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A NOTE ON THE HYPOTHETICAL PROJECTIVE PLANE OF ORDER 39

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Abstract

An idea for the construction of finite projective planes, proposed by Z. Janko, is investigated in the case of a hypothetical plane of order 39. It is shown that such a plane cannot exist under the assumption of the action of a certain collineation group which includes both elations and homologies.

Key words: *projective plane, collineation group*

1. Introduction

One of the central open problems in the field of finite geometries is the existence of finite projective planes of order n which is not a power of a prime number. Investigations of hypothetical planes for the smallest "undecided" values of n, that is 12 and 15, have not left much hope for solution by any of the usual methods, partly because of insufficient power of computers. Thus it seems too extravagant to "attack" a plane with n as big as 39, where the number of points and lines equals $v = n^2 + n + 1 = 1561$. However, 39 is chosen as the smallest product of two different primes which, according to the known results about collineations of finite projective planes (Hughes' theorems, see eg. [2,p.82, eq.(3)]), could allow a plane of that order possessing both types of central colineations - elations and homologies. Therefore, Z. Janko has proposed investigation of the plane **P** of order 39 under the action of an appropriately large group $G \leq$ Aut **P** which comprises elations and homologies. Let us assume that $G \cong (Z_{39} \times Z^*_{39})\, Z_6$, whereby $Z_{39} = <\alpha>$ is a cyclic group of collineations fixing exactly one point and one line, $Z^*_{39} = <\beta>$ is a cyclic group of elations and Z_6 is a cyclic group generated by an involutory homology τ and an element φ of order 3 which keeps invariant all the points and lines of a subplane of order 3. Furthermore, τ normalizes $<\alpha>$ and $<\beta>$. The collineation φ will not be taken into account in the following analysis, because later on it turns out to be superfluous (although its fixed subplane makes the whole situation more interesting and does not collide with other assumptions).

Before we continue with examination of the action of given collineations, let us mention that the existence of an elation of order 39 is not contradictory in itself, but it is well known that nontrivial elations with a common axis and different centres must be of the same order, which is necessarily a prime.

2. The action of central collineations

Denote the centre and the axis of β by C and l, respectively. Clearly, these are also the only elements fixed by α. Besides C, the line l contains a full $<\alpha>$-orbit of points, which will be denoted by C_i, i = 0, 1, ..., 38, with α acting in the obvious manner, increasing the indices by 1, mod 39. The set of points outside of l is partitioned into 39 orbits which we denote by "big" numbers 0, 1, ..., 38, whereby particular points are determined by indices with values in the same set of integers. Thus α maps I_j onto I_{j+1}, index taken mod 39. Without loss of generality, we may assume that the orbit of lines passing through C, different from l, is represented by a line h containing the points I_0, $I = 0, 1, ..., 38$, while the elation β is given by $I_j \to (I+1)_j$ with $I+1$ taken mod 39.

Let us now examine the homology τ. We will show that it acts invertingly on $< \alpha >$, commutes with β and that C_0 and h may be chosen as its centre and axis, respectively. The line l is τ-invariant and there is at least one point fixed by τ lying outside of l. We may choose 0_0 for that point. Furthermore, we choose C_0 to be fixed by τ so that the lines l, h and 0_0C_0 form the triangle with sides fixed by τ.

Assume that $\tau\alpha\tau = \alpha^k$. Then, for any integer i, $\alpha^i\tau = \tau\alpha^{ki}$. This relation will be used several times.

Now $(0_i)\tau = (0_0)\alpha^i\tau = (0_0)\tau\alpha^{ki} = 0_{ki}$. It follows that $(0_{ki})\tau = O_{k^2 i}$. The action of τ on the points C_i is given in the same form, as multiplication of indices by k. Since $\tau^2 = 1$, we have $k^2 i \equiv i$ (mod 39). This holds for any i, hence $k^2 - 1 = 0$ (mod 39). The solutions in the set of residues mod 39 are 1,14,25 and 38, that is 1,-1,14 and -14 (mod 39). If $k = 1$, then τ commutes with α and τ keeps fixed all the points from the orbits C and 0, which is impossible. If $k = 14$ or -14, fixed points in these orbits are determined by the solutions of the equation $(k-1)i \equiv 0$ (mod 39). In both cases, τ would have more than two fixed points on the lines l and 0_0C_0, given by values of i divisible by 3, resp. 13, which is impossible. Thus $k = -1$ and $\tau\alpha\tau = \alpha^{-1}$. Previous argument also shows that the line 0_0C_0 cannot be the axis of τ.

Acting by $\alpha\tau = \tau\alpha^{-1}$ on the point C_0 we see that the line l is not the axis of τ, either. Hence, h is the axis, so that each point I_0 is fixed by τ.

As τ keeps each $< \alpha >$-orbit invariant, it interchanges the points I_j and I_{-j} for $I = 0, 1, 2, ..., 38$ and for $I = C$, whereby we write $-j$ (mod 38) instead of $38 - j$. Now the crucial part of the construction consists of building 39 lines which join the point C_0 to the points $0_0, 1_0, ..., 38_0$ on the line h, because these lines represent different $< \alpha >$-orbits. These lines are also $< \tau >$- invariant, which implies that, besides the points fixed by τ, every one of them contains 19 pairs of the form I_jI_{-j}.

Finally, consider the normalizing action of τ on the subgroup $< \beta >$. Assuming $\tau\beta\tau = \beta^s$ for some integer s and applying this relation to the point 0_0, we get $(0_0)\beta^s = 1_0$, implyng $s = 1$. Hence τ centralizes $< \beta >$.

3. The orbit structure

Now let us consider the associated orbit structure, i.e. a matrix M of order 39 such that its rows are in one-to-one correspondence with the lines C_0I_0, $I = 0, 1, ..., 38$, and its columns correspond to orbit symbols 0, 1, ..., 38 (the orbit C may be momentarily neglected). Positions in the matrix M are filled by 0, 1 or 2, according to the number of times that a particular symbol occurs on a certain line. Each row contains a unique symbol 1, whereas the remaining 38 positions are occupied by 0's and 2's, in equal number.

It is easy to prove that the inner product of different rows (obtained in fact by counting the intersection points of a representative of one line orbit with all the lines from another orbit) must have constant value, equal to 38. (Recall that the last remaining intersection occurs in C_0). It is important to notice that this value may be only realized in such a manner that any two distinct rows have exactly nine time the symbol 2 in the same column, while 1 and 2 appear together in exactly one column. This follows from $39 \equiv 3 \pmod 4$, whereas for $n \equiv 1 \pmod 4$ there would exist different possibilities. Finally, we replace all non-zero symbols in M by a unique symbol 1. Then M obviously turns into an incidence matrix for a symmetric design with parameters (39, 20, 10), hence the complementary design of a Hadamard design with (39, 19, 9).

Such a design does exist, but in our case it would have a regular automorphism group induced by the action of $<\beta>$. It is impossible, according to the following.

Theorem. (Hall - Ryser, see. e.g. [1, p. 272-274])

Let G be a group of order v with a normal subgroup of index w. If G contains a difference set for parameters (v, k, λ), then the diophantine equation $nx^2 + (-1)^{(w-1)/2} wy^2 = z^2$ has a nontrivial solution.

Since any group of order 39 has a normal subgroup of order 13 and the equation $10x^2 - 3y^2 = z^2$ has no nontrivial solution in Z^3, our assertion is proved, so that the assumed orbit structure does not exist.

Taking into account that in a projective plane of order 39 any involutory collineation is necessarily a homology, we can state the following

Theorem 1. There is no projective plane of order 39 with a collineation group G given by $G = <\alpha, \beta, \tau \mid \alpha^{39} = \beta^{39} = \tau^2 = 1, \alpha\beta = \beta\alpha, \tau\alpha\tau = \alpha^{-1}, \tau\beta = \beta\tau>$, where α has a unique fixed point and β is an elation.

4. Concluding remarks

If we take into account the action of the collineation φ, which keeps invariant a subplane of order 3, the points of that subplane are given by C_j and I_j, where both the orbit symbol I and the index j take values 0, 13 and 26. This is derived from the fact that if φ has order 3 and normalizes $<\alpha>$ of order 39, then φ permutes $<\alpha>$-orbits as the multiplier 16 or 22. Hence 0, 13 and 26 are invariant numbers (mod 39).

Let us finish with a few more remarks. In general, for $n \equiv 3 \pmod 4$, the assumptions corresponding to those in Janko's problem (if n fulfills other known conditions) imply an orbit structure which, on its turn, induces the complementary design of a Hadamard design with parameters (n, (n-1)/2, (n-3)/4). Conversely, starting from such a Hadamard design, one should try to reconstruct the orbit matrix for the action of the dihedral group D_n on a projective plane of order n, excluding the additional action of a group of elations. This actually means keeping the symbol 1 in just the right positions in the matrix and replacing all the other 1's by 2.

It is not very likely that such a procedure could yield interesting new results, but a positive outcome may be illustrated by the case of $n = 7$. A plane of order 7 can be constructed starting from a Hadamard design for (7, 3, 1), i.e. a plane of order 2.

References

[1] Th. Beth, D. Jungnickel and H. Lenz: Design Theory, Cambridge University Press, London 1986.
[2] P. Dembowski: Finite Geometrics, Springer, Berlin-Heidelberg-New York 1968.

O hipotetičkoj projektivnoj ravnini reda 39

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SAŽETAK

Jedna zamisao Z. Janka o konstrukciji konačnih projektivnih ravnina istražena je u slučaju hipotetičke projektivne ravnine reda 39. Pokazano je da takva ravnina ne postoji uz pretpostavku djelovanja određene grupe kolineacije, koja sadrži i elacije i homologije.

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