm op} \otimes H$ is expressed in terms of a heap-Sweedler notation as

$$\tau(hg) = \sum (hg)^{(1)} \otimes (hg)^{(2)} \otimes (hg)^{(3)} = \sum h^{(1)} g^{(1)} \otimes g^{(2)} h^{(2)} \otimes h^{(3)} g^{(3)} = \tau(h)\tau(g)$$

3.3 A morphism of quantum heaps is a homomorphism ϕ of unital algebras such that $\tau(\phi(h)) = (\phi \otimes \phi \otimes \phi)\tau(h)$. Quantum heaps make a category $\underline{QHeaps}$.

3.4 We define a covariant functor $QHeap$ from the category of Hopf algebras $\underline{Hopf - Alg}$ to $\underline{QHeaps}$. The underlying associative algebra of the object is the same, and the quantum heap operation is given by

$$\tau(h) = \sum h_{(1)} \otimes S h_{(2)} \otimes h_{(3)},$$

This functor is identity on morphisms. However, not all morphisms of quantum heaps are in the image of this functor (for example, take a coordinate ring of $SL(n)$ and permute the rows of the matrix of generators – it will induce a morphism between the canonical and the obvious "permuted" quantum heap structures).

Let us prove that this functor has the required codomain, i.e. indeed the output of functor $QHeap$ is in $\underline{QHeaps}$:

$$\begin{aligned} (\mathrm{id} \otimes \mathrm{id} \otimes \tau)\tau(h) &= \sum h_{(1)} \otimes S h_{(2)} \otimes (h_{(3)} \otimes S h_{(4)} \otimes h_{(5)}) \\ &= \sum (h_{(1)} \otimes S h_{(2)} \otimes h_{(3)}) \otimes S h_{(4)} \otimes h_{(5)} \\ &= (\tau \otimes \mathrm{id} \otimes \mathrm{id})\tau(h) \end{aligned}$$

$$\begin{aligned}(\mathrm{id}\otimes\mu)\tau(h) &= \sum h_{(1)}\otimes Sh_{(2)}h_{(3)} = h\otimes 1\\ (\mu\otimes\mathrm{id})\tau(h) &= \sum h_{(1)}Sh_{(2)}\otimes h_{(3)} = 1\otimes h\end{aligned}$$

$$\begin{aligned}\tau(hg) &= \sum (hg)_{(1)}\otimes S((hg)_{(2)})\otimes (hg)_{(3)}\\ &= \sum h_{(1)}g_{(1)}\otimes (S(g_{(2)})\cdot_H S(h_{(2)}))\otimes h_{(3)}g_{(3)}\\ &= \sum h_{(1)}g_{(1)}\otimes (S(h_{(2)})\cdot_{H_{\mathrm{op}}} S(g_{(2)}))\otimes h_{(3)}g_{(3)}\\ &= \tau(h)\tau(g).\end{aligned}$$

3.5 Let now $\mathbf{A} = (A,\mu,\eta,\tau)$ be a quantum heap and $\epsilon : A\to\mathbf{k}$ any unital $\mathbf{k}$-algebra homomorphism. Such a pair $(\mathbf{A},\epsilon)$ is called a **copointed quantum heap**. A morphism of copointed quantum heaps is a morphism $\phi:\mathbf{A}\to\mathbf{A}'$ of quantum heaps such that $\epsilon'\circ\phi=\epsilon$. Copointed quantum heaps make a category $\underline{Q\star Heaps}$ (notation from the view of them as quantum "pointed heaps").

Now we define a functor $\chi : \underline{Q\star Heaps}\to\underline{Hopf-Alg}$.

For a given heap H thus define

$$\Delta : H\to H\otimes\mathbf{k}\otimes H\cong H\otimes H \quad\text{by}\quad \Delta = (\mathrm{id}\otimes\epsilon\otimes\mathrm{id})\tau,$$

Then Δ is coassociative:

$$(\mathrm{id}\otimes\Delta)\Delta(h) = \sum h^{(1)}\otimes\epsilon(h^{(2)})h^{(3)}\epsilon(h^{(4)})\otimes h^{(5)} = (\Delta\otimes\mathrm{id})\Delta(h).$$

Moreover, ϵ becomes a counit for coalgebra (H,Δ,ϵ). Indeed,

$$\begin{aligned}(\mathrm{id}\otimes\epsilon)\Delta(h) &= h^{(1)}\otimes\epsilon(\epsilon(h^{(2)})h^{(3)}) = h^{(1)}\otimes\epsilon(h^{(2)})\epsilon(h^{(3)})\\ &= h^{(1)}\otimes\epsilon(h^{(2)}h^{(3)}) = (\mathrm{id}\otimes\epsilon)(h^{(1)}\otimes h^{(2)}h^{(3)})\\ &= (\mathrm{id}\otimes\epsilon)(h\otimes 1) = h\otimes\epsilon(1)\cong h,\end{aligned}$$

$$\begin{aligned}(\epsilon\otimes\mathrm{id})\Delta(h) &= \epsilon(h^{(1)}\epsilon(h^{(2)}))\otimes h^{(3)} = \epsilon(h^{(1)})\epsilon(h^{(2)})\otimes h^{(3)}\\ &= \epsilon(h^{(1)}h^{(2)})\otimes h^{(3)} = (\epsilon\otimes\mathrm{id})(h^{(1)}h^{(2)}\otimes h^{(3)})\\ &= (\mathrm{id}\otimes\epsilon)(1\otimes h) = h\otimes\epsilon(1)\cong h.\end{aligned}$$

As τ is an algebra map, we can easily see that the map Δ is an algebra homomorphism too, so we have a bialgebra:

$$\begin{aligned}\Delta(h)\Delta(g) &= \sum (h^{(1)}\otimes\epsilon(h^{(2)})h^{(3)})(g^{(1)}\otimes\epsilon(g^{(2)})g^{(3)})\\ &= \sum h^{(1)}g^{(1)}\otimes\epsilon(h^{(2)})h^{(3)}\epsilon(g^{(2)})g^{(3)}\\ &= \sum h^{(1)}g^{(1)}\otimes\epsilon(g^{(2)}h^{(2)})h^{(3)}g^{(3)}\\ &= \sum (hg)^{(1)}\otimes\epsilon((hg)^{(2)})(hg)^{(3)}\\ &= \Delta(hg).\end{aligned}$$

We can also define the antipode

$$Sh = \sum \epsilon(h^{(1)})h^{(2)}\epsilon(h^{(3)}).$$

Indeed,

$$\begin{aligned}
\cdot(\mathrm{id}\otimes S)\Delta(h) &= h^{(1)}S(\epsilon(h^{(2)})h^{(3)}) = h^{(1)}\epsilon(h^{(2)})S(h^{(3)}) \\
&= h^{(1)}\epsilon(h^{(2)})\epsilon(h^{(3)})h^{(4)}\epsilon(h^{(5)}) \\
&= [(\mathrm{id}\otimes\epsilon)(h^{(1)}\otimes h^{(2)}h^{(3)})]h^{(4)}\epsilon(h^{(5)}) \\
&= [(\mathrm{id}\otimes\epsilon)(h^{(1)}\otimes 1)]h^{(2)}\epsilon(h^{(3)}) \\
&= h^{(1)}h^{(2)}\epsilon(h^{(3)}) \\
&= (\mathrm{id}\otimes\epsilon)(h^{(1)}h^{(2)}\otimes h^{(3)}) \\
&= (\mathrm{id}\otimes\epsilon)(1\otimes h) \\
&= \epsilon(h)1_H
\end{aligned}$$

and similarly for S at the left:

$$\begin{aligned}
\cdot(S\otimes\mathrm{id})\Delta(h) &= S(h^{(1)}\epsilon(h^{(2)}))h^{(3)} = Sh^{(1)}\epsilon(h^{(2)})h^{(3)} \\
&= \epsilon(h^{(1)})h^{(2)}\epsilon(h^{(3)})\epsilon(h^{(4)})h^{(5)} \\
&= \epsilon(h^{(1)})h^{(2)}[(\epsilon\otimes\mathrm{id})(h^{(3)}h^{(4)}\otimes h^{(5)})] \\
&= \epsilon(h^{(1)})h^{(2)}[(\epsilon\otimes\mathrm{id})(1\otimes h^{(3)})] \\
&= \epsilon(h^{(1)})h^{(2)}h^{(3)} \\
&= (\epsilon\otimes\mathrm{id})(h^{(1)}\otimes h^{(2)}h^{(3)}) \\
&= (\epsilon\otimes\mathrm{id})(h\otimes 1) \\
&= \epsilon(h)1_H
\end{aligned}$$

Thus we have obtained correspondence from copointed quantum heaps into Hopf algebras where the underlying set is the same. We leave to the reader to check that a map of underlying sets is a map of copointed quantum heaps iff it is a map of Hopf algebras obtained via this correspondence. Thus the correspondence extends to a functor.

3.6 Main theorem. *The two functors constructed above are mutually inverse isomorphisms of categories: copointed quantum heaps ⇔ Hopf algebras.*

Proof. We need to show that the two functors are inverse. The underlying algebra is identical, so we have to show that one composition of the two functors does not change the coproduct Δ and the other composition does not change the cooperation τ.

Start with a copointed heap $(H, \mu, \eta, \tau, \epsilon)$. Then

$$\begin{aligned}
\tau'(h) &= (\mathrm{id} \otimes S \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta(h) \\
&= (\mathrm{id} \otimes S \otimes \mathrm{id})(\sum h^{(1)} \otimes \epsilon(h^{(2)})h^{(3)}\epsilon(h^{(4)}) \otimes h^{(5)}) \\
&= \sum h^{(1)} \otimes \epsilon(h^{(2)})[\epsilon(h^{(3)})h^{(4)}\epsilon(h^{(5)})]\epsilon(h^{(6)}) \otimes h^{(7)} \\
&= \sum h^{(1)} \otimes \epsilon(h^{(2)}h^{(3)})h^{(4)}\epsilon(h^{(5)}h^{(6)}) \otimes h_{(7)} \\
&= \sum [(\mathrm{id} \otimes \epsilon)(h^{(1)} \otimes h^{(2)}h^{(3)})]h^{(4)}[(\epsilon \otimes \mathrm{id})(h^{(2)}h^{(3)} \otimes h^{(4)})] \\
&= \sum [(\mathrm{id} \otimes \epsilon)(h^{(1)} \otimes 1)] \cdot h^{(2)} \cdot [(\epsilon \otimes \mathrm{id})(1 \otimes h^{(3)})] \\
&= \sum h^{(1)} \otimes h^{(2)} \otimes h^{(3)} = \tau(h).
\end{aligned}$$

Start with a Hopf algebra $(H, \mu, \eta, \Delta, \epsilon)$. Then

$$\begin{aligned}
\Delta'(h) &= \sum h^{(1)} \otimes \epsilon(h^{(2)})h^{(3)} = \sum h_{(1)} \otimes \epsilon(Sh_{(2)})h_{(3)} \\
&= \sum h_{(1)} \otimes \epsilon(h_{(2)})h_{(3)} = \sum h_{(1)} \otimes h_{(2)} = \Delta(h).
\end{aligned}$$

□

4. The context and the work of Grunspan

The present author has discovered the notion of a quantum heap and proved the main theorem of this article in Spring 2000, and this entered then as Ch. 9 in his thesis [6] on coset spaces for quantum groups. The coset spaces were constructed there using coactions of Hopf algebras and gluing using noncommutative localizations ([6, 7, 10, 9, 8]). This included nonaffine generalization of torsors for Hopf algebras.

Later, and independently, Grunspan ([2]) considered an approach to torsors via paralelogram approach and dualized this to a noncommutative setup. He cites Kontsevich ([3]) for using earlier this approach in commutative affine case. In other words, he studied certain coheap monoids in the symmetric monoidal category of bimodules over a fixed commutative "base" algebra over a ground field. When the base algebra is the ground field, this is the same as our earlier introduced concept of quantum heap. There are few differences however, in this case as well. The first is minor, namely Grunspan introduced the axiomatics with one additional axiom, but Schauenburg [5] proved later that this axiom is superfluous. The second difference is in the scope of work. Our main theorem concerns the relation to Hopf algebras, namely the role of forgetting and then reintroducing an algebra character. Grunspan overlooks this theorem, but proceeds with a study of bitorsor picture, with a construction of a left and right "automorphism" quantum heaps, without a need to specify a character.

Schauenburg ([5]) proves that Grunspan's torsors are essentially Hopf-Galois extensions. My approach ([7, 10]), to glue Hopf-Galois extensions along coaction compatible noncommutative biflat localizations is more general, in the sense that it gives a larger class of objects which have the right to be called noncommutative torsors. One can extend this localization picture to Grunspan's formalism by introducing the localizations compatible, with the ternary (co)operation τ, to develop a sort of gluing theory as well.

5. Heapy categories

Given a quantum heap (H,μ,η,τ), the category of representations of its underlying algebra inherits an additional structure: a ternary product on objects. Namely, the tensor product of three H-modules A,B,C also has an H-action: if $a\otimes b\otimes c\in A\otimes B\otimes C$, then $h(a\otimes b\otimes c):=h^{(1)}a\otimes h^{(2)}b\otimes h^{(3)}c$. This triple tensor product of H-modules is the object part of a categorical ternary product, which is a functor $\mathcal{C}\times\mathcal{C}^{\rm op}\times\mathcal{C}\to\mathcal{C}$. We denote this product by $(A,B,C)\mapsto A\Diamond B\Diamond C$. Then

$$(Q_1\Diamond Q_2\Diamond Q_3)\Diamond Q_4\Diamond Q_5 = Q_1\Diamond(Q_2\Diamond Q_3\Diamond Q_4)\Diamond Q_5 = Q_1\Diamond Q_2\Diamond(Q_3\Diamond Q_4\Diamond Q_5)$$

In fact, the equalities above are true only after natural identifications, which are analogous to the MacLane's coherences for monoidal categories. If it were a small category and if we neglect the coherences, we see that essentially the equality $(Q_1\Diamond Q_2\Diamond Q_3)\Diamond Q_4\Diamond Q_5 = Q_1\Diamond Q_2\Diamond(Q_3\Diamond Q_4\Diamond Q_5)$ just says that this category is a cop in the category of categories. In general, appropriate coherence isomorphisms are introduced, and we call such structures heapy categories. We will discuss their coherences properly in the forthcoming work [11] as well as their connections to a PRO for (co)heaps. An important notion in this context is the notion of a unit for a heapy category. It is an object $\mathbf{1}$ such that the objects $\mathbf{1}\Diamond Q\Diamond Q$, $\mathbf{1}$ and $Q\Diamond Q\Diamond\mathbf{1}$ are isomorphic for each Q (again, we should require coherent isomorphisms with certain dinaturality properties).

A PRO is a strict monoidal category whose object part is the set of natural numbers with the addition as the tensor product. The addition of morphisms does not need to be commutative though. The PRO for coheap monoids is generated by morphisms $t:1\to 3$, $e:0\to 1$ and $d:2\to 1$ which satisfy the relations $(1+t+1)t=(2+t)t=(t+2)t$, $(d+1)t=e+1=1+e=(1+d)t$, $d(d+1)=d(1+d)$, $d(1+e)=d(e+1)=1$. Clearly, usual heaps correspond to those strict monoidal functors from its *opposite PRO* (=for heaps) to the cartesian category of sets, for which $d^{\rm op}:1\to 2$ maps to the usual diagonal $a\mapsto(a,a)$ and $e^{\rm op}:1\to 0$ to the cancelling map $a\mapsto()$. Considering nonstrict monoidal functors to the category of categories, or instead, the techniques of [4], one can systematically introduce coherences in this setup.

As expected from the main theorem of this article, each rigid monoidal category (having duals $Q\mapsto Q^*$) gives rise to a unital heapy category via $Q_1\Diamond Q_2\Diamond Q_3 := (Q_1\otimes Q_2^*)\otimes Q_3$; and the unit of the rigid monodial category may be equipped in canonical way with coherences for the unit of a heapy category. Conversely, a unital heapy category may be made monoidal via $Q_1\otimes Q_2 := Q_1\Diamond\mathbf{1}\Diamond Q_2$, again with appropriate coherences ([11]). In this way, a category of rigid monoidal categories is equivalent to the category of unital heapy categories. It is interesting to further study how much the rigid monoidal category depends on the choice of unit; and which heapy categories have unit at all. The torsor picture suggests that nonunital heapy categories may be of much more interest than the nonunital monoidal categories are.

About the language: co in *cop* mimics co in coalgebra; furthermore, in the dialect of Kent, according to OED, a cop is *a small heap of hay or straw*.

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