

The perturbation bound for the solution of the Lyapunov equation*

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Abstract. We present the first order error bound for the Lyapunov equation $AX + XA^* = -GG^*$, where A is perturbed to $A + \delta A$. We use the structure of the solution of the Lyapunov equation $X = \sum_{k=1}^m W_k W_k^*$, where W_k is the k -th matrix obtained by the Low Rank Cholesky Factor ADI (LRCF-ADI) algorithm using the set of ADI parameters equal to exact eigenvalues of A , that is with ADI parameters $\{p_1, \dots, p_m\} = \sigma(A)$. Our bound depends on the structure of the right-hand side G of the Lyapunov equation, and sometimes it can be sharper than the classical error bounds.

Key words: Lyapunov equation, perturbation theory, perturbation bound, low rank Cholesky factor ADI method

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1. Introduction

Throughout this paper we will consider the following continuous-time Lyapunov equation

$$AX + XA^* = -GG^*, \quad (1.1)$$

perturbed to

$$(A + \Delta A)\tilde{X} + \tilde{X}(A + \Delta A)^* = -GG^*, \quad (1.2)$$

where A and $A + \Delta A \in \mathbb{C}^{m \times m}$ are assumed to be stable and $G \in \mathbb{C}^{m \times s}$ with $\text{rank}(G) = s \ll n$.

The main aim of this paper is to give answers to the following questions:

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How good one can bound $\|\tilde{X} - X\|$, where X and \tilde{X} are the solutions of the Lyapunov equations (1.1) and (1.2), respectively? Can we derive a bound which depends on the structure of the right-hand side G of the Lyapunov equation?

As the illustration and motivation one can consider the following simple example:

Example 1. How sensitive is the solution of the following Lyapunov equation

$$AX + XA^* = -GG^*, \text{ where } A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad (1.3)$$

under the perturbation of the matrix A , such that

$$\tilde{A} = A + \Delta A = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix},$$

with $\sigma(A_{11}) \cap \sigma(A_{22}) = \emptyset$ and $\sigma(\tilde{A}_{11}) \cap \sigma(\tilde{A}_{22}) = \emptyset$?

One of the possible answers to the posed question asserts (see for example [4])

$$\frac{\|\tilde{X} - X\|}{\|X\|} \leq 2 \|\tilde{A} - A\| \cdot \|H\|, \quad (1.4)$$

where H is the solution of the following Lyapunov equation $AH + HA^* = -I$.

One can easily see that sometimes the bound (1.4) is too pessimistic. This is the case, for instance, if the right-hand side in (1.3) has the following form

$$GG^* = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where the blocks of the matrix GG^* correspond with the blocks of A . Now it is obvious that the solution X as well as its perturbation \tilde{X} depends only on the (1, 1)-block, that is

$$X = \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} \tilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

where X_{11} and \tilde{X}_{11} are the solutions of

$$A_{11}X_{11} + X_{11}A_{11}^* = -I, \quad \text{and} \quad \tilde{A}_{11}\tilde{X}_{11} + \tilde{X}_{11}\tilde{A}_{11}^* = -I, \quad (1.5)$$

respectively.

The problem of the influence of the right-hand side on the solution of linear equations has been considered in [2]. Recently in [10] it has been shown that sometimes the structure of the right-hand side B of the Lyapunov equation $AX + XA^* = B$ can greatly influence the eigenvalue decay rate of the solution.

Under the influence of this result we will show that the first order perturbation bound for the solution of Lyapunov equation (1.1) perturbed as in (1.2) can depend on the structure of the right-hand side.

Throughout the paper we assume that the unperturbed and the perturbed quantities are of the same order, and $\|\cdot\|$ denotes the spectral matrix norm.

The rest of the paper is organized as follows. In the first part of *Section 2*, we describe some properties of the Low Rank Cholesky Factor ADI (LRCF-ADI) for

solving the Lyapunov equation. The first subsection of *Section 2*. contains the first order perturbation bound which considers the *influence of the perturbation of the eigenvalues* while in the second subsection we present the first order perturbation bound which considers the *influence of the perturbation of eigenvectors*. The last theorem in *Section 2*. contains our main result and it is a combination of the above two perturbation bounds.

Finally, *Section 3*. contains an example which illustrates the usage of our main result, that is which shows that sometimes our bound can be better than existing ones which depends on the structure of the right-hand side of the Lyapunov equation.

2. The main result

As we have mentioned in the introduction, in this paper we will consider Lyapunov equations (1.1) and (1.2), with additional assumption that A and the corresponding perturbed matrix $A + \Delta A$ are diagonalizable, that is, we have

$$\begin{aligned} A &= SAS^{-1}; \quad S \in \mathbb{C}^{m \times m}, \quad \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_m\} \\ \tilde{A} &= \tilde{S}\tilde{\Lambda}\tilde{S}^{-1}; \quad \tilde{S} \in \mathbb{C}^{m \times m}, \quad \tilde{\Lambda} = \text{diag}\{\tilde{\lambda}_1, \dots, \tilde{\lambda}_m\}, \end{aligned} \quad (2.1)$$

where the eigenvalue decomposition of the matrix A and $A + \Delta A$, respectively. Here the eigenvalues are ordered in such a way that $i < j$ implies $|\lambda_i| < |\lambda_j|$, $|\tilde{\lambda}_i| < |\tilde{\lambda}_j|$.

Before we continue, let us briefly review some auxiliary results which are the basis for our calculations.

First, we will use the following decomposition of the solution X of the Lyapunov equation (1.1) (for more details see [1] or [8]):

$$X = \sum_{k=1}^m W_k W_k^*, \quad (2.2)$$

where W_k is the k -th matrix obtained by the Low Rank Cholesky Factor ADI (LRCF-ADI) algorithm using the set of ADI parameters equal to exact eigenvalues of A , that is with ADI parameters $\{p_1, \dots, p_m\} = \sigma(A)$.

The LRCF-ADI was proposed in [5] (see also [6]) and implemented in [7]. For the purpose of completeness, we will present the basic code for the LRCF-ADI taken from [7]:

Algorithm 1 (Low rank Cholesky factor ADI (LRCF-ADI))

INPUT: $A, G, \{p_1, p_2, \dots, p_{i_{max}}\}$

OUTPUT: $V = V_{i_{max}} \in \mathbb{C}^{m \times s_{i_{max}}}$, such that $VV^* \approx X$.

1. $W_1 = \sqrt{-2\text{Re}(p_1)} (A + p_1 I_m)^{-1} G$
 2. $V_1 = W_1$
- FOR: $i = 2, 3, \dots, i_{max}$
3. $W_i = \sqrt{\text{Re}(p_i)/\text{Re}(p_{i-1})} (W_{i-1} - (p_i + p'_{i-1})(A + p_i I_m)^{-1} W_{i-1})$
 4. $V_i = [V_{i-1} \ W_i]$
- END

Here p' denotes the conjugation of p .

The second result which we are going to use is taken from [10]. It can be shown that W_k from the (2.2) can be written as

$$W_k = SD_k S^{-1}G, \quad (2.3)$$

where

$$D_k = \sqrt{-2 \operatorname{Re}(\lambda_k)} \cdot \operatorname{diag}(\sigma(k, 1), \sigma(k, 2), \dots, \sigma(k, m)), \quad (2.4)$$

and $\sigma(k, i)$ are defined as

$$\sigma(k, 1) = \frac{1}{\lambda_k + \lambda_1}, \quad \sigma(k, j) = \frac{1}{\lambda_k + \lambda_1} \prod_{t=1}^{j-1} \frac{\lambda_k - \lambda'_t}{\lambda_k + \lambda_{t+1}} \quad \text{for } j > 1. \quad (2.5)$$

Indeed, from **Algorithm 1** (for more details see the proof of [10, Theorem 2.1]), it follows that

$$\begin{aligned} W_j &= \sqrt{-2 \operatorname{Re}(\lambda_j)} S \cdot (I - (\lambda_j + \lambda'_{j-1})(\Lambda + \lambda_j I_m)^{-1}) \\ &\quad \times (I - (\lambda_{j-1} + \lambda'_{j-2})(\Lambda + \lambda_{j-1} I_m)^{-1}) \\ &\quad \times \dots \times (I - (\lambda_2 + \lambda'_1)(\Lambda + \lambda_2 I_m)^{-1}) \cdot (\Lambda + \lambda_1 I_m)^{-1} S^{-1}G, \end{aligned}$$

which together with fact that in the above equality we have a $(j-1)$ -diagonal matrix of the form

$$(I - (\lambda_k + \lambda'_{k-1})(\Lambda + \lambda_k I_m)^{-1}) = \operatorname{diag} \left(\frac{\lambda_i - \lambda'_{k-1}}{\lambda_i + \lambda_k} \right)_i$$

$i = 1, \dots, m, k = 2, \dots, j$, gives (2.3).

Now, we can continue with our considerations. Let

$$\widehat{X} = \sum_{k=1}^m \widehat{W}_k \widehat{W}_k^*$$

be the solution of the Lyapunov equation,

$$\widehat{A} \widehat{X} + \widehat{X} \widehat{A}^* = -G G^*, \quad \text{where} \quad \widehat{A} = S \widetilde{\Lambda} S^{-1},$$

obtained by LRCF-ADI algorithm with ADI parameters $p = \sigma(\widehat{A}) = \{\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_m\}$.

Similarly, let

$$\widetilde{X} = \sum_{k=1}^m \widetilde{W}_k \widetilde{W}_k^*$$

be the solution of the Lyapunov equation

$$\widetilde{A} \widetilde{X} + \widetilde{X} \widetilde{A}^* = -G G^*, \quad \text{where} \quad \widetilde{A} = \widetilde{S} \widetilde{\Lambda} \widetilde{S}^{-1},$$

obtained by LRCF-ADI algorithm with ADI parameters $p = \sigma(\tilde{A}) = \{\tilde{\lambda}_1, \dots, \tilde{\lambda}_m\}$.

Similarly as in (2.4) we can write:

$$\begin{aligned} W_k &= SD_k S^{-1}G, \quad D_k = \sqrt{-2 \operatorname{Re}(\lambda_k)} \cdot \operatorname{diag}(\sigma(k, 1), \dots, \sigma(k, m)), \\ \widehat{W}_k &= S\tilde{D}_k S^{-1}G, \quad \tilde{D}_k = \sqrt{-2 \operatorname{Re}(\tilde{\lambda}_k)} \cdot \operatorname{diag}(\tilde{\sigma}(k, 1), \dots, \tilde{\sigma}(k, m)), \\ \widetilde{W}_k &= \tilde{S}\tilde{D}_k\tilde{S}^{-1}G. \end{aligned} \quad (2.6)$$

Thus we have

$$\delta\widehat{W}_k = \widehat{W}_k - W_k = S \left(\tilde{D}_k - D_k \right) S^{-1}G. \quad (2.7)$$

If we assume that $\|E_S\| < 1$, then

$$\delta W_k = \widetilde{W}_k - \widehat{W}_k = S \left(E_S \tilde{D}_k - \tilde{D}_k E_S \right) S^{-1}G + \mathcal{O}(\|E_S\|^2), \quad (2.8)$$

where

$$S^{-1}\tilde{S} = I + E_S. \quad (2.9)$$

From the inequality

$$\|X - \tilde{X}\| \leq \|X - \widehat{X}\| + \|\widehat{X} - \tilde{X}\|, \quad (2.10)$$

follows

$$\begin{aligned} \|X - \tilde{X}\| &\leq 2 \sum_{k=1}^m \|W_k\| \|\delta\widehat{W}_k\| + 2 \sum_{k=1}^m \|\widehat{W}_k\| \|\delta W_k\| \\ &\quad + \sum_{k=1}^m \left(\mathcal{O}(\|\delta\widehat{W}_k\|^2) + \mathcal{O}(\|\delta W_k\|^2) \right). \end{aligned} \quad (2.11)$$

Note that from (2.11) it follows that we have to bound $\|\delta\widehat{W}_k\|$ which depends on eigenvalue perturbation and $\|\delta W_k\|$ which depends on perturbation of eigenvectors of the matrix A .

2.1. Influence of the eigenvalue perturbation

In this section we will derive the bound for the $\|\delta\widehat{W}_k\|$.

The matrix \tilde{D}_k from (2.6) has the following form

$$\tilde{D}_k = \sqrt{-2 \operatorname{Re}(\tilde{\lambda}_k)} \cdot \operatorname{diag}(\tilde{\sigma}(k, 1), \tilde{\sigma}(k, 2), \dots, \tilde{\sigma}(k, m)) \quad (2.12)$$

where $\tilde{\sigma}(k, i)$ are defined by

$$\tilde{\sigma}(k, 1) = \frac{1}{\tilde{\lambda}_k + \tilde{\lambda}_1}, \quad \tilde{\sigma}(k, j) = \frac{1}{\tilde{\lambda}_k + \tilde{\lambda}_1} \prod_{t=1}^{j-1} \frac{\tilde{\lambda}_k - \tilde{\lambda}_t}{\tilde{\lambda}_k + \tilde{\lambda}_{t+1}}, \quad (2.13)$$

for $j > 1$.

Since $\delta\widehat{W}_k = S(\widetilde{D}_k - D_k)S^{-1}G$, it follows that we have to bound terms

$$(\widetilde{D}_k)_{ii} - (D_k)_{ii} = \sqrt{-2\operatorname{Re}(\widetilde{\lambda}_i)} \cdot \widetilde{\sigma}(k, i) - \sqrt{-2\operatorname{Re}(\lambda_i)} \cdot \sigma(k, i). \quad (2.14)$$

Let eigenvalue

$$\widetilde{\lambda}_i = \lambda_i + \delta\lambda_i, \quad (2.15)$$

be i -th eigenvalue of the matrix $A + \Delta A$. If we assume that all eigenvalues of A and $A + \Delta A$ are simple, then for $\delta\lambda_i$ from (2.15) we have the following bound (see [3] or [9])

$$|\delta\lambda_k| \leq \frac{t_k^* \Delta A s_k}{t_k^* s_k} = \varepsilon_k, \quad (2.16)$$

where s_k and t_k are right and left eigenvectors belonging to λ_k normalized so that $\|s_k\| = \|t_k\| = 1$ and $|t_k^* s_k| = t_k^* s_k$.

Note that from the fact that

$$|\operatorname{Re}(\delta\lambda_k)| \leq |\delta\lambda_k|$$

it follows that for ε_k small enough we can write

$$\sqrt{-2\operatorname{Re}(\widetilde{\lambda}_k)} \leq \sqrt{-2\operatorname{Re}(\lambda_k)} \cdot \left(1 + \frac{\varepsilon_k}{|\operatorname{Re}(\lambda_k)|}\right). \quad (2.17)$$

Further, from (2.13) and (2.5) it follows

$$\widetilde{\sigma}(k, 1) - \sigma(k, 1) \cdot \frac{\frac{\delta\lambda_k + \delta\lambda_1}{\lambda_k + \lambda_1}}{1 + \frac{\delta\lambda_k + \delta\lambda_1}{\lambda_k + \lambda_1}}.$$

Similarly one can write

$$\frac{1}{\widetilde{\lambda}_k + \widetilde{\lambda}_j} - \frac{1}{\lambda_k + \lambda_j} = -\frac{1}{\lambda_k + \lambda_j} \cdot \frac{\frac{\delta\lambda_k + \delta\lambda_j}{\lambda_k + \lambda_j}}{1 + \frac{\delta\lambda_k + \delta\lambda_j}{\lambda_k + \lambda_j}}.$$

Note that in the both of the above equalities the term which determines a magnitude of perturbation can be bound with

$$\eta(k, j) \equiv \left| \frac{\delta\lambda_k + \delta\lambda_j}{\lambda_k + \lambda_j} \right| \leq \frac{\varepsilon_k + \varepsilon_j}{|\lambda_k + \lambda_j|} \leq \frac{\varepsilon_{kj}}{|\lambda_k + \lambda_j|} \quad \text{for } j < k, \quad (2.18)$$

where $\varepsilon_{kj} = 2 \max\{\varepsilon_k, \varepsilon_j\}$.

Now, for $\eta(k, t)$ small enough we can write

$$\begin{aligned} |\widetilde{\sigma}(k, 1)| &\leq \sigma(k, 1) \cdot \left(1 + \frac{\eta(k, 1)}{1 + \eta(k, 1)}\right) \approx \sigma(k, 1) \cdot (1 + \eta(k, 1)), \\ |\widetilde{\sigma}(k, j)| &\leq \frac{1}{|\lambda_k + \lambda_1|} \left(1 + \frac{\eta(k, 1)}{1 + \eta(k, 1)}\right) \cdot \prod_{t=1}^{j-1} \left| \frac{\widetilde{\lambda}_k - \widetilde{\lambda}'_t}{\lambda_k + \lambda_{t+1}} \right| \cdot \left(1 + \frac{\eta(k, t+1)}{1 + \eta(k, t+1)}\right) \\ &\approx \frac{1 + \eta(k, 1)}{|\lambda_k + \lambda_1|} \cdot \prod_{t=1}^{j-1} \left| \frac{\widetilde{\lambda}_k - \widetilde{\lambda}'_t}{\lambda_k + \lambda_{t+1}} \right| \cdot (1 + \eta(k, t+1)) \quad \text{for } j > 1. \end{aligned}$$

Now, the first order approximation of the right-hand sides of the above inequalities gives

$$\begin{aligned} |\tilde{\sigma}(k, j)| &\approx \frac{1}{|\lambda_k + \lambda_1|} \cdot \prod_{t=1}^{j-1} \left| \frac{\tilde{\lambda}_k - \tilde{\lambda}'_t}{\lambda_k + \lambda_{t+1}} \right| \cdot \left(1 + \sum_{t=1}^j \eta(k, t) \right) \\ &= \frac{1}{|\lambda_k + \lambda_1|} \cdot \prod_{t=1}^{j-1} \left| \frac{\tilde{\lambda}_k - \tilde{\lambda}'_t}{\lambda_k + \lambda_{t+1}} \right| \cdot (1 + \eta_j(k)), \end{aligned}$$

where $\eta_j(k) = \sum_{t=1}^j \eta(k, t)$.

Finally, using (2.15) we obtain

$$|\tilde{\sigma}(k, j)| \approx \frac{1}{|\lambda_k + \lambda_1|} \cdot \prod_{t=1}^{j-1} \left| \frac{\lambda_k - \lambda'_t}{\lambda_k + \lambda_{t+1}} + \frac{\delta\lambda_k - \delta\lambda'_t}{\lambda_k + \lambda_{t+1}} \right| \cdot (1 + \eta_j(k)).$$

Now, all above imply

$$|\tilde{\sigma}(k, j)| \approx \frac{1}{|\lambda_k + \lambda_1|} \cdot \prod_{t=1}^{j-1} \left| \frac{\lambda_k - \lambda'_t}{\lambda_k + \lambda_{t+1}} + \frac{\delta\lambda_k - \delta\lambda'_t}{\lambda_k + \lambda_{t+1}} + \frac{\lambda_k - \lambda'_t}{\lambda_k + \lambda_{t+1}} \cdot \eta_j(k) \right|,$$

which gives

$$|\tilde{\sigma}(k, j)| \approx |\sigma(k, j)| \cdot \prod_{t=1}^{j-1} \left| 1 + \frac{\lambda_k + \lambda_{t+1}}{\lambda_k - \lambda'_t} \frac{\delta\lambda_k - \delta\lambda_t}{\lambda_k + \lambda_{t+1}} + \eta_j(k) \right|. \quad (2.19)$$

If we write $\tilde{\sigma}(k, j) = \sigma(k, j) + \delta\sigma(k, j)$ then as the first order approximation bound for $|\delta\sigma(k, j)|$ we have

$$\begin{aligned} |\delta\sigma(k, j)| &\lesssim |\sigma(k, j)| \cdot \sum_{t=1}^{j-1} \left(\left| \frac{\lambda_k + \lambda_{t+1}}{\lambda_k - \lambda'_t} \right| \left| \frac{\delta\lambda_k - \delta\lambda_t}{\lambda_k + \lambda_{t+1}} \right| + \eta_j(k) \right) \\ &\leq |\sigma(k, j)| \cdot \left(\max_t \frac{|\lambda_k + \lambda_{t+1}|}{|\lambda_k - \lambda'_t|} \cdot \vartheta_j(k) + (j-1)\eta_j(k) \right), \end{aligned}$$

where

$$\sum_{t=1}^{j-1} \left| \frac{\delta\lambda_k - \delta\lambda_t}{\lambda_k + \lambda_{t+1}} \right| \leq \sum_{t=1}^{j-1} \frac{\varepsilon_k + \varepsilon_t}{|\lambda_k + \lambda_{t+1}|} \doteq \vartheta_j(k)$$

Here symbol \lesssim stands for upper bound on approximate value, that is if $a \approx b$ and $b \leq c$ we write $a \lesssim c$.

The above gives

$$|\delta\sigma(k, j)| \lesssim |\sigma(k, j)| \cdot \left(\frac{\vartheta_j(k)}{\text{rg}(\lambda_k, \lambda_t)} + (j-1)\eta_j(k) \right), \quad (2.20)$$

$$\text{where} \quad \text{rg}(\lambda_k, \lambda_t) = \min_t \frac{|\lambda_k - \lambda'_t|}{|\lambda_k + \lambda_{t+1}|}.$$

From (2.14), (2.17) it follows that we have the following first order bound

$$|(\tilde{D}_k)_{ii} - (D_k)_{ii}| \leq \sqrt{-2 \operatorname{Re}(\lambda_k)} \left(|\delta\sigma(k, i)| + \frac{\varepsilon_i |\sigma(k, i)|}{|\operatorname{Re}(\lambda_i)|} \right). \quad (2.21)$$

Now from (2.7), (2.14) and (2.21) one can write

$$\delta\widehat{W}_k \approx \sqrt{-2 \operatorname{Re}(\lambda_k)} S \begin{bmatrix} \left(|\delta\sigma(k, 1)| + \varepsilon_1 \frac{|\sigma(k, 1)|}{|\operatorname{Re}(\lambda_1)|} \right) \cdot \widehat{g}_1 \\ \left(|\delta\sigma(k, 2)| + \varepsilon_2 \frac{|\sigma(k, 2)|}{|\operatorname{Re}(\lambda_2)|} \right) \cdot \widehat{g}_2 \\ \vdots \\ \left(|\delta\sigma(k, m)| + \varepsilon_m \frac{|\sigma(k, m)|}{|\operatorname{Re}(\lambda_m)|} \right) \cdot \widehat{g}_m \end{bmatrix}. \quad (2.22)$$

where

$$\widehat{G} = S^{-1}G \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1s} \\ g_{21} & g_{22} & \cdots & g_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{ms} \end{bmatrix} = \begin{bmatrix} \widehat{g}_1 \\ \widehat{g}_2 \\ \vdots \\ \widehat{g}_m \end{bmatrix} \quad (2.23)$$

Now we can state our first result.

Theorem 1. *Let S be the eigenvector matrix and $\tilde{\Lambda}$ be the eigenvalue matrix of matrices A and $\tilde{A} = A + \Delta A$, respectively defined in (2.1). Let \tilde{X} be the solution of the Lyapunov equation*

$$\widehat{A}\widehat{X} + \widehat{X}\widehat{A}^* = -GG^*, \quad \widehat{A} = S\tilde{\Lambda}S^{-1} \quad (2.24)$$

obtained by **Algorithm 1** with the set of ADI parameters which correspond to the spectrum of the matrix \tilde{A} . Then the following first order bound holds:

$$\|X - \widehat{X}\| \lesssim 2 \|S\| \sum_{j=1}^m \|W_j\| \sqrt{-2 \operatorname{Re}(\lambda_j)} \left(\sum_{k=1}^m \left| \delta\sigma(j, k) + \varepsilon_k \frac{\sigma(j, k)}{|\operatorname{Re}(\lambda_k)|} \right|^2 \cdot \|\widehat{g}_k\|^2 \right)^{1/2}, \quad (2.25)$$

where

$$|\delta\sigma(k, j)| \lesssim |\sigma(k, j)| \cdot \left(\frac{\vartheta_j(k)}{\operatorname{rg}(\lambda_k, \lambda_t)} + (j-1)\eta_j(k) \right),$$

$$\operatorname{rg}(\lambda_k, \lambda_t) = \min_{k,t} \frac{|\lambda_k - \lambda'_t|}{|\lambda_k + \lambda_{t+1}|}, \quad \text{and} \quad \vartheta_j(k) = \sum_{t=1}^{j-1} \frac{\varepsilon_k + \varepsilon_t}{|\lambda_k + \lambda_{t+1}|},$$

and where $\eta_j(k) = \sum_{t=1}^j \eta(k, t)$ and $\eta(k, t)$ is defined in (2.18), ε_k is defined in (2.16) and \widehat{g}_k is defined in (2.23).

Proof. The proof follows from the considerations given above. \square

2.2. Influence of the eigenvectors' perturbation

We continue with deriving the bound for $\|\delta W_k\|$.

From (2.7) it follows that

$$\|\delta W_k\| \lesssim \|S\| \left(\|E_S \tilde{D}_k \hat{G}\| + \|\tilde{D}_k E_S \hat{G}\| \right). \quad (2.26)$$

Note that the right-hand side of the above expression obviously depends on the structure of the matrix $\hat{G} = S^{-1}G$.

Without loss of generality, we will assume that \hat{G} has the following structure,

$$\hat{G} = \begin{bmatrix} \hat{G}_1 \\ 0 \end{bmatrix},$$

and

$$E_S = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad \tilde{D}_k = \begin{bmatrix} (\tilde{D}_k)_{11} & 0 \\ 0 & (\tilde{D}_k)_{22} \end{bmatrix}, \quad (2.27)$$

where E_{ij} and $(\tilde{D}_k)_{ii}$, $i, j = 1, 2$, correspond with the structure of \hat{G} .

From the equality

$$S\Lambda S^{-1} - \tilde{S}\tilde{\Lambda}\tilde{S}^{-1} = -\Delta A$$

follows that

$$\Lambda E_S - E_S \Lambda \approx \delta \Lambda - S^{-1} \Delta A \tilde{S}. \quad (2.28)$$

If write $T^* = S^{-1}$, then $E_S = T^* \tilde{S} - I$.

Note that from (2.7) it follows that we only need to bound E_{11} and E_{21} , which can be done according to (2.28). It is easy to see that for off diagonal entries of the matrix E_{11} the following bounds holds

$$|(E_{11})_{ij}| \leq \frac{|t_i^* \Delta A \tilde{s}_j|}{\min_{i \neq j} |\lambda_i - \lambda_j|} \equiv (\Psi_1)_{ij} \quad \text{where } \lambda_i, \lambda_j \in \Lambda_{11}, \quad (2.29)$$

for $i \neq j$ where Λ_{11} is a diagonal matrix with the same dimension as the block \hat{G}_1 .

For $i = j$ we can assume that $(E_S)_{ii} \approx 0$, since assumption that $\|E_S\|$ has the modest magnitude is equivalent to the assumption that $\sin \angle(t_i, \tilde{s}_i)$ is small. This further implies $t_i^* \tilde{s}_i \approx 1$, that is $t_i^* \tilde{s}_i = 1 + \mathcal{O}(\sin \angle(t_i, \tilde{s}_i)^2)$.

The bound for $|(E_{21})_{ij}|$ is simpler to obtain, indeed we have

$$|(E_{21})_{ij}| \leq \frac{|t_i^* \Delta A \tilde{s}_j|}{\min_{i,j} |\lambda_i - \lambda_j|} \equiv (\Psi_2)_{ij} \quad \text{where } \lambda_i \in \Lambda_{11}, \lambda_j \in \Lambda_{22}. \quad (2.30)$$

The bound (2.30) is a point-wise bound for the entries of the matrix E_{21} , which means that we will be able to bound only $\|E_{21}\|_F$ (where $\|\cdot\|_F$ stands for the Frobenius norm).

Note that

$$\|\tilde{D}_k E_S \hat{G}\| \leq \|(\tilde{D}_k)_{11} E_{11} \hat{G}_1\| + \|(\tilde{D}_k)_{22} E_{21} \hat{G}_1\|. \quad (2.31)$$

Now we can state our second result.

Theorem 2. *Let \tilde{X} be the solution of the Lyapunov equation*

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^* = -G G^*, \quad \tilde{A} = \tilde{S}\tilde{\Lambda}\tilde{S}^{-1} \quad (2.32)$$

obtained by **Algorithm 1** with the set of ADI parameters which correspond to the spectrum of the matrix $A + \Delta A$. Then the following first order bound holds:

$$\begin{aligned} \|\hat{X} - \tilde{X}\| &\lesssim 2 \|S\| \sum_{j=1}^m \|\widehat{W}_j\| \left(\|\Psi_1(\tilde{D}_j)_{11} \hat{G}_1\|_F + \|\Psi_2(\tilde{D}_j)_{11} \hat{G}_1\|_F \right) \\ &\quad + 2 \|S\| \sum_{j=1}^m \|\widehat{W}_j\| \left(\|(\tilde{D}_j)_{11}\| \|\Psi_1 \hat{G}_1\|_F + \|(\tilde{D}_j)_{22}\| \|\Psi_2 \hat{G}_1\|_F \right) \end{aligned} \quad (2.33)$$

Proof. The bound (2.33) follows from (2.26), (2.29), (2.30), (2.31) and the fact that

$$\begin{aligned} \|E_{i1}(\tilde{D}_k)_{11} \hat{G}_1\| &\leq \|\Psi_i(\tilde{D}_k)_{11} \hat{G}_1\|_F \text{ for } i = 1, 2, \\ \|(\tilde{D}_k)_{ii} E_{i1} \hat{G}_1\| &\leq \|(\tilde{D}_k)_{ii}\| \|E_{i1} \hat{G}_1\|_F \leq \|(\tilde{D}_k)_{ii}\| \|\Psi_i \hat{G}_1\|_F \text{ for } i = 1, 2. \end{aligned}$$

□

Now we will state our main result.

Theorem 3. *Let X and \tilde{X} be the solutions of the Lyapunov equations*

$$AX + XA^* = -GG^* \quad \text{and} \quad (A + \Delta A)\tilde{X} + \tilde{X}(A + \Delta A)^* = -GG^*, \quad (2.34)$$

respectively, where A and $A + \Delta A \in \mathbb{C}^{m \times m}$ are assumed to be stable and diagonalizable, and $G \in \mathbb{C}^{m \times s}$ with $\text{rank}(G) = s$. Then the following first order bound holds:

$$\begin{aligned} \|X - \tilde{X}\| &\lesssim 2 \|S\| \sum_{j=1}^m \|W_j\| \sqrt{-2 \text{Re}(\lambda_j)} \left(\sum_{k=1}^m \left| \delta\sigma(j, k) + \frac{\varepsilon_k}{\text{Re}(\lambda_k)} \right|^2 \cdot \|\hat{g}_k\|^2 \right)^{1/2} \\ &\quad + 2 \|S\| \sum_{j=1}^m \|\widehat{W}_j\| \left(\|\Psi_1(\tilde{D}_j)_{11} \hat{G}_1\|_F + \|\Psi_2(\tilde{D}_j)_{11} \hat{G}_1\|_F \right) \\ &\quad + 2 \|S\| \sum_{j=1}^m \|\widehat{W}_j\| \left(\|(\tilde{D}_j)_{11}\| \|\Psi_1 \hat{G}_1\|_F + \|(\tilde{D}_j)_{22}\| \|\Psi_2 \hat{G}_1\|_F \right) \end{aligned} \quad (2.35)$$

where $\delta\sigma(j, k)$ are defined by (2.19), \hat{g}_k by (2.23) and $\hat{G} = [\hat{G}_1 \ 0]^*$.

The quality of the above bound depends on the numbers ε_k defined in (2.16) and Ψ_i , $i = 1, 2$ defined in (2.29) and (2.30) whose magnitude strongly depends on the structure of the matrix A and its perturbation ΔA . The bound (2.35) also depends on the structure of the right-hand side of the Lyapunov equation (due to the existence of \hat{G}_1 in the bound). The following section contains one possible case when the above bound can be better than the existing ones.

3. Application of a new bound

As it has been already pointed out, the application of our bound strongly depends on the structure of matrices A , ΔA and G involved in Lyapunov equations (1.1) and (1.2). As an illustration of the possible case when the bound (2.35) can be better than existing ones, we will consider again *Example 1* from *Introduction*.

Thus, let $A = SAS^{-1}$ and $\tilde{A} = \tilde{S}\tilde{\Lambda}\tilde{S}^{-1}$ be eigenvalue decompositions of

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \text{and} \quad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \quad (3.36)$$

respectively. Note that S and \tilde{S} are upper block diagonal.

We consider the following Lyapunov equation:

$$AX + XA^* = -GG^*, \quad \text{where} \quad G = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad (3.37)$$

is decomposed according to A and \tilde{A} . Now, using the structure of A and \tilde{A} it is obvious that $E_S = S^{-1}\tilde{S} - I$ is upper block diagonal, that is $E_{21} = 0$ (where E_{21} is defined in (2.27)).

Also note that

$$\hat{G} = S^{-1}G = \begin{bmatrix} \hat{G}_1 \\ 0 \end{bmatrix}. \quad (3.38)$$

Let s be the rank of the matrix \hat{G}_1 . Note that for A of the form given above we have $W_k = 0$, for $k = s + 1, \dots, m$ where W_k are generated by **Algorithm 1**. This follows from the fact that $\sigma(i, j) = 0$, for $j > i$, and similar holds for perturbed quantities, that is $\tilde{W}_k = 0$, for $k = s + 1, \dots, m$. Note also that all entries of matrices W_k , \tilde{W}_k and then of $\delta\tilde{W}_k$ (where $\delta\tilde{W}_k$ is defined in (2.22)), from the $s + 1$ -th row up to the m -th row have zero entries.

All this together implies that we have the following first order bound

$$\begin{aligned} \|X - \tilde{X}\| &\lesssim 2 \|S\| \sum_{j=1}^s \|W_j\| \sqrt{-2 \operatorname{Re}(\lambda_j)} \left(\sum_{k=1}^s \left| \delta\sigma(j, k) + \frac{\varepsilon_k}{\operatorname{Re}(\lambda_k)} \right|^2 \cdot \|\hat{g}_k\|^2 \right)^{1/2} \\ &\quad + 2 \|S\| \sum_{j=1}^s \|\hat{W}_j\| \left(\|\Psi_1(\tilde{D}_j)_{11} \hat{G}_1\|_F + \|(\tilde{D}_j)_{11}\| \|\Psi_1 \hat{G}_1\|_F \right). \end{aligned}$$

Also, we see that the above bound depends only on the structure of the matrices A_{11} and the corresponding perturbed one \tilde{A}_{11} .

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