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A NOTE ON THE COSINE EQUATION FOR PROBABILITY MEASURES ON LOCALLY COMPACT SEMIGROUPS

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Abstract

In this note we study the cosine equation (5) where μ_t ($t \in \mathbf{R}$) is probability measure on locally compact semigroup. If, in addition, the function $t \mapsto \mu_t$ ($t \in \mathbf{R}$) satisfies the condition (6) it is shown that the solution of (5) is of the form $\mu_t = \cos^* t\,\mu$ ($t \in \mathbf{R}$), where $\mu \in M^1(S)$ is unique.

Key words and phrases: cosine equation, probability measure on semigroup, convolution vague topology.

Let S be a locally compact (Hausdoff) second countable semigroup with identity e. By a mesaure on S, we mean a finite regular non-negative measure on the class B_S of all Borel sets in S. $P(S)$ denotes the set of al regular probabilty measures defined on S. Let $K(S)$ be the space, of all (real-valued) continuous functions with comapct support. $K(S)$ can be normed by $\|f\| = \sup_{x\in S}|f(x)|(f \in K(S))$.

A net (μ_α) of a measures converges **vaguely** to a measure μ if

$$\lim_\alpha \int_S f\, d\mu_\alpha = \int_S f\, d\mu, f \in K(S). \tag{1}$$

Then we write $\mu = (v)\lim_\alpha \mu_\alpha$ or $\mu_\alpha \xrightarrow{v} \mu$.

The convolution $\mu * \nu$ of two measures μ, ν is defined by

$$\int_S f\, d(\mu * \nu) = \int_S \int_S f(xy)\, d\mu(x)\, d\nu(y),\ f \in K(S). \tag{2}$$

If $\mu, \nu \in P(S)$, then $\mu * v \in P(S)$. Moreover $P(S)$ is a topological semigroup with respect to the convolution (i.e. the mapping $* : P(S) \times P(S) \to P(S)$ is jointly continuous in the vague topology).

$\mu * \mu \ldots * \mu$ (with n terms) we denote by μ^n. We also put $\mu^0 = \delta_e$, where $\delta_e \in P(S)$ is concentrated at the identity e.

For $\mu \in P(S)$ and $t \in \mathbf{R}$ put

$$\mu_t = \cos^* t\,\mu = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\mu^n. \tag{3}$$

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The measure μ_t is determined uniquely by relation

$$\int_S f\, d\mu_t = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \int_S f\, d\mu^n, f \in K(S). \tag{4}$$

We assert that the function $t \mapsto \mu_t\ (t \in R)$ satisfies the cosine equation

$$\mu_{t+s} + \mu_{t-s} = 2\,\mu_t * \mu_s\ t, s \in \mathbf{R}. \tag{5}$$

Indeed, for $f \in K(S)$ and $t, s \in \mathbf{R}$ we have

$$\int_S f\, d(\mu_{t+s} + \mu_{t-s}) =$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left[\sum_{k=0}^{2n} \binom{2n}{k} t^k s^{2n-k} + \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} t^k s^{2n-k} \right] \int_S f\, d\mu^n =$$

$$= 2 \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{t^{2k} s^{2n-2k}}{(2k)!(2n-2k)!} \right] \int_S f\, d\mu^n,$$

and

$$\int_S f\, d(\mu_t * \mu_s) = \int_S \left[\sum_{k=0}^{\infty} \frac{s^{2k}}{(2k)!} \int_S f(xy)\, d\mu^k(y) \right] d\mu_t(x) =$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^{2i} s^{2k}}{(2i)!(2k)!} \int_S f\, d\mu^{k+i} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{t^{2k} s^{2n-2k}}{(2k)!(2n-2k)!} \right] \int_S f\, d\mu^n$$

Therefore (5) holds.

Now we shall consider the following problem. If we have a function $t \mapsto \mu_t$ from $\mathbf{R}$ into $P(S)$ which satisfies (5) does there exist a measure μ, which is not in $P(S)$, such that (3) and (4) hold.

For this purpose, by $M^1(S)$ we denote the set of all regular (real-valued) signed measures on B_S.

Theorem. *Let S be a locally compact (Hausdorff) second countable semigroup with identity e and let* $t \mapsto \mu_t$ *be a function from* $\mathbf{R}$ *into P(S) which satisfies cosine equation (5) and*

$$\mu_0 = \delta_e, \lim_{t \to 0} \mu_t(e) = 1. \tag{6}$$

Then there exists an unique $\mu \in M^1(S)$ *such that* $\mu_t = cos^*\, t\, \mu$, $t \in \mathbf{R}$ *in the sense of (3) and* (4).

Proof. By the Riesz representation theorem there is a one-to-one correspondence between the set $M^1(S)$ and the dual of $K(S)$. Elements from $P(S)$ correspond to positive functionals with norm one. By [1], VIII. 3.1, p.142, $M^1(S)$ is Banach algebra with convolution as multiplication and norm defined by

$$\|\mu\| = |\mu|(S), \ \mu \in M^1(S), \tag{7}$$

where $|\mu|$ is the total variation of μ. δ_e is identity in $M^1(S)$.

Since $P(S) \subset M^1(S)$ function $t \mapsto \mu_t$ ($t \in \mathbf{R}$) is the cosine function from $\mathbf{R}$ into $M^1(S)$. Moreover we have

$$\|\mu_t - \delta_e\| = |\mu_t - \delta_e|(S) = 1 - \mu_t(e) + \mu_t(\{e\}^c) = 2[1 - \mu_t(e)].$$

It follows from (6) that

$$\lim_{t \to 0} \mu_t = \delta_e \ (in \, M^1(S)). \tag{8}$$

Thus the function $t \mapsto \mu_t$ from $\mathbf{R}$ into $M^1(S)$ is continuous, and therefore measurable cosine function. Then by [3] (Theorem 1) there exists the unique $\mu \in M^1(S)$ such that

$$\mu_t = \cos t \, \mu = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \mu^n, \ t \in \mathbf{R}, \tag{9}$$

where the series in (9) is convergent in $M^1(S)$ for every $t \in \mathbf{R}$.

Since

$$\left| \int_S f(x) \, d\mu(x) \right| \leq \|\mu\| \|f\|, \ , f \in K(S), \ \mu \in M^1(S) \tag{10}$$

we have $\mu_t = \cos^* t \, \mu$ in the sense of (3) and (4).

Q.E.D.

References

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O kosinusnoj jednadžbi za vjerojatnosne mjere na lokalno kompaktnim polugrupama

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SAŽETAK

U članku se proučava kosinusova jednadžba (5), gdje je $\mu_t (t \in \mathbf{R})$ vjerojatnosna mjera na lokalno kompaktnoj polugrupi. Ako, povrh toga, funkcija $t \mapsto \mu_t (t \in \mathbf{R})$ zadovoljava uvjet (6), pokazano je da je rješenje jednadžbe (5) oblika $\mu_t = \cos^* t\mu (t \in \mathbf{R})$, gdje je $\mu \in M^1(S)$ jedinstven.

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