UDK 517.51
517.958
Izvorni znanstveni članak

Prihvaćeno za tisak: 3.10.2005.

SOME DISCRETE INEQUALITIES OF GRÜSS TYPE AND APPLICATIONS IN GUESSING THEORY

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Abstract

Some discrete inequalities of Grüss type and their applications in estimating the p-moments of guessing mapping are given.

Key words and phrases: Discrete Inequalities, Guessing Mapping, *p*-Moments

1. Introduction

In 1935, G. Grüss proved the following integral inequality which gives an approximation of the integral of the product in terms of the product of the integrals as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma) \tag{1.1}$$

where f, g: $[a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfy the condition

$$\varphi \leq f(x)\leq \Phi, \gamma \leq g(x) \leq \Gamma \tag{1.2}$$

for each $x \in [a, b]$ where φ, Φ, γ, Γ, are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For a simple proof of (1.1) as well as some other integral inequalities of Grüss' type see Chapter X of the recent book [1] by Mitrinović, Pečarić and Fink.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardzewski [1, Ch. X] established the following discrete version of Grüss' inequality:

Theorem 1. *Let $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ be two n-tuples of real numbers such that $r \leq a_i \leq R$ and $s \leq b_i \leq S$ for $i = 1, ..., n$. Then one has:*

$$\left|\frac{1}{n}\sum_{i=1}^{n}a_ib_i-\frac{1}{n}\sum_{i=1}^{n}a_i\cdot\frac{1}{n}\sum_{i=1}^{n}b_i\right| \tag{1.3}$$

Date: March, 1999.
1991 *Mathematics Subject Classification*. 26D15, 94Axx.

$$\leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s)$$

where $[x]$ is the integer part of x, $x \in R$.

A weighted version of Grüssn discrete inequality was proved by J.E. Pečarić in 1979, [1, Ch. X]:

Theorem 2. *Let a and b be two monotonic n-tuples and p a positive one. Then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i b_i\right| \tag{1.4}$$

$$\leq |a_n - a_1||b_n - b_1| \max_{1\leq k\leq n-1}\left(\frac{P_k \overline{P}_{k+1}}{P_n^2}\right),$$

where $P_n := \sum_{i=1}^{n} p_i$, $\overline{P}_{k+1} = P_n - P_{k+1}$.

In 1981, A. Lupaş [1, Ch. X] proved some similar results for the first difference of a as follows:

Theorem 3. *Let a, b be two monotonic n-tuples in the same sense and p a positive n-tuple. Then*

$$\min_{1\leq i\leq n-1}|a_{i+1} - a_i| \min_{1\leq i\leq n-1}|b_{i+1} - b_i| \left[\frac{1}{P_n}\sum_{i=1}^{n} i^2 p_i - \left(\frac{1}{P_n}\sum_{i=1}^{n} i p_i\right)^2\right] \tag{1.5}$$

$$\leq \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i b_i$$

$$\leq \max_{1\leq i\leq n-1}|a_{i+1} - a_i| \max_{1\leq i\leq n-1}|b_{i+1} - b_i| \left[\frac{1}{P_n}\sum_{i=1}^{n} i^2 p_i - \left(\frac{1}{P_n}\sum_{i=1}^{n} i p_i\right)^2\right].$$

If there exist the numbers $\bar{a}$, $\bar{a}_1$, r, r_1 ($rr_1 > 0$) such that $a_k = \bar{a} + kr$ and $b_k = \bar{a}_1 + kr_1$, then in (1.5) *the equality holds.*

For some generalizations of Grüss' inequality for isotonic linear functionals defined on certain spaces of mappings see Chapter X of the book [1] where further references are given.

In the recent paper [2], S.S. Dragomir and G.L. Booth obtained the following inequality of Grüss' type:

Theorem 4. *Let α_i, x_i be real numbers and $p_i \geq 0$* (i = 1, ..., n) *so that* $\sum_{i=1}^{n} p_i = 1$.

Then we have the inequality:

$$\left|\sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \cdot \sum_{i=1}^{n} p_i x_i\right| \tag{1.6}$$

$$\leq \max_{1\leq j\leq n-1}\left|\alpha_{j+1}-\alpha_j\right| \max_{1\leq j\leq n-1}\left|x_{j+1}-x_j\right| \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right].$$

The inequality (1.6) *is sharp in the sense that the constant C = 1 in the right side can not be replaced by a smaller one.*

Note that in paper [2] they proved the inequality (1.6) in the general case of normed linear spaces.

The main aim of this paper is to point out another inequality of Grüss' type and apply it to estimate the moments of guessing mappings as in the papers [5]-[7].

The following results holds:

Theorem 5. *Let a_i, b_i ($i = 1, ..., n$) be a real number and p_i ($i = 1, ..., n$) a probability distribution. Define*

$$a_{ij}^{(\alpha)} := \left(\sum_{k=j+1}^{i}\left|a_k - a_{k-1}\right|^{\alpha}\right)^{\frac{1}{\alpha}}, \quad b_{ij}^{(\beta)} := \left(\sum_{k=j+1}^{i}\left|b_k - b_{k-1}\right|^{\beta}\right)^{\frac{1}{\beta}}$$

for all $1 \leq j < i \leq n$ *and* $\alpha > 1, \frac{1}{\alpha}+\frac{1}{\beta} = 1$. *Then we have the inequality*

$$\left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i b_i\right| \tag{2.1}$$

$$\leq \left(\sum_{1\leq j<i\leq n}(i-j)p_i p_j \left[a_{ij}^{(\alpha)}\right]^{\alpha}\right)^{\frac{1}{\alpha}} \times \left(\sum_{1\leq j<i\leq n}(i-j)p_i p_j \left[b_{ij}^{(\beta)}\right]^{\beta}\right)^{\frac{1}{\beta}}$$

$$\leq a_{n1}^{(\alpha)}\, b_{n1}^{(\beta)} \sum_{1\leq j<i\leq n}(i-j)p_i p_j .$$

The first inequality in (2.1) *is sharp.*

Proof. It is well known that the following equality holds:

$$\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i b_i$$

$$= \sum_{1\le j<i\le n} p_i p_j (a_i - a_j)(b_i - b_j).$$

Using the simple observation which asserts that

$$a_i - a_j = \sum_{k=j+1}^{i}(a_k - a_{k-1}), b_i - b_j = \sum_{k=j+1}^{i}(b_k - b_{k-1})$$

and the generalized triangle inequality, we get

$$(2.2) \quad \left|\sum_{i=1}^{n} p_i a_i b_1 - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i b_i\right|$$

$$\le \sum_{1\le j<i\le n} p_i p_j \sum_{k,l=j+1}^{i} \left|(a_k - a_{k-1})(b_l - b_{l-1})\right|.$$

Using Hölder's discrete ineequality for double sums, we have

$$(2.3) \quad \sum_{k,l=j+1}^{i} \left|(a_k - a_{k-1})(b_l - b_{l-1})\right|$$

$$\le \left(\sum_{k,l=j+1}^{i} \left|a_k - a_{k-1}\right|^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{k,l=j+1}^{i} \left|b_k - b_{k-1}\right|^{\beta}\right)^{\frac{1}{\beta}}$$

$$= (i-j)^{\frac{1}{\alpha}} \left(\sum_{k=j+1}^{i} \left|a_k - a_{k-1}\right|^{\alpha}\right)^{\frac{1}{\alpha}} \times (i-j)^{\frac{1}{\beta}} \left(\sum_{k=j+1}^{i} \left|b_k - b_{k-1}\right|^{\beta}\right)^{\frac{1}{\beta}}$$

$$= (i-j) a_{ij}^{(\alpha)}\, b_{ij}^{(\beta)}$$

for all $1 \le j < i \le n$.

Now, by (2.2) and (2.3) and by Hölder's inequality, we have:

$$\left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i b_i\right|$$

$$\le \sum_{1\le j<i\le n} (i-j) p_i p_j a_{ij}^{(\alpha)} b_{ij}^{(\beta)}$$

$$\leq\left(\sum_{1\leq j<i\leq n}(i-j)p_ip_j\left[a_{ij}^{(\alpha)}\right]^{\alpha}\right)^{\frac{1}{\alpha}}\times\left(\sum_{1\leq j<i\leq n}(i-j)p_ip_j\left[b_{ij}^{(\beta)}\right]^{\beta})\right)^{\frac{1}{\beta}}$$

and the first inequality in (2.1) is proved.

For the second inequality, let observe that

$$a_{ij}^{(\alpha)}\leq a_{n1}^{(\alpha)} \text{ and } b_{ij}^{(\beta)}\leq b_{n1}^{(\beta)} \text{ for all} 1\leq j\leq i\leq n.$$

Then

$$\left(\sum_{1\leq j<i\leq n}(i-j)p_ip_j\,[a_{ij}^{(\alpha)}]^{\alpha}\right)^{\frac{1}{\alpha}}\times\left(\sum_{1\leq j<i\leq n}(i-j)p_ip_j[b_{ij}^{(\beta)}]^{\beta}\right)^{\frac{1}{\beta}}$$

$$\leq a_{n1}^{(\alpha)}\left(\sum_{1\leq j<i\leq n}(i-j)p_ip_j\right)^{\frac{1}{\alpha}}\times b_{n1}^{(\beta)}\left(\sum_{1\leq j<i\leq n}(i-j)p_ip_j\right)^{\frac{1}{\beta}}$$

$$\leq a_{n1}^{(\alpha)}b_{n1}^{(\beta)}\sum_{1\leq j<i\leq n}(i-j)p_ip_j$$

and the second inequality is also proved.

For the sharpness of the first inequality in (2.1), let choose $a_i = a_1 + (i-1)a$, $b_i = b_1 + (i-1)b$ with $a, b > 0$ and $i = 2, ..., n$. Then we have:

$$\sum_{i=1}^{n}p_ia_ib_i-\sum_{i=1}^{n}p_ia_i\cdot\sum_{i=1}^{n}p_ib_i$$

$$=ab\sum_{1\leq j<i\leq n}(i-j)^2\,p_ip_j\ .$$

Also, we have

$$a_{ij}^{(\alpha)}=a(i-j)^{\frac{1}{\alpha}},b_{ij}^{(\beta)}=b(i-j)^{\frac{1}{\beta}}$$

and then

$$\left(\sum_{1\leq j<i\leq n}(i-j)p_ip_j\,[a_{ij}^{(\alpha)}]^{\alpha}\right)^{\frac{1}{\alpha}}\times\left(\sum_{1\leq j<i\leq n}(i-j)p_ip_j[b_{ij}^{(\beta)}]^{\beta}\right)^{\frac{1}{\beta}}$$

$$= a\left(\sum_{1\le j<i\le n}(i-j)^2\, p_i p_j\right)^{\frac{1}{\alpha}} \times b\left(\sum_{1\le j<i\le n}(i-j)^2\, p_i p_j\right)^{\frac{1}{\beta}}$$

$$\le ab\sum_{1\le j<i\le n}(i-j)^2\, p_i p_j$$

and the sharpness of the first inequality in (2.1) is proved.

The following corollary holds:

Corollary 1. *With the above assumptions we have:*

$$\text{(2.4)}\quad \left|\frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \cdot \frac{1}{n}\sum_{i=1}^{n} b_i\right|$$

$$\le \frac{1}{n^2}\left(\sum_{1\le j<i\le n}(i-j)[a_{ij}^{(\alpha)}]^{\alpha}\right)^{\frac{1}{\alpha}} \times \left(\sum_{1\le j<i\le n}(i-j)[b_{ij}^{(\beta)}]^{\beta}\right)^{\frac{1}{\beta}}$$

$$= \frac{n^2-1}{6n}\left(\sum_{k=2}^{n}\left|a_k - a_{k-1}\right|^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{k=2}^{n}\left|b_k - b_{k-1}\right|^{\beta}\right)^{\frac{1}{\beta}}.$$

The first inequality in (2.4) *is sharp.*

Proof. The first inequality is obvious by the above theorem.

Let us compute

$$\sum_{1\le j<i\le n}(i-j)$$

$$= \sum_{1\le j\le 2}(2-j) + \sum_{1\le j\le 3}(3-j) + \ldots + \sum_{1\le j\le n}(n-j)$$

$$= 2\cdot 2-(1+2)+3\cdot 3-(1+2+3)+\ldots+n\cdot n-(1+2+\ldots+n)$$

$$= 1^2+2^2+\ldots+n^2-1-(1+2)-(1+2+3)-\ldots-(1+2+\ldots+n)$$

$$= \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n}\frac{k(k+1)}{2} = \frac{1}{2}\left(\sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k\right) = \frac{n\,(n^2-1)}{6}$$

and the corollary is thus proved.

3. Applications for the Moments of Guessing Mapping

J.L. Massey in the paper [3] considered the problem of guessing the value of realization of a random variable *X* by asking questions of the form: "Is *X* equal to *x*?" until the answer is "Yes".

Let *G (X)* denote the number of guesses required by a particular guessing strategy when $X = x$.

Massey observed that *E (G (X))*, the average number of guesses is minimized by a guessing strategy that guesses the possible values of *X* in decreasing order of probability.

We begin by giving a formal and generalized statement of the above problem by following E. Arikan [4].

Let *(X, Y)* be a pair of random variable with *X* taking values in a finite set χ of size *n, Y* taking values in a countable set *Y*. Call a function *G (X)* of the random variable X a guessing function in *X* if $G: \chi \to \{1, ..., n\}$ in one-to-one. Call a function G (X | Y) a guessing function for X given Y if for any fixed value Y = y, *G (X | y)* is a guessing function for *X*. *G (X | y)* will be thought of as the number of guessing required to determine *X* when the value of Y is given.

The following inequalities on the moments of *G (X)* and *G (X | Y*) were proved by E. Arikan in the recent paper [4].

Theorem 6. *For an arbitrary guessing function G (X) and G (X | Y) and any* $p \geq 0$, *we have:*

$$E(G(X)^p) \geq (1+\ln n)^{-p} \left[\sum_{x \in \chi} P_X(x)^{\frac{1}{1+p}} \right]^{1+p} \tag{3.1}$$

and

$$E(G(X|Y)^p) \geq (1+\ln n)^{-p} \sum_{y \in Y} \left[\sum_{x \in \chi} P_{X,Y}(x,y)^{\frac{1}{1+p}} \right]^{1+p} \tag{3.2}$$

where $P_{X,Y}$ *and* P_X *are probability distributions of (X, Y) and X, respectively.*

In paper [7], S.S. Dragomir and J. van der Hoek have proved the following estimation result for the moments of guessing mapping:

Theorem 7. *Assume that*

$$P_M := \max\{p_i \mid i = 1, ..., n\} \text{ an } P_m := \min\{p_i \mid i = 1, ..., n\}$$

and $P_M \neq P_m$. *Then we have the estimates:*

$$G_p(n)\left[P_m n^{p+1} + \frac{1}{(P_M - P_m)^p}(1 - nP_m)^{p+1}\right]$$

$$\leq E\,(G\,(X)^p)$$

$$\leq G_p(n)\left[P_M n^{p+1} + \frac{1}{(P_M - P_m)^p}(nP_M - 1)^{p+1}\right]$$

for $p \geq 1$, where

$$G_p\,(n) := \frac{S_p(n)}{n^{p+1}}$$

and

$$S_p(n) := \sum_{i=1}^{n} i^p.$$

Corollary 2. *With the above assumption, we have:*

$$\frac{1}{2}\left(1+\frac{1}{n}\right)\frac{P_m P_M n^2 - 2nP_m + 1}{P_M - P_m}$$

$$\leq E(G(X)) \leq \frac{1}{2}\left(1+\frac{1}{n}\right)\frac{-P_m P_M n^2 + 2P_M n - 1}{P_M - P_m}.$$

For other estimations of $E\,(G\,(X)^p)$ see the papers [5]-[7].

Now, let us introduce the notations

$$h_{ij}^{(\alpha)}(p) := \left[\sum_{k=j+1}^{i}(k^p - (k-1)^p)^\alpha\right]^{\frac{1}{\alpha}}$$

where $1 \leq j < i \leq n$, $p > 0$, $\alpha > 1$.

The following proposition holds.

Proposition 1. Let $p, q > 0$. Under the above assumptions, we have

$$\left| E(G(X)^{p+q}) - E(G(X)^p) \right|$$

$$\leq \left(\sum_{1\leq j<i\leq n}(i-j)p_i p_j\,[h_{ij}^{(\alpha)}(p)]^\alpha\right)^{\frac{1}{\alpha}} \times \left(\sum_{1\leq j<i\leq n}(i-j)p_i p_j\,[h_{ij}^{(\beta)}(q)]^\beta\right)^{\frac{1}{\beta}}$$

$$\leq \left[\sum_{k=2}^{n}(k^p - (k-1)^p)^\alpha\right]^{\frac{1}{\alpha}}\left[\sum_{k=2}^{n}(k^q - (k-1)^q)^\beta\right]^{\frac{1}{\beta}}\sum_{1\leq j<i\leq n}(i-j)p_i p_j$$

where $\alpha, \beta > 1$ *and* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

The proof follows by Theorem 5 applied for $a_i = i^p$, $b_i = i^q$ $(i = 1, ..., n)$. We shall omit the details.

Now, starting to the probability distribution $p_i \geq 0$ $(i = 1, ..., n)$ we define:

$$p_{ij}^{(\beta)} := \left(\sum_{k=j+1}^{i} \left| p_k - p_{k-1} \right|^{\beta} \right)^{\frac{1}{\beta}}$$

where $1 \leq j < i \leq n, \beta > 1$.

Using Corollary 1 we have the following estimation of the moments of guessing mapping:

Proposition 2. *Let* $p > 0$. *Then we have the estimate.*

$$\left| E(G(X)^p) - S_n(p) \right|$$
$$\leq \frac{1}{n} \left(\sum_{1 \leq j < i \leq n} (i-j) [h_{ij}^{(\alpha)}(p)]^{\alpha} \right)^{\frac{1}{\alpha}} \times \left(\sum_{1 \leq j < i \leq n} (i-j) [p_{ij}^{(\beta)}]^{\beta} \right)^{\frac{1}{\beta}}$$
$$\leq \frac{n^2 - 1}{6} \left[\sum_{k=2}^{n} (k^p - (k-1)^p)^{\alpha} \right]^{\frac{1}{\alpha}} \left(\sum_{k=2}^{n} \left| p_k - p_{k-1} \right|^{\beta} \right)^{\frac{1}{\beta}}.$$

In particular, we have the following bound for the average number of guesses:

$$\left| E(G(X)) - \frac{n+1}{2} \right|$$
$$\leq \frac{1}{n} \left[\frac{n^2 (n^2 - 1)}{12} \right]^{\frac{1}{\alpha}} \left(\sum_{1 \leq j \leq i \leq n} (i-j) [p_{ij}^{(\beta)}]^{\beta} \right)^{\frac{1}{\beta}}$$
$$\leq \frac{(n^2 - 1)(n-1)^{\frac{1}{\alpha}}}{6} \left(\sum_{k=2}^{n} \left| p_k - p_{k-1} \right|^{\beta} \right)^{\frac{1}{\beta}}.$$

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Diskretne nejednakosti Grüssovog tipa i primjene u tepriji odlučivanja

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SAŽETAK

Dokazane su neke nejednakosti Grüssovog tipa i dane su njihove primjene pri procjenjivanju p-momenata preslikavanja u teoriji odlučivanja

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