The equivalence between Krasnoselskij, Mann, Ishikawa, Noor and multistep iterations

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Abstract. *We prove that Krasnoselskij, Mann, Ishikawa, Noor and multistep iterations are equivalent when applied to quasi-contractive operators.*

Key words: *Krasnoselskij iteration, Mann iteration, Ishikawa iteration, quasi-contractive operators*

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1. Introduction

Let X be a real Banach space, D a nonempty, convex subset of X, and T a selfmap of D, let $x_0 = u_0 \in D$. The Mann iteration, (see [5]), is defined by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n, \tag{1}$$

where $\{\alpha_n\} \subset (0, 1)$. The Krasnoselskij iteration, (see [4]), is defined by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \tag{2}$$

where $\lambda \in (0, 1)$.

Definition 1. [7] *The operator $T : X \to X$ satisfies condition Z (or is a quasi-contraction) if and only if there exist real numbers a, b, c satisfying $0 < a < 1,\ 0 < b, c < 1/2$ such that for each pair x, y in X, at least one condition is true*

- (z_1) $\|Tx - Ty\| \leq a\ \|x - y\|$,
- (z_2) $\|Tx - Ty\| \leq b\ (\|x - Tx\| + \|y - Ty\|)$,
- (z_3) $\|Tx - Ty\| \leq c\ (\|x - Ty\| + \|y - Tx\|)$.

It is known, see Rhoades [8], that (z_1), (z_2) and (z_3) are independent conditions. Note that a map satisfying condition Z is independent, see Rhoades [7], of the class of strongly pseudocontractive maps.

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In [9, **?**] the following conjecture was given: "if the Mann iteration converges, then so does the Ishikawa iteration". In a series of papers [9], [10], [11], [12], [13], Professor B. E. Rhoades and the author, we have given a positive answer to this Conjecture, showing the equivalence between Mann and Ishikawa iterations for strongly and uniformly pseudocontractive maps.

In [2], the following open question was given: "are Krasnoselskij iteration and Mann iteration equivalent (in the sense of [9]) for enough large classes of mappings?" We shall give a positive answer to this question: if Krasnoselskij iteration converges, then Mann (and the corresponding Ishikawa iteration) also converges and conversely, dealing with maps satisfying condition Z. Note that Professor B. E. Rhoades and the author have already given a positive answer in [15] for the class of pseudocontractive maps.

Lemma 1 [[18]]. *Let $\{a_n\}$ be a nonnegative sequence which satisfies the following inequality*

$$a_{n+1} \leq (1-\lambda_n)a_n + \sigma_n, \tag{3}$$

where $\lambda_n \in (0,1)$, $\forall n \geq n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main results

Let $F(T)$ denote the fixed point set with respect to D for the map T. Suppose that $x^* \in F(T)$.

Theorem 1. *Let X be a normed space, D a nonempty, convex, closed subset of X and $T : D \to D$ an operator satisfying condition Z. If $u_0 = x_0 \in D$, then the following are true: if the Mann iteration (1) converges to x^*, then the Krasnoselskij iteration (2) converges to x^*. Conversely, if the Krasnoselskij iteration (2) converges to x^*, then the Mann iteration (1) converges to x^*, provided that $\alpha_n \geq A > 0, \forall n \in \mathbb{N}$.*

Proof. Consider $x, y \in D$. Since T satisfies condition Z, at least one of the conditions from $(z_1), (z_2)$ and (z_3) is satisfied. If (z_2) holds, then

$$\begin{aligned} \|Tx - Ty\| &\leq b\ (\|x - Tx\| + \|y - Ty\|) \\ &\leq b\left(\|x - Tx\| + (\|y - x\| + \|x - Tx\| + \|Tx - Ty\|)\right), \end{aligned}$$

thus

$$(1-b)\|Tx - Ty\| \leq b\|x - y\| + 2b\|x - Tx\|.$$

From $0 \leq b < 1$ one obtains,

$$\|Tx - Ty\| \leq \frac{b}{1-b}\|x - y\| + \frac{2b}{1-b}\|x - Tx\|.$$

If (z_3) holds, then one gets,

$$\begin{aligned} \|Tx - Ty\| &\leq c\ (\|x - Ty\| + \|y - Tx\|) \\ &\leq c\left(\|x - Tx\| + \|Tx - Ty\| + \|x - y\| + \|x - Tx\|\right), \end{aligned}$$

hence,

$$(1-c)\|Tx-Ty\| \leq c\|x-y\|+2c\|x-Tx\| \; i.e.$$
$$\|Tx-Ty\| \leq \frac{c}{1-c}\|x-y\|+\frac{2c}{1-c}\|x-Tx\|.$$

Denote

$$\delta:=\max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\},$$

to obtain

$$0 \leq \delta<1.$$

Finally, we get

$$\|Tx-Ty\| \leq \delta\|x-y\|+2\delta\|x-Tx\|, \forall x, y \in D. \tag{4}$$

Formula (4) was obtained as in [1].

We will prove the implication $(i) \Rightarrow (ii)$. Use (1) (2) and (4) with

$$x:=u_n,$$
$$y:=y_n,$$

to obtain

$$\begin{aligned}
&\|u_{n+1}-x_{n+1}\|=\|x_{n+1}-u_{n+1}\| \\
&=\|x_n-u_n-\lambda x_n+\lambda u_n-\lambda u_n+\alpha_n u_n+\lambda Tx_n-\lambda Tu_n+\lambda Tu_n-\alpha_n Tu_n\| \\
&\leq(1-\lambda)\|u_n-x_n\|+|\alpha_n-\lambda|\|u_n-Tu_n\|+\lambda\|Tu_n-Tx_n\| \\
&\leq(1-\lambda)\|u_n-x_n\|+|\alpha_n-\lambda|\|u_n-Tu_n\|+\lambda\delta\|u_n-x_n\|+2\lambda\delta\|u_n-Tu_n\| \\
&=(1-\lambda(1-\delta))\|u_n-x_n\|+(|\alpha_n-\lambda|+2\lambda\delta)\|u_n-Tu_n\|.
\end{aligned}$$

Denote

$$\begin{aligned}
a_n &:=\|u_n-x_n\|, \\
\lambda_n &:=\lambda(1-\delta) \subset(0,1), \\
\sigma_n &:=(|\alpha_n-\lambda|+2\lambda\delta)\|u_n-Tu_n\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|u_n-x^*\|=0,$ T satisfies condition Z, and $x^* \in F(T)$, from (4) one has

$$\begin{aligned}
0 &\leq\|u_n-Tu_n\| \\
&\leq\|u_n-x^*\|+\|x^*-Tu_n\| \\
&\leq(\delta+1)\|u_n-x^*\| \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

Hence $\lim_{n\to\infty}\|u_n-Tu_n\|=0$; that is $\sigma_n=o(\lambda_n)$. *Lemma 1* leads to $\lim_{n\to\infty}\|u_n-x_n\|=0$. Use

$$0 \leq\|x^*-x_n\| \leq\|u_n-x^*\|+\|x_n-u_n\|$$

to deduce
$$\lim_{n\to\infty} x_n = x^*.$$

We will prove $(ii) \Rightarrow (i)$. That is, if Krasnoselskij iteration converges, then Mann iteration does converge. Use (4) with

$$\begin{aligned} x &:= x_n, \\ y &:= u_n, \end{aligned}$$

to obtain

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\| \\
&= \|x_n - u_n - \alpha_n x_n + \alpha_n u_n + \alpha_n x_n - \lambda x_n + \lambda T x_n - \alpha_n T x_n + \alpha_n T x_n - \alpha_n T u_n\| \\
&= \|(1-\alpha_n)(x_n - u_n) + (\alpha_n - \lambda) x_n - (\alpha_n - \lambda) x_n T x_n + \alpha_n (T x_n - T u_n)\| \\
&\le (1-\alpha_n)\|x_n - u_n\| + |\alpha_n - \lambda| \|x_n - T x_n\| + \alpha_n \|T x_n - T u_n\| \\
&\le (1-\alpha_n)\|x_n - u_n\| + |\alpha_n - \lambda| \|x_n - T x_n\| + \alpha_n \delta \|x_n - u_n\| + 2\alpha_n \delta \|x_n - T x_n\| \\
&= (1 - \alpha_n (1-\delta)) \|x_n - u_n\| + (|\alpha_n - \lambda| + 2\alpha_n \delta) \|x_n - T x_n\|.
\end{aligned}$$

Denote

$$\begin{aligned}
a_n &:= \|x_n - u_n\|, \\
\lambda_n &:= \alpha_n (1-\delta) \subset (0,1), \\
\sigma_n &:= (|\alpha_n - \lambda| + 2\alpha_n \delta) \|x_n - T x_n\|.
\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x^*\| = 0$, T satisfies condition Z, and $x^* \in F(T)$, from (4) one has,

$$\begin{aligned}
0 &\le \|x_n - T x_n\| \\
&\le \|x_n - x^*\| + \|x^* - T x_n\| \\
&\le (\delta + 1) \|x_n - x^*\| \to 0 \text{ as } n \to \infty,
\end{aligned}$$

Hence $\lim_{n\to\infty} \|x_n - T x_n\| = 0$, that is $\sigma_n = o(\lambda_n)$. Lemma 1 leads to $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Thus,

$$\|x^* - u_n\| \le \|x_n - u_n\| + \|x_n - x^*\| \to 0 \text{ as } n \to \infty.$$

□

The Ishikawa iteration is defined (see [3]) by

$$\begin{aligned}
x_{n+1} &= (1-\alpha_n) x_n + \alpha_n T y_n, \\
y_n &= (1-\beta_n) x_n + \beta_n T x_n,
\end{aligned} \tag{5}$$

where $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset [0,1)$.

The following result is from [17].

Theorem 2 [[17]]. *Let X be a normed space, D a nonempty, convex, closed subset of X and $T : D \to D$ an operator satisfying condition Z. If $u_0 = x_0 \in D$, then the following are equivalent:*

(i) the Mann iteration (1) *converges to x^*,*

(ii) the Ishikawa iteration (5) *converges to* x^*.

Theorems 1 and *2* lead to the following corollary.

Corollary 1. *Let* X *be a normed space,* D *a nonempty, convex, closed subset of* X *and* $T : D \to D$ *an operator satisfying condition Z. If* $u_0 = x_0 \in D$, $\alpha_n \geq A > 0, \forall n \in \mathbb{N}$, *then the following are equivalent:*

(i) the Mann iteration (1) *converges to* x^*,

(ii) the Ishikawa iteration (5) *converges to* x^*.

(iii) the Krasnoselskij iteration (2) *converges to* x^*.

3. Further results

For $v_1 \in D$, Noor introduced in [6] the following three-step procedure,

$$\begin{aligned} t_n &= (1-\gamma_n)v_n + \gamma_n T v_n, \\ w_n &= (1-\beta_n)v_n, +\beta_n T t_n, \\ v_{n+1} &= (1-\alpha_n)v_n + \alpha_n T w_n. \end{aligned} \tag{6}$$

The multi-step procedure of arbitrary fixed order $p \geq 2$, see [14], is defined by

$$\begin{aligned} y_n^{p-1} &= (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T x_n, \\ y_n^i &= (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1},\ i = 1, ..., p-2; \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n^1, \end{aligned} \tag{7}$$

where $\{\alpha_n\} \subset (0,1), \{\beta_n^i\} \subset [0,1)$, $1 \leq i \leq p-1$.

We shall generalize the above *Theorem 2*, see also [17], by proving that (7) and (1) are equivalent.

Theorem 3. *Let* X *be a normed space,* D *a nonempty, convex, closed subset of* X *and* $T : D \to D$ *an operator satisfying condition Z. If* $u_0 = x_0 \in D$, *then the following are equivalent:*

(i) the Mann iteration (1) *converges to* x^*,

(ii) the iteration (7) *converges to* x^*.

Proof. We shall use (4) :

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\|, \forall x, y \in D.$$

We will prove the implication $(i) \Rightarrow (ii)$. Suppose that $\lim_{n\to\infty} u_n = x^*$. Using $\lim_{n\to\infty} \|x_n - u_n\| = 0$, and $0 \leq \|x^* - x_n\| \leq \|u_n - x^*\| + \|x_n - u_n\|$ we get

$$\lim_{n\to\infty} x_n = x^*.$$

Using now (1) (7) and (4) with

$$\begin{aligned} x &:= u_n, \\ y &:= y_n^1, \end{aligned}$$

we have

$$\begin{aligned}\|u_{n+1}-x_{n+1}\| &\leq \left\|(1-\alpha_n)(u_n-x_n)+\alpha_n\left(Tu_n-Ty_n^1\right)\right\| \\ &\leq (1-\alpha_n)\|u_n-x_n\|+\alpha_n\left\|Tu_n-Ty_n^1\right\| \\ &\leq (1-\alpha_n)\|u_n-x_n\|+\alpha_n\delta\left\|u_n-y_n^1\right\|+ \\ &\quad +2\alpha_n\delta\|u_n-Tu_n\|.\end{aligned} \tag{8}$$

Using (4) with $x:=u_n,\ y:=y_n^1$, we have

$$\begin{aligned}\left\|u_n-y_n^1\right\| &\leq \left\|\left(1-\beta_n^1\right)(u_n-x_n)+\beta_n^1(u_n-Tx_n)\right\| \\ &\leq \left(1-\beta_n^1\right)\|u_n-x_n\|+\beta_n^1\|u_n-Tx_n\| \\ &\leq \left(1-\beta_n^1\right)\|u_n-x_n\|+\beta_n^1\|u_n-Tu_n\|+ \\ &\quad +\beta_n^1\|Tu_n-Tx_n\| \\ &\leq \left(1-\beta_n^1\right)\|u_n-x_n\|+\beta_n^1\|u_n-Tu_n\|+ \\ &\quad +\beta_n^1\delta\|u_n-x_n\|+2\delta\beta_n^1\|u_n-Tu_n\| \\ &= \left(1-\beta_n^1(1-\delta)\right)\|u_n-x_n\|+ \\ &\quad +\beta_n^1\|u_n-Tu_n\|(1+2\delta).\end{aligned} \tag{9}$$

Relations (8) and (9) lead to

$$\begin{aligned}\|u_{n+1}-x_{n+1}\| &\leq (1-\alpha_n)\|u_n-x_n\|+ \\ &\quad +\alpha_n\delta\left(1-\beta_n^1(1-\delta)\right)\|u_n-x_n\|+ \\ &\quad +\alpha_n\beta_n^1\delta\|u_n-Tu_n\|(1+2\delta)+ \\ &\quad +\alpha_n\delta\|u_n-y_n\| \\ &= \left(1-\alpha_n\left(1-\delta\left(1-\beta_n^1(1-\delta)\right)\right)\right)\|u_n-x_n\|+ \\ &\quad +\alpha_n\delta\|u_n-Tu_n\|\left(\beta_n^1(1+2\delta)+2\delta\right).\end{aligned} \tag{10}$$

Denote by

$$\begin{aligned}a_n &:= \|u_n-x_n\|, \\ \lambda_n &:= \alpha_n\left(1-\delta\left(1-\beta_n^1(1-\delta)\right)\right)\subset(0,1), \\ \sigma_n &:= \alpha_n\delta\|u_n-Tu_n\|\left(\beta_n^1(1+2\delta)+2\delta\right).\end{aligned}$$

Since $\lim_{n\to\infty}\|u_n-x^*\|=0,\ T$ satisfies condition Z, and $x^*\in F(T)$, from (4) we obtain

$$\begin{aligned}0 &\leq \|u_n-Tu_n\| \\ &\leq \|u_n-x^*\|+\|x^*-Tu_n\| \\ &\leq (\delta+1)\|u_n-x^*\|\to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence $\lim_{n\to\infty}\|u_n-Tu_n\|=0$; that is $\sigma_n=o(\lambda_n)$. Lemma 1 leads to $\lim_{n\to\infty}\|u_n-x_n\|=0$.

We will prove now that if multistep iteration converges then Mann iteration does. Using (4) with

$$x := y_n^1, \\ y := u_n,$$

we obtain

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &\leq \left\|(1-\alpha_n)(x_n - u_n) + \alpha_n\left(Ty_n^1 - Tu_n\right)\right\| \\ &\leq (1-\alpha_n)\|x_n - u_n\| + \alpha_n\left\|Ty_n^1 - Tu_n\right\| \\ &\leq (1-\alpha_n)\|x_n - u_n\| + \alpha_n\delta\left\|y_n^1 - u_n\right\| + \\ &\quad +2\alpha_n\delta\left\|y_n^1 - Ty_n^1\right\|. \end{aligned} \tag{11}$$

The following relation holds

$$\begin{aligned} \left\|y_n^1 - u_n\right\| &\leq \left\|\left(1-\beta_n^1\right)(x_n - u_n) + \beta_n^1(Tx_n - u_n)\right\| \\ &\leq \left(1-\beta_n^1\right)\|x_n - u_n\| + \beta_n^1\|Tx_n - u_n\| \\ &\leq \left(1-\beta_n^1\right)\|x_n - u_n\| + \beta_n^1\|Tx_n - x_n\| + \\ &\quad +\beta_n^1\|x_n - u_n\| \\ &\leq \|x_n - u_n\| + \beta_n^1\|Tx_n - x_n\|. \end{aligned} \tag{12}$$

Substituting (12) in (11), we obtain

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &\leq (1-\alpha_n)\|x_n - u_n\| + \\ &\quad +\alpha_n\delta\left(\|x_n - u_n\| + \beta_n^1\|Tx_n - x_n\|\right) + \\ &\quad +2\alpha_n\delta\left\|y_n^1 - Ty_n^1\right\| \\ &\leq (1-(1-\delta)\alpha_n)\|x_n - u_n\| + \alpha_n\beta_n^1\delta\|Tx_n - x_n\| + \\ &\quad +2\alpha_n\delta\left\|y_n^1 - Ty_n^1\right\|. \end{aligned} \tag{13}$$

Denote by

$$\begin{aligned} a_n &:= \|x_n - u_n\|, \\ \lambda_n &:= \alpha_n(1-\delta) \subset (0,1), \\ \sigma_n &:= \alpha_n\beta_n^1\delta\|Tx_n - x_n\| + 2\alpha_n\delta\left\|y_n^1 - Ty_n^1\right\|. \end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - x^*\| = 0,$ T satisfies condition Z, and $x^* \in F(T)$, from (4) we obtain

$$\begin{aligned} 0 &\leq \|x_n - Tx_n\| \\ &\leq \|x_n - x^*\| + \|x^* - Tx_n\| \\ &\leq (\delta+1)\|x_n - x^*\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Note that $\beta_n^i \in [0,1)$, $\forall n \geq 1$, $1 \leq i \leq p-1$, and use (4) to obtain

$$
\begin{aligned}
0 &\leq \left\| y_n^1 - Ty_n^1 \right\| \\
&\leq \left\| y_n^1 - x^* \right\| + \left\| x^* - Ty_n^1 \right\| \\
&\leq (\delta+1) \left\| y_n^1 - x^* \right\| \leq (\delta+1) \ \left[\left(1-\beta_n^1\right) \|x_n - x^*\| + \beta_n^1 \left\| Ty_n^2 - x^* \right\| \right] \\
&\leq (\delta+1) \ \left[\|x_n - x^*\| + \delta \left\| y_n^2 - x^* \right\| \right] \\
&\leq (\delta+1) \ \left[\|x_n - x^*\| + \left\| y_n^2 - x^* \right\| \right] \\
&\leq (\delta+1) \ \left[\|x_n - x^*\| + \left(1-\beta_n^2\right) \|x_n - x^*\| + \beta_n^2 \left\| Ty_n^3 - x^* \right\| \right] \\
&\leq (\delta+1) \ \left[\|x_n - x^*\| + \|x_n - x^*\| + \left\| Ty_n^3 - x^* \right\| \right] \\
&\leq (\delta+1) \ \left[2\|x_n - x^*\| + \delta \left\| y_n^3 - x^* \right\| \right] \\
&\leq (\delta+1) \ \left[2\|x_n - x^*\| + \left\| y_n^3 - x^* \right\| \right] \ldots \\
&\leq (\delta+1) \ \left[(p-2)\|x_n - x^*\| + \left\| y_n^{p-1} - x^* \right\| \right] \\
&\leq (\delta+1) \ \left[(p-2)\|x_n - x^*\| + \left(1-\beta_n^{p-1}\right) \|x_n - x^*\| + \beta_n^{p-1} \|Tx_n - x^*\| \right] \\
&\leq (\delta+1) \ \left[(p-1)\|x_n - x^*\| + \|Tx_n - x^*\| \right] \\
&\leq (\delta+1) \ \left[(p-1)\|x_n - x^*\| + \delta \|x_n - x^*\| \right] \\
&= (\delta+1) \ \|x_n - x^*\| \ \left[(p-1) + \delta \right] \to 0 \text{ as } n \to \infty,
\end{aligned}
$$

Hence $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \left\| y_n^1 - Ty_n^1 \right\| = 0$ that is $\sigma_n = o(\lambda_n)$. Lemma 1 and (13) lead to $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Thus, we get $\|x^* - u_n\| \leq \|x_n - u_n\| + \|x_n - x^*\| \to 0$. □

Theorem 3 and *Corollary 1* lead to the following result.

Corollary 2. *Let X be a normed space, D a nonempty, convex, closed subset of X and $T : D \to D$ an operator satisfying condition Z. If the initial point is the same for all iterations, $\alpha_n \geq A > 0, \forall n \in \mathbb{N}$, then the following are equivalent:*

(i) the Mann iteration (1) *converges to x^*;*

(ii) the Ishikawa iteration (5) *converges to x^*;*

(iii) the iteration (7) *converges to x^*.*

(iii) the Noor iteration (6) *converges to x^*,*

(iv) the Krasnoselskij iteration (2) *converges to x^*.*

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References

[1] V. Berinde, *On the convergence of the Ishikawa iteration in the class of quasi contractive operators*, Acta Math. Univ. Comenianae **LXXIII**(2004), 119-126.

[2] V. Berinde, M. Berinde, *The fastest Krasnoselskij iteration for approximating fixed points of strictly pseudo-contractive mappings*, Carpatian J. Math. **21**(2005), 13-20.

[3] S. ISHIKAWA, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44**(1974), 147-150.

[4] M. A. KRASNOSELSKIJ, *Two remarks on the method of succesive approximations*, Uspehi Mat. Nauk. **10**(1955), 123-127.

[5] W. R. MANN, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4**(1953), 506-510.

[6] M. A. NOOR, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. **251**(2000), 217-229.

[7] B. E. RHOADES, *Fixed point iterations using infinite matrices*, Trans. Amer. Math. Soc. **196**(1974), 161-176.

[8] B. E. RHOADES, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226**(1977), 257-290.

[9] B. E. RHOADES, Ş. M. ŞOLTUZ, *On the equivalence of Mann and Ishikawa iteration methods*, Int. J. Math. Math. Sci. **2003**(2003), 451-459.

[10] B. E. RHOADES, Ş. M. ŞOLTUZ, *The equivalence of the Mann and Ishikawa iteration for non-Lipschitzian operators*, Int. J. Math. Math. Sci. **42**(2003), 2645-2652.

[11] B. E. RHOADES, Ş. M. ŞOLTUZ, *The equivalence between the convergences of Ishikawa and Mann iterations for asymptotically pseudocontractive map*, J. Math. Anal. Appl. **283**(2003), 681-688.

[12] B. E. RHOADES, Ş. M. ŞOLTUZ, *The equivalence of Mann and Ishikawa iteration for a Lipschitzian psi-uniformly pseudocontractive and psi-uniformly accretive maps*, Tamkang J. Math., **35**(2004), 235-245.

[13] B. E. RHOADES, Ş. M. ŞOLTUZ, *The equivalence between the convergences of Ishikawa and Mann iterations for asymptotically nonexpansive in the intermediate sense and strongly successively pseudocontractive maps*, J. Math. Anal. Appl. **289**(2004), 266-278.

[14] B. E. RHOADES, Ş. M. ŞOLTUZ, *The equivalence between Mann-Ishikawa iterations and multistep iteration*, Nonlinear Analysis **58**(2004), 219-228.

[15] B. E. RHOADES, Ş. M. ŞOLTUZ, *The equivalence between the Krasnoselskij, Mann and Ishikawa iterations*, Revue d'analyse numerique et de theorie de l'approximation **36**(2006), to appear.

[16] B. E. RHOADES, Ş. M. ŞOLTUZ, *The equivalence between the T-stabilities of Mann and Ishikawa iterations* J. Math. Anal. Appl. **318**(2006), 472-475.

[17] Ş. M. ŞOLTUZ, The *equivalence of Picard, Mann and Ishikawa iterations dealing with quasi-contractive operators*, Math. Comm. **10**(2005), 81-89.

[18] X. WENG, *Fixed point iteration for local strictly pseudocontractive mapping*, Proc. Amer. Math. Soc. **113**(1991), 727-731.