UDK 512.7
514.112
Izvorni znanstveni članak

Prihvaćeno za tisak: 2.10.2001.

ON ALGEBRAIC EQUATIONS CONCERNING CHORDAL AND TANGENTIAL POLYGONS

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Abstract

Some properties of the equations (11) and (12) are proved (Theorem 1-5) and established that the positive roots of these equations are radii of a sequance of chordal or tangential polygons with the same lenths of sides or the same lengths of tangents.

Key words: algebraic equation, chordal polygon, tangential polygon

1. Preliminaries

A polygon with vertices $A_1, \ldots, A_n$ (in this order) will be denoted by $A_1 \ldots A_n$ shortly by $\boldsymbol{A}$. The lenghts of its sides will be denoted by $|A_1A_2|, \ldots, |A_nA_1|$ or $a_1, \ldots, a_n$, and the interior angle at the vertex A_i by α_i or $\angle A_i$, i.e.

$$\angle A_i = \angle A_{n-1+i}A_iA_{i+1},\ i = 1, \ldots, n.$$

Of course, indices are calculated modulo n.

A polygon $\boldsymbol{A} = A_1 \ldots A_n$ is a chordal polygon if there exists a circle $\mathscr{C}$ such that each vertex of $\boldsymbol{A}$ lie on $\mathscr{C}$.

A polygon $\boldsymbol{A} = A_1 \ldots A_n$ is a tangential polygon if there exists a circle $\mathscr{C}$ such that each side of $\boldsymbol{A}$ lie on a tangent line of $\mathscr{C}$.

In this paper we shall use the following notation used in [1] and [2].

If it is a question of a chordal polygon $A_1 \ldots A_n$, then C is the centre of its circumcircle, and $\beta_1, \ldots, \beta_n$ are the angles such that

$$\beta_i = \angle CA_iA_{i+1},\ i = 1, \ldots, n. \tag{1a}$$

If it is a question of a tangential polygon $A_1 \ldots A_n$, then C is the centre of its inscribed circle, and the angles $\beta_1, \ldots, \beta_n$ are such that $\beta_i = \frac{1}{2}\angle A_i$, that is

$$\beta_i = \angle CA_iA_{i+1},\ i = 1, \ldots, n. \tag{1b}$$

AMS subject classifications: 51E12

If $A_1 \dots A_n$ is a chordal polygon or a tangential polygon, then by $\phi_1, \dots, \phi_n$ will be denoted the angles such that

$$\phi_i = \angle A_i C A_{i+1},\ i = 1, \dots, n. \tag{2}$$

Symbol $\left[\frac{n-1}{2}\right]$. If n is a positive integer, then

$$\left[\frac{n-1}{2}\right] = \frac{n-1}{2} \text{ if n is odd, } \left[\frac{n-1}{2}\right] = \frac{n-2}{2} \text{ if n is even.}$$

Symbol P_j^n. If j and n are positive integers and $j \le n$, then P_j^n is the sum of $\binom{n}{j}$ products of the form

$$\cos\beta_{i_1} \dots \cos\beta_{i_j}\ \sin\beta_{i_{j+1}} \dots \sin\beta_{i_n}$$

where $(i_1, i_2, \dots, i_n)$ is a permutation of $\{1, 2, \dots, n\}$. For example:

$$P_1^3 = \cos\beta_1 \sin\beta_2 \sin\beta_3 + \sin\beta_1 \cos\beta_2 \sin\beta_3 + \sin\beta_1 \sin\beta_2 \cos\beta_3$$

$$P_3^4 = \cos\beta_1 \cos\beta_2 \cos\beta_3 \sin\beta_4 + \cos\beta_1 \cos\beta_2 \sin\beta_3 \cos\beta_4 +$$

$$+ \cos\beta_1 \sin\beta_2 \cos\beta_3 \cos\beta_4 + \sin\beta_1 \cos\beta_2 \cos\beta_3 \cos\beta_4 .$$

Symbol $S_j(x_1, \dots, x_n)$. Let $x_1, \dots, x_n$ be real numbers, and let j be an integer such that $1 \le j \le n$. Then $S_j(x_1, \dots, x_n)$ is the sum of all $\binom{n}{j}$ products of the form $x_{i_1} \dots x_{i_j}$, where $i_1, \dots, i_j$ are different elements of the set $\{1, \dots, n\}$, that is

$$S_j(x_1, \dots, x_n) = \sum_{1 \le i_1 < \dots < i_j \le n} x_{i_1} \dots x_{i_j} .$$

Symbol T_j^n. The sum $S_j(\tan\beta_1, \dots, \tan\beta_n)$ will be briefly written as T_j^n.

Symbol C_j^n. The sum $S_j(\cot\beta_1, \dots, \cot\beta_n)$ will be briefly written as C_j^n.

Notice 1. In the following where $t_1, \dots, t_n$ will be lenghts of tangents we shall, for brevity, write S_j^n instead of $S_j(t_1, \dots, t_n)$.

For example: $S_1^3 = t_1 + t_2 + t_3$, $S_2^3 = t_1 t_2 + t_2 t_3 + t_3 t_1$, $S_3^3 = t_1 t_2 t_3$.

One property of the angles β_i and φ_i given by (1a), (1b) and (2).

In this paper we shal use oriented angles. As it is known, an angle $\angle PQR$ is positively or negatively oriented if from QP to QR is going counter-clockwise or clockwise.

It is very important to remark that the angles β_i and φ_i have opposit orientation, that is, if β_i is positively oriented, then φ_i is negatively oriented, and vice versa. For example, see Fig. 1.

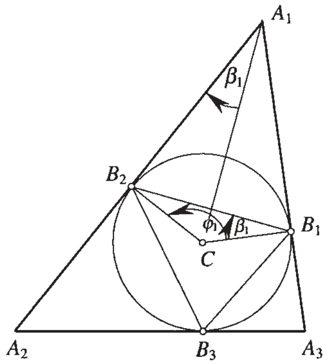

Fig. 1

The measure of an oriented angle will be taken with sign + or – depending on whether the angle is positively or negatively oriented.

2. Connection between chordal and tangential polygons

It is easy to see that every tangential polygon determines a chordal polygon. So, if $\boldsymbol{A}=A_1...A_n$ is any given tangential polygon and if $B_1, ..., B_n$ are its tangential points with the circle inscribed into A such that $B_i \in A_iA_{n+i-1}$, $i=1, ..., n$, then the polygon B is a chordal polygon determined by the polygon $\boldsymbol{A}$ and it is valid

$$\beta_i = \angle CA_iA_{i+1} = \angle CB_iB_{i+1},\ i=1,\dots,n \tag{3}$$

$$b_i^2 = \frac{4r^2t_i^2}{r^2+t_i^2},\ i=1,\dots,n. \tag{4}$$

where r is the radius of the inscribed circle into $\boldsymbol{A}$, $b_i = |B_iB_{i+1}|$, $i=1,\dots,n$, and t_1, ..., t_n are the lenghts of the tangents of $\boldsymbol{A}$, i.e.

$$t_i = |A_iB_i|,\ i=1,\dots,n.$$

Let us remark that (4) follows from

$$b_i = 2r\cos\beta_i,\ b_i = 2t_i\sin\beta_i. \tag{5}$$

Consequently, for any relation which is valid for a tangential polygon there is a relation which is valid for the corresponding chordal polygon. Therefore, the relations (3) and (4) will play important role in the following considerations.

Notice 2. In order that every chordal polygon be the corresponding of a tangential polygon, we shall permit that some of the angles $b_1, \ldots, b_n$ may be zero, but we shall exclude that some of the angles $b_1, \ldots, b_n$ may be $\frac{\pi}{2}$ (two consecutive vertices the same).

In the following considerations we shall use only oriented angles. So, if it is a question of a chordal (tangential) polygon $A_1 \ldots A_n$, then $\beta_1, \ldots, \beta_n$ will be the measures of the oriented angles

$$\angle CA_iA_{i+1},\ i = 1, \ldots, n..$$

Definition 1. Let $\boldsymbol{A}=A_1 \ldots A_n$ be a chordal polygon, and let k be a positive integer such that $k \le \left[\frac{n-1}{2}\right]$. Then the polygon A will be called k-chordal polygon if

$$\beta_1 + \ldots \beta_n = (n-2k)\frac{\pi}{2}, \tag{6}$$

$$\beta_i > 0,\ i = 1, \ldots, n \text{ or } \beta_i < 0,\ i = 1, \ldots, n.$$

It is easy to see that the chordal polygon $A_1 \ldots A_n$ is a k-chordal polygon iff

$$\varphi_i + \ldots + \varphi_n = 2k\pi, \tag{7}$$

$$\varphi_i > 0,\ i=1, \ldots, n \text{ or } \varphi_i < 0,\ i=1, \ldots, n.$$

So, for example, if $\varphi_i > 0$, $i=1, \ldots n$, then $\beta_i < 0$, $i=1, \ldots, n$, and from (7) it follows that

$$(\pi - 2\beta_i) + \ldots + (\pi - 2\beta_n) = 2k\pi,$$

which can be written as (6). Conversely, from (6) it follows that

$$2\beta_1 + \ldots + 2\beta_n = n\pi - 2k\pi$$

or

$$(\pi - \varphi_1) + \ldots + (\pi - \varphi_n) = n\pi - 2k\pi,$$

which can be written as (7).

Definition 2. Let $\boldsymbol{A} = A_1 \ldots A_n$ be a tangential polygon, and let k be a positive integer such that $k \le \left[\frac{n-1}{2}\right]$. Then the polygon A will be called k-tangential polygon if

$$\beta_1 + \ldots + \beta_n = (n-2k)\frac{\pi}{2},$$

$$\beta_i > 0,\ i=1, \ldots, n \text{ or } \beta_i < 0,\ i=1, \ldots, n.$$

It is easy to see that a k-tangential polygon has the property that any two of its consecutive sides have only one point in common.

In [2] we have proved the following result (Theorem 2 and its Corollary 3):

(**Theorem 2**) Let $\boldsymbol{A} = A_1 \ldots A_n$ be a given 1-tangential polygon. If $t_1, \ldots t_n$ are the lengths of its tangents and r the radius of its inscribed circle, then r is a solution of the equation

$$S_1^n x^{n-1} - S_3^n x^{n-3} + S_5^n x^{n-5} - \ldots + (-1)^s\ S_n^n = 0, \tag{8}$$

or

$$S_1^{n+1} x^{n-1} - S_3^n x^{n-3} + S_5^{n+1} x^{n-5} - \ldots + (-1)^s\ S_n^{n+1} = 0, \tag{9}$$

where n is an odd number, and $s = (1+3+5+\ldots+n)+1$.

(**Corollary 3**) Let m be a positive integer such that $m = \frac{n-1}{2}$, where $n \ge 3$ is odd, and let r_k, $k = 1, \ldots, m$, be the radius of k-tangential polygon whose lengths of its tangents are $t_1, \ldots, t_n$. Then every r_k is a solution of the equation (8), i.e.

$$S_1^n r_k^{n-1} - S_3^n r_k^{n-3} + S_5^n r_k^{n-5} - \ldots + (-1)^s\ S_n^n = 0. \tag{10}$$

Similarly holds for the equation (9) where $n+1$ is an even number. (In [2], Theorem 1, it is proved that for each $k=1, \ldots, \left[\frac{n-1}{2}\right]$ there exists k-tangential polygon whose tangents have the lengths $t_1, \ldots, t_n$.

The following theorem is a generalization of Theorem 2 and its Corollary 3 given in [2].

Theorem 1. *Let $\boldsymbol{A} = A_1 \ldots A_n$ be a tangential polygon, and let its tangents have the lengths $t_1, \ldots t_n$. If $\beta_1, \ldots, \beta_n$ are such that*

$$\beta_u > 0,\ u=i_1, \ldots, i_j \text{ and } \beta_v < 0,\ v=i_{j+1}, \ldots, i_n$$

then the radius r of the inscribed circle into $\boldsymbol{A}$ is a solution of the equation

$$S_1^n x^{n-1} - S_3^n x^{n-3} + S_5^n x^{n-5} - \ldots + (-1)^{s_1}\ S_n^n = 0,\ n \text{ is odd} \tag{11}$$

or

$$S_1^n x^{n-2} - S_3^n x^{n-4} + S_5^n x^{n-6} - \ldots + (-1)^{s_2} S_{n-1}^n = 0, n \text{ is even} \tag{12}$$

where

$$s_1 = (1+3+5+\ldots+n)+1,\ s_2 = (1+3+5+\ldots+(n\text{-}1))+1 \tag{13}$$

and where instead of t_v, $v=i_{j+1}$, ..., i_n is written $-t_v$.

Of course, here S_j^n is not referred to the numbers t_1, ..., t_n but to the numbers t_{i_1}, ..., t_{i_j}, $-t_{i_{j+1}}$, ..., $-t_{i_n}$, that is

$$S_j^n = S_j\,(t_{i_1}, \ldots, t_{i_j}, -t_{i_{j+1}}, \ldots, -t_{i_n}),\ i = 1,3,5,\ldots$$

Proof. First we shall prove the following lemma.

Lemma 1. *If $\boldsymbol{B} = B_1 ... B_n$ is the chordal polygon determined by $\boldsymbol{A}$, then*

$$\left|\beta_1 + \ldots + \beta_n\right| = h \cdot \frac{\pi}{2}, h \in \{0, 1, 2, \ldots, n-2\}. \tag{14}$$

Proof. Since

$$\left|\varphi_1 + \ldots + \varphi_n\right| = 2k\pi,\ k \in \left\{0, 1, \ldots, \left[\frac{n-1}{2}\right]\right\}$$

$$\varphi_i = \begin{cases} -(\pi - 2\beta_i)\ if\ \beta_i \geq 0 \\ \pi + 2\beta_i\ if\ \beta_i < 0 \end{cases}$$

if follows that $\left|\varphi_1 + \ldots + \varphi_n\right| = \left|n\pi - 2(\beta_1 + \ldots + \beta_n)\right|$ or

$$2\left|\beta_1 + \ldots + \beta_n\right| = |n\pi - 2k\pi|$$

which can be written as (14)

Lemma 2. If $\left|\beta_1 + \ldots + \beta_n\right| = j\pi, j \in \{0, 1, \ldots, n-2\}$, *then*

$$T_1^n - T_3^n + T_5^n - \ldots + (-1)^{s_1} T_n^n = 0,\ \ n \text{ is odd} \tag{15}$$

$$T_1^n - T_3^n + T_5^n - \ldots + (-1)^{s_2} T_{n-1}^n = 0,\ \ n \text{ is even} \tag{16}$$

and if $\left|\beta_1 + \ldots + \beta_n\right| = (2j+1)\frac{\pi}{2}, j \in \{0, 1, \ldots, n-2\}$ *then*

$$C_1^n - C_3^n + C_5^n - \ldots + (-1)^{s_1} C_n^n = 0, \text{n, is odd} \tag{17}$$

$$C_1^n - C_3^n + C_5^n - \ldots + (-1)^{s_2}\ C_{n-1}^n = 0,\ n \text{ is even} \tag{18}$$

where s_1 *and* s_2 *are as in* (13).

Proof. In [2], Lemma 1, it is proved that (17) and (18) is valid if

$$0 < \beta_i < \frac{\pi}{2},\ i = 1, \ldots, n$$

$$\beta_1 + \ldots + \beta_n = (n-2k)\frac{\pi}{2},\ k = 1, \ldots, \left[\frac{n-1}{2}\right].$$

In exactly the same way can be proved that (17) and (18) are valid for the given condition here in Lemma 2. Also, in like manner can be proved (15) and (16).

Let us remark that from (17) and (18) (dividing with $(-1)^{s_1} C_n^n$ or $(-1)^{s_2} C_{n-1}^n$) we get

$$1 - T_2^n + T_4^n - \ldots + (-1)^{s_1} T_{n-1}^n = 0, \tag{19}$$

$$1 - T_2^n + T_4^n - \ldots + (-1)^{s_2} T_{n-2}^n = 0, \tag{20}$$

Now, if $\cot\beta_i$ in the equations (17) and (18) is replaced by $\frac{t_i}{r}$ if $\beta_i > 0$ and by $\frac{-t_i}{r}$ if $\beta_i < 0$, we shall get the equtions

$$S_1^n r^{n-1} - S_3^n r^{n-3} + S_5^n r^{n-5} - \ldots + (-1)^{s_1}\ S_n^n = 0 \quad n \text{ is odd}$$

$$S_1^n r^{n-2} - S_3^n r^{n-4} + S_5^n r^{n-6} - \ldots + (-1)^{s_2}\ S_{n-1}^n = 0,\ \ n \text{ is even.}$$

So, Theorem 1 is proved.

Corollary 1.1. *If some of* $\beta_1, \ldots, \beta_n$ *are equals zero, then we can use the equation with the terms* T_i^n *instead of* C_i^n.

Some special cases will be discussed in more detail in the following examples which may be interesting.

Example 1. The triangle $A_1A_2A_3$ in Fig 2a is a tangential triangle and $B_1B_2B_3$ is its coresponding chordal triangle. Since

$$|A_1B_1| = |A_1B_3| = t_1,\ |A_2B_1| = |A_2B_2| = t_2,\ |A_3B_2| = |A_3B_3| = t_3$$

and $\beta_1 < 0$, we can use the equation $1 - T_2^3 = 0$, that is

$$1 - (\tan\beta_1 \tan\beta_2 + \tan\beta_2 \tan\beta_3 + \tan\beta_3 \tan\beta_1) = 0$$

or

$$1-\left(\frac{r}{-t_1}\cdot\frac{r}{t_2}+\frac{r}{t_2}\cdot\frac{r}{t_3}+\frac{r}{t_3}\cdot\frac{r}{-t_1}\right)=0.$$

In the case when $\beta_1 = 0$ (Fig. 2b), then $t_1 = \infty$, and we have the equation $r^2 = t_2t_3$.

Of course, the same result can be obtained from the equation $S_1^3 r^2 - S_3^3 = 0$ or (since $\beta < 0$)

$$(-t_1+t_2+t_3)r^2-(-t_1)t_2t_3=0.$$

Namely, if $t_1 = \infty$, then $r^2 = t_2t_3$.

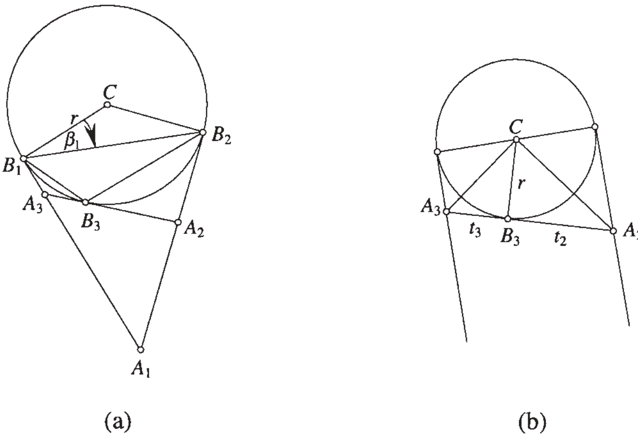

Fig. 2

Example 2. Let $A_1A_2A_3A_4$ be a tangential quadrangle where $\beta_1 < 0$, and let $\beta_1 \to 0$. Then the equations $T_1^4 - T_3^4 = 0$ for $\beta_1 = 0$ can be written as

$$tan\,\beta_2 + tan\,\beta_3 + tan\,\beta_4 - tan\,\beta_2\,tan\,\beta_3\,tan\,\beta_4 = 0.$$

Using the expression $\tan\beta_i = \frac{r}{t_i}$, we get $r^2 = t_2t_3 + t_3t_4 + t_4t_2$ (Fig. 3).

Of course, the same result can be obtained using the equation $S_1^4 r^2 - S_3^4 = 0$ taking $t_1 = \infty$.

Example 3. Let $\boldsymbol{B} = B_1...B_5$ be a chordal pentagon like this in Fig 4. The corresponding tangential pentagon has property that $t_1 = t_5 = \infty$. Using the equation $1-T_2^5+T_4^5 = 0$, we get

$$t_2t_3t_4 - (t_2 + t_3 + t_4)\, r^2 = 0.$$

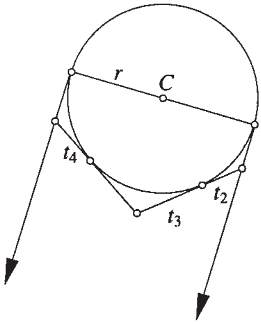

Fig. 3

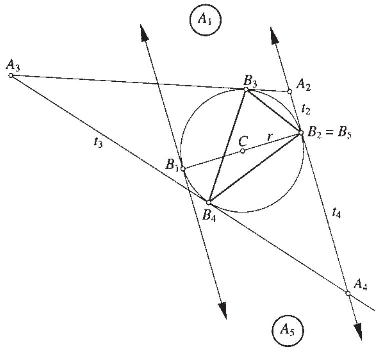

Fig 4.

So, if $t_2 = t_3 = t_4 = 1$, then $r = \frac{\sqrt{3}}{3}$.

Theorem 2. *Let $t_1, ..., t_n$ be any given lengths and let m be any given integer such* that $1 \le m \le \left[\frac{n-3}{2}\right]$. *Then any m of the angles $\beta_1, ..., \beta_n$ may be negative.*

Proof. Since

$$n - 2\left[\frac{n-3}{2}\right] = 3 \text{ if } n \text{ is odd,}$$

$$n - 2\left[\frac{n-3}{2}\right] = 4 \text{ if } n \text{ is even,}$$

that is

$$n - 2m \ge 3 \text{ if } n \text{ is odd,}$$

$$n - 2m \ge 4 \text{ if } n \text{ is even,}$$

it is easy to see that is a lenght r such that

$$-\sum_{i=1}^{m} \arctan\frac{r}{t_i} + \sum_{i=m+1}^{n} \arctan\frac{r}{t_i} = (n - 2(m+1))\frac{\pi}{2}. \tag{21}$$

And Theorem 2 is proved.
In the following corollary will be used the symbol $\sigma(n)$ given by

$$\sigma(n) = \left[\frac{n-1}{2}\right] + \binom{n}{1}\left[\frac{n-3}{2}\right] + \binom{n}{2}\left[\frac{n-5}{2}\right] + \ldots + \binom{n}{u}\left[\frac{n-2u-1}{2}\right], \tag{22}$$

where

$$n-2u-1=3 \text{ if } n \text{ is odd,}$$

$$n-2u-1=4 \text{ if } n \text{ is even.}$$

Corollary 2.1. *Let t_1, ... t_n be any given lenghts. Then there are at least $\sigma(n)$ tangential polygons whose tangents have the given lengths.*

Proof. If all of the angles $\beta_1, .., \beta_n$ are positive then there are $\left[\frac{n-1}{2}\right]$ tangential polygons whose tangents have the given lengths. If one of the angles $\beta_1, ..., \beta_n$ is negative, then there are $\binom{n}{1}\left[\frac{n-3}{2}\right]$ tangential polygons whose tangents have the given lengths. And so on.

From the following two corollaries will be clear that the number of tangential polygons whose tangents have the given lengths may be greater than $\sigma(n)$.

Corollary 2.2. *Let S_j^n , j = 1, 3, 5, ... be referred to the numbers $-t_1, t_2, ..., t_n$. Then the equation*

$$S_1^n x^{\frac{n-1}{2}} - S_3^n x^{\frac{n-3}{2}} + S_5^n x^{\frac{n-5}{2}} - \dots + (-1)^{s_1} S_n^n = 0, \; n \text{ is odd}$$

or

$$S_1^n x^{\frac{n-2}{2}} - S_3^n x^{\frac{n-4}{2}} + S_5^n x^{\frac{n-6}{2}} - \dots + (-1)^{s_2} S_{n-1}^n = 0, \; n \text{ is even}$$

where s_1 and s_2 are as in (13), has no less then $\left[\frac{n-1}{2}\right]-1$ positive solutions.

Corollary 2.3. *If $t_1 > t_2 + \dots + t_n$, then there are $\left[\frac{n-1}{2}\right]$ tangential polygons whose tangents have the lenghts t_1, ..., t_n. In other words, the equations in Corollary 2.2 have all positive solutions.*

Proof. Let n be odd. Then

$$S_1^n = -t_1 + S_1^{n-1}$$
$$S_3^n = -t_1 S_2^{n-1} + S_3^{n-1}$$
$$S_5^n = -t_1 S_4^{n-1} + S_5^{n-1}$$
$$\cdots\cdots\cdots$$
$$S_n^n = -t_1 S_{n-1}^{n-1}$$

where S_i^{n-1}, $i = 1, 2, \dots, n-1$ is referred to the numbers t_2, ... t_n.

It is easy to see that the numbers $S_1^n, S_3^n, \ldots, S_n^n$ are all negative, and by Vietas formulas it is clear that the equation can not has a negative solution (since the all other by Corollary 2.2. are positive). In the same way we find it is valid if n is an even number.

For example, if $n = 5$, $t_1 = 5$, $t_2 = t_3 = t_4 = t_5 = 1$, then

$$-x^2 - (-5\cdot 6+4)x + (-5) = 0 \quad \text{or} \quad x^2 - 26x+5 = 0$$

$$x_1 = 13+\sqrt{164}, \qquad x_2 = 13-\sqrt{164}$$

$$r_1^2 = x_1, \quad r_2^2 = x_2.$$

$$r_1 \approx 5.07\,998, \qquad r_2 \approx 0.44\,017$$

The pentagons are as in Fig. 5.

In Fig. 6 we see another two tangential pentagon whose tangents have the lengths $t_1 = 5$, $t_2 = t_3 = t_4 = t_5 = 1$. The radii of their inscribed circles are the solutions of the equation $S_1^5 r^4 - S_3^5 r^2 + S_5^5 = 0$ or $9r^2 - 34r^2 + 5 = 0$, where now S_i^5, $i = 1, 3, 5$ is referred to the numbers 5, 1, 1, 1, 1. We find that

$$r_1 \approx 1.90\,381, \qquad r_2 \approx 0.39\,151.$$

Let us remark that there are others tangential pentagons whose tangents have the lengths 5, 1, 1, 1, 1. The following combinations are possible:

5,	–1,	1,	1,	1
5,	1,	–1,	1,	1
. . .	. . .	. . .	. . .	. . .
–5,	–1,	1,	1,	1
–5,	1,	–1,	1,	1
. . .	. . .	. . .	. . .	. . .
5,	1,	–1,	–1,	–1
5,	–1,	1,	–1,	–1

But there are no tangential pentagons for the combinations

5,	1,	1,	–1,	–1
5,	1,	–1,	1,	–1

Notice 3. As it can be seen, it is very difficult to say with precision about the number of all together tangential polygons whose tangents have the given lengths. It is because the expressions S_i^n are not so simple. So, for example, if $n = 7$, $j = 3$, then

$$S_3^7 = (-t_1)\, S_2^6 + S_3^6 \;\; if - t_1, t_2, \ldots, t_7$$

$$S_3^7 = (-t_1)(-t_2) S_1^5 + [(-t_1)+(-t_2)]\, S_2^5 + S_3^5 \;\; if - t_1, -t_2, t_3, \ldots, t_7$$

$$S_3^7 = (-t_1)(-t_2)(-t_3) + [(-t_1)(-t_2) + (-t_1)(-t_3) + (-t_2)(-t_3)]\, S_1^4 +$$
$$+ [(-t_1) + (-t_2) + (-t_3)]\, S_2^4 + S_3^4 \ \mathit{if} \ -t_1, -t_2, -t_3, t_4, \ldots, t_7.$$

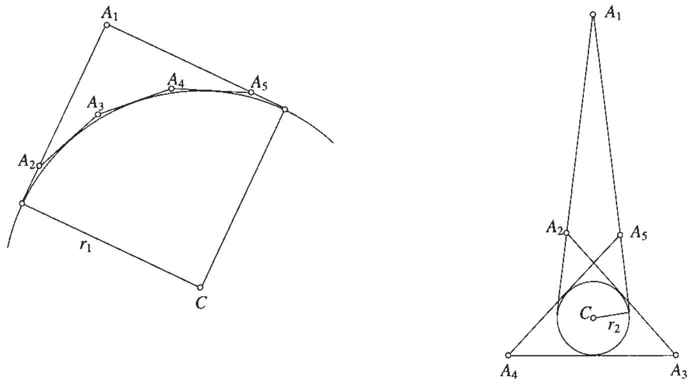

Fig. 5

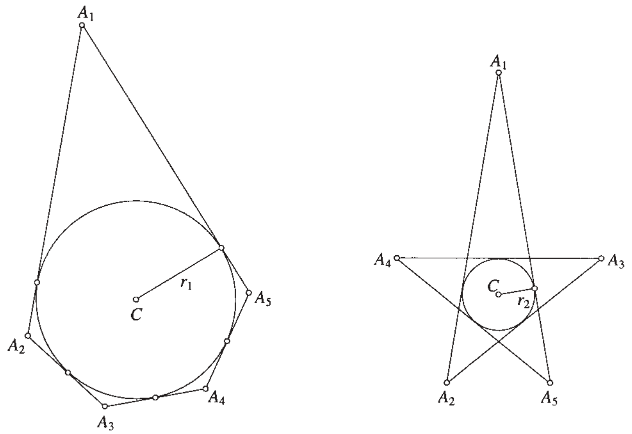

Fig. 6

In the case when n is an even number, then it is possible to be infinity tangential polygons whose tangents have the given lengths. So, for example, if $n = 6$, $t_1 = \ldots = t_6 = 1$, and S_i^6 is referred to the numbers 1, 1, 1, –1, –1, –1 then $S_i^6 = 0$, $i = 1, 3, 5$. Thus, in this case we have the equation

$$0r^4 + 0r^2 + 0 = 0.$$

The situation is like this in Fig. 7. But in this case such a tangential polygon has two consecutive vertices the same, and we do not consider (by Notice 2) such cases in this paper.

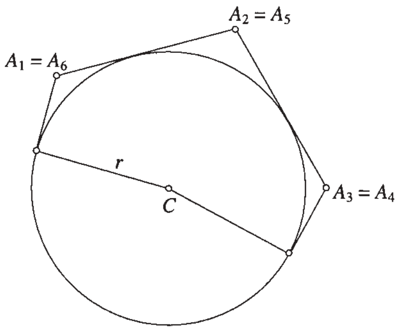

Fig. 7

The following theorem is concerning the question: If $b_1, ..., b_n$ are the lengths of the sides of a 1-chordal polygon, are there the k-chordal polygons with k>1 and the same lenghts of sides?

In [1], Corollary 1.2, we have proved: If $b_1, ..., b_n$ are the lengths of the sides of a k-chordal polygon, then

$$\sum_{i=1}^{n} b_i > 2kb_j, \; j = 1, \dots, n. \tag{23}$$

But the converse may not be valid fok k>1. Namely, the above inequality may be fullfilled, but may not be a k-chordal polygon for k>1 whose sides have the lengths $b_1, ..., b_n$.

In the folowing, for brevity, we use the symbol $S(x_1, ..., x_n, y)$ defined as follows: If $x_1, ..., x_n, y$ are some real numbers, then

$$S(x_1, ..., x_\mathrm{n}, y) = 0 \qquad 24)$$

is the equation

$$S_1^n y^{\bar{n}-1} - S_3^n y^{\bar{n}-3} + S_5^n y^{\bar{n}-5} - \ldots + (-1)^s S_{\bar{n}}^n = 0,$$

where

$$\bar{n} = \begin{cases} n & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases}$$

$$s = (1+3+5+\ldots+\bar{n})+1$$

$$S_j^n = S_j(x_1, \dots, x_n), \; j = 1, 3, \dots, \bar{n}.$$

Theorem 3. *Let* $\boldsymbol{B} = B_1 \dots B_n$ *be a 1-chordal polygon and let* $\boldsymbol{A} = A_1 \dots A_n$ *be its corresponding tangential polygon. Further, let* $b_1, \dots, b_n$ *be the lengths of the sides of* $\boldsymbol{B}$ *and* $t_1, \dots, t_n$ *be the lenghts of the tangets of* $\boldsymbol{A}$*. If in the equation*

$$S\,(t_1, \dots, t_n, r) = 0$$

we put $\frac{rb_i}{\sqrt{4r^2 - b_i^2}}$ *instead of* t_i, $i = 1, \dots, n$ *we shall get the equation with the property that for each its positive root* r_k *there is a k-chordal polygon whose sides have the lengths* $b_1, \dots, b_n$.

Proof. Let r_k be a positive root of the such obtained equation. Then there are positive numbers $t_i^{(k)}$, $i=1, \dots, n$ such that

$$t_i^{(k)} = \frac{r_k b_i}{\sqrt{4r_k^2 - b_i^2}}, \text{ i=1, ..., n.}$$

Hence, according to the relation (5), it follows that there is a k-tangential polygon with the property that its corresponding k-chordal polygon has $b_1, \dots, b_n$ as the lengths of its sides.

Corollary 3.1. *Let* $b_1, \dots, b_n$ *be given lengths (in fact some positive numbers), and let* $x_1, \dots, x_n$ *be as follows*

$$x_i = \frac{yb_i}{\sqrt{4y^2 - b_i^2}}, \text{ i = 1, ..., n.}$$

If the equations $S(x_1, \dots, x_n, y) = 0$ *has not a positive root, then there is not a positive integer k such that there is a k-chordal polygon whose lenghts of sides are* $b_1, \dots, b_n$*. In the case when the equation has m positive roots, then for each k=1, ..., m there is the k-chordal polygon whose sides have the lengths* $b_1, \dots, b_n$.

Corollary 3.2. *Let* $t_1, \dots, t_n$ *be given lengths. Since the equation* $S(t_1, \dots, t_n, r) = 0$ has $\left[\frac{n-1}{2}\right]$ *positive roots* r_k, $k=1, \dots, \left[\frac{n-1}{2}\right]$*, which are all different, let*

$$r_1 > r_2 > \dots > r_m,$$

where $m = \left[\frac{n-1}{2}\right]$*. Then for each positive integer* $k \le m$ *there is k-chordal polygon whose lengths of sides are given by*

$$b_i^2 = \frac{4r_k^2 + t_i^2}{r_k^2 + t_i^2}, i = 1, \dots, n.$$

For example, if $n = 5$ and r_2 is the less positive root the equation $S_1^5 r^4 - S_3^5 r^2 + S_5^5 = 0$, i.e.

$$r_2^2 = \frac{S_3^5 - \sqrt{(S_3^5)^2 - 4S_1^5 S_5^5}}{2S_1^5},$$

then there are 1-chordal pentagon and 2-chordal pentagon with property that their lenghts of sides are given by

$$b_i^2 = \frac{4r_2^2 + t_i^2}{r_2^2 + t_i^2}, i = 1, \ldots, 5.$$

In this way we may find the lengths $b_1, \ldots, b_5$ which may be the lenths of the sides of a 2-chordal pentagon.

Corollary 3.3. If $b_1 = \ldots = b_n = b$, *then for each* $k = 1, \ldots, \left[\frac{n-1}{2}\right]$ *there is the equailateral k-chordal polygon whose side has length b. The radii of their circumcricles are the solutions of the equation*

$$S\left(\frac{b}{\sqrt{4r^2 - b^2}}, \ldots, \frac{b}{\sqrt{4r^2 - b^2}}, r\right) = 0. \tag{25}$$

For example, if $n = 7$, then we have the equation

$$\binom{7}{1}x - \binom{7}{3}x^3 + \binom{7}{5}x^5 - \binom{7}{7}x^7 = 0 \tag{26}$$

where $x = \frac{b}{\sqrt{4r^2 - b^2}}$. Its positive roots are

$$r_k = \frac{b}{2\sin\frac{k\pi}{n}}, k = 1, 2, 3.$$

It is easy to prove that the equation (26) can be written as

$$P_1^6 - P_3^6 + P_5^6 = (-1)^{k+1} \cos\beta(k), k = 1, 2, 3$$

where

$$\beta(k) = (n - 2k)\frac{\pi}{14}, \quad P_1^6 = \binom{6}{1}\cos\beta(k)\sin^5\beta(k),$$

$$P_3^6 = \binom{6}{3}\cos^3\beta(k)\sin^3\beta(k),\ P_5^6 = \binom{6}{5}\cos^5\beta(k)\sin\beta(k),$$

$$\cos\beta(k) = \frac{b}{2r_k},\ \sin\beta(k) = \sqrt{1-\left(\frac{b}{2r_k}\right)^2}.$$

Corollary 3.4. *Let* $\boldsymbol{B} = B_1 ... B_n$ *be a chordal polygon such that*

$$\beta_u > 0, u = i_1, \dots, i_k,\ \beta_v < 0, v = i_{k+1}, \dots, i_n.$$

If $|\beta_1 + \dots + \beta_n| = j\pi, j \in \{0, 1, \dots, n-2\}$, *let* $\tan\beta_u$ *and* $\tan\beta_v$ *in the equation (15) or (16) be replaced by* $\frac{b_u}{\sqrt{4r^2 - b_u^2}}$ *and* $\frac{b_v}{\sqrt{4r^2 - b_v^2}}$ *respectively. The positive roots of the such a obtained equation are the radii of the circucircles of the chordal polygons whose lengths of sides are* $b_1, \dots, b_n$.

Analogously holds in the case when $|\beta_1 + \dots + \beta_n| = (2j-1)\frac{\pi}{2}$. *Then can be used (19) or (20).*

Of course, if some of $\beta_1, \dots \beta_n$ are zero, say $\beta_1 = 0$, then $2r = b_1$. But the problem may be as follows. If the lengths $b_1, \dots, b_{n-1}$ are given, then the length b_n may be wanted so that $\beta_n = 0$.

The following exaples will be considered in more detail.

Example 1. Let $\boldsymbol{B} = B_1B_2B_3B_4$ be a chordal quadrangle such that $\beta_1+\beta_2+\beta_3+\beta_4=0$. If the situation is like this in Fig. 8, then

$$-\beta_1 - \beta_2 = |\beta_1 + \beta_2| = \beta_3 + \beta_4$$

since $\beta_1 < 0, \beta_2 < 0$. Using the equatin $T_1^4 - T_3^4 = 0$ we find that

$$r^2 = \frac{(b_1b_2 - b_3b_4)(b_1b_3 - b_2b_4)(b_2b_3 - b_1b_4)}{D}, \tag{27}$$

where $D = (4 \text{ area of } B)^2$, that is

$$D = b_1^4 + b_2^4 + b_3^4 + b_4^4 - 8b_1b_2b_3b_4 - 2b_1^2b_2^2 - 2b_1^2b_3^2 - 2b_1^2b_4^2 - 2b_2^2b_3^2 - 2b_2^2b_4^2 - 2b_3^2b_4^2.$$

If $b_1 = b_3, b_2 = b_4$, then $r^2 = \frac{0}{0}$ or $0r^2 = 0$. Thus, in this case there are infinity chordal quandrangles whose lenghts of sides sre b_1, b_2, b_3, b_4.

In the case when each of $\beta_1, \beta_2, \beta_3, \beta_4$ is positive or each is negative, then (as it is well knowen)

$$r^2 = \frac{(b_1b_2+b_3b_4)(b_1b_3+b_2b_4)(b_2b_3+b_1b_4)}{D_1}, \tag{28}$$

where $D_1 = (4 \text{ area of } B)^2$, that is

$$D=-b_1^4-b_2^4-b_3^4-b_4^4+8b_1b_2b_3b_4+2b_1^2b_2^2+2b_1^2b_3^2+2b_1^2b_4^2+2b_2^2b_3^2+2b_3^2b_4^2+2b_3^2b_2^4.$$

The similarity between (27) and (28) may be interesting.

Example 2. Let b_1, b_2, b_3 be given lengths and let b_4 be wanted so that $\beta_i < 0$, $i = 1, 2, 3$, $\beta_4 = 0$. Thus, b_4 must be $2r$. It is easy to prove that r is the positive root of the equation

$$4r^3-(b_1^3+b_2^2+b_3^2)r-b_1b_2b_3=0.$$

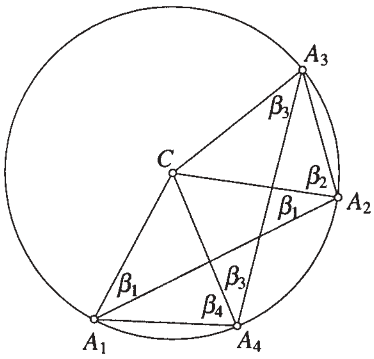

Fig. 8

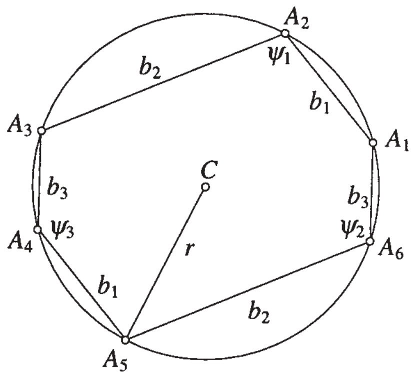

Fig. 9

The situation is as in Fig. 9, where $b_4 = |A_1A_4|$. It may be interesting that the abgles $\psi 1, \psi 2, \psi 3$, are required in the following problem:

Let b_1, b_2, b_3 be given distances from a point 0 (Fig. 10). The angles $\psi 1, \psi 2, \psi 3$ are required to be so that the area of ΔPQR = maximal.

It is easy to prove thet the angles must be equal to the angles $\psi 1, \psi 2, \psi 3$ in Fig. 9.

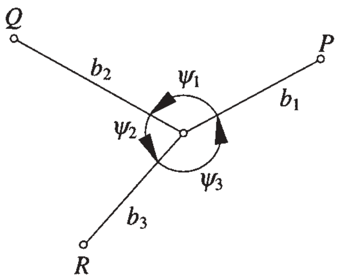

Fig. 10

Corollary 3.5. *Let* $\mathbf{A} = A_1 \ldots A_n$ *be a tangential polygon such that no one of* $\beta_1, \ldots, \beta_n$ *is zero. Then the area of* $\mathbf{A} = \left|rS_1^n\right|$, *where in* S_1^n *stand* $-t_i$ *instead of* t_i *for each* $\beta_i < 0$.

If $A^{(k)} = A_1^{(k)} \ldots A_n^{(k)}$, $k = 1, \ldots, \left[\frac{n-1}{2}\right]$, *is the k-tangential polygon whose tangents have the lengths* $t_1, \ldots, t_n$, *and if* $y_1, \ldots, y_m$, *where* $m = \left[\frac{n-1}{2}\right]$, *are the positive roots of the equation obtained multiplying the equation*

$$S_1^n x^{\bar{n}-1} - S_3^n x^{\bar{n}-3} + S_5^n x^{\bar{n}-5} + \ldots + (-1)^s S_{\bar{n}}^n = 0$$

by $(S_1^n)^{\bar{n}-2}$ *and putting* $y = xS_1^n$, *then*

$$\text{area } \mathbf{A}^{(k)} = y_k,\ k = 1, \ldots, m$$

where $y_1 > y_2 > \ldots > y_m$.

Also, by Vietas formulas, it is valid $y_1^2 + \ldots + y_m^2 = S_1^n S_3^n$.

Corollary 3.6. *Let m, n, q be positive integers such that* $n \geq 3$, $mq = n$, *and let* $t_1, \ldots, t_n$ *be positive numbers sucht that*

$$t_{i+jm} = t_i,\ i = 1, \ldots, m,\ j = 1, \ldots, q-1.$$

Then $S(t_1, \ldots, t_n, r)$ *is divisible by* $S(t_1, \ldots, t_m, r)$.

Proof. Without loss of generality we may, for simplicity, take an example, say, $n = 15$, $m = 5$, $q = 3$, since in all other cases it is analogous.

So, in this case we have

$t_1 = t_6 = t_{11}$
$t_2 = t_7 = t_{12}$
$t_3 = t_8 = t_{13}$
$t_4 = t_9 = t_{14}$
$t_5 = t_{10} = t_{15}$

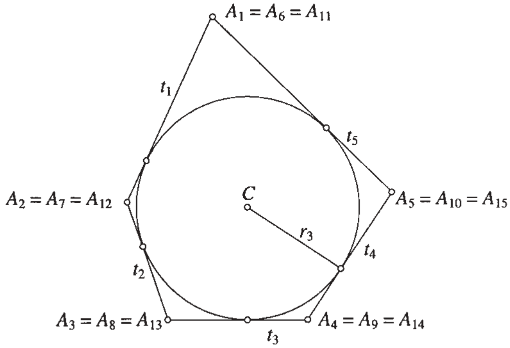

Fig. 11

Since the equation

$$S_1^{15} r^{14} - S_3^{15} r^{12} + S_5^{15} r^{10} - S_7^{15} r^8 + S_9^{15} r^6 - S_{11}^{15} r^4 + S_{13}^{15} r^2 - S_{15}^{15} = 0 \qquad (29)$$

has 7 positive roots r_k, $k = 1, ..., 7$ there are 7 tangential polygons such that for each $k \leq 7$ there is one k-tangential 15-gon whose tangent have the lengths $t_1, ..., t_{15}$.

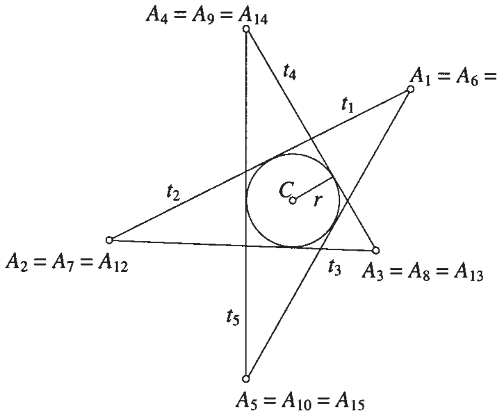

Fig. 12

In Fig. 11 and Fig 12 are showed 3-tangential 15-gon and 5-tangential 15-gon whose tangents have lengths $t_1, ..., t_{15}$. Their radii are the positive roots of the equation

$$S_1^5 r^4 - S_3^5 r^2 + S_5^5 = 0, \tag{30}$$

where S_1^5, S_3^5, S_5^5 are referred to the lengths $t_1, ..., t_5$.

Accordingly, the polynomial

$$S_1^{15}x^7 - S_3^{15}x^6 + S_5^{15}x^5 - S_7^{15}x^4 + S_9^{15}x^3 - S_{11}^{15}x^2 - S_{13}^{15}x + S_{15}^{15}x$$

is divisible by the polynomial $S_1^5 x^2 - S_3^5 x + S_5^5$.

Notice 4. The above corollary is a special case of the following theorem. But it may be useful for better comprehension of the geometrical interpretation of the following theorem.

Theorem 4. *Let m, n, q be positive integers such that $n \geq 3$, $mq = n$, and let $t_1, ..., t_n$ be real numbers different form zero such that*

$$t_{i+jm} = t_i, \; i = 1, ..., m, \; j = 1, ..., q-1. \tag{31}$$

Then $S(t_1, ..., t_n, r)$ is divisible by $S(t_1, ..., t_m, r)$.

Proof. Let $\beta_1, ..., \beta_n$ be different from zero and such that

$\beta_{i+jm} = \beta_i$, $i = 1, ..., m$, $j = 1, ..., q-1$.

Then

$$\frac{\sin(\beta_1+\ldots+\beta_n)}{\sin\beta_1\ldots\sin\beta_n}=\frac{\sin q(\beta_1+\ldots+\beta_m)}{(\sin\beta_1+\ldots+\beta_n)^q}.$$

Using induction on n it can be seen that

$$\frac{\sin(\beta_1+\ldots+\beta_n)}{\sin\beta_1\ldots\sin\beta_n)}=C_1^n-C_3^n+C_5^n-\ldots+(-1)^s C_{\bar{n}}^n.$$

Thus, it is sufficient to show that

$$\frac{\sin q(\beta_1+\ldots+\beta_m)}{(\sin\beta_1\ldots\beta_m)^q}:\frac{\sin(\beta_1+\ldots+\beta_m)}{\sin\beta_1\ldots\beta_m}=F(\cot\beta_1,\ldots\cot\beta_m),$$

where $F(\cot\beta_1, ..., \cot\beta_m)$ is a whole function of $\cot\beta_1, ..., \cot\beta_m$.
First it is clear that

$$\frac{\sin q\psi}{\sin(\beta_1\ldots\beta_m)^q}:\frac{\sin\psi}{\sin\beta_1\ldots\beta_m}=\frac{\binom{q}{1}\cos^{q-1}\psi-\binom{q}{3}\cos^{q-3}\psi\sin^2\psi+\ldots}{\sin(\beta_1\ldots\beta_m)^{q-1}},$$

where $\psi=\beta_1+\ldots+\beta_m$.
Also it is clear that $\binom{q}{1}\cos^{q-1}\psi-\binom{q}{3}\cos^{q-3}\psi\sin^2\psi+\ldots$ can be written as a sum of the products of the form

$$(\cos^{k_1}\beta_{i_1}\ldots\cos^{k_j}\beta_{i_j}\cdot\sin^{k_{j+1}}\beta_{i_{j+1}}\ldots\sin^{k_m}\beta_{i_m})^{q-1}$$

where $i_1, ..., i_m \in \{1, ..., m\}$, $k_1+\ldots+k_m = m$.
This complete the prof of Theorem 4.
In the following corollaries we shall use some symbols which will be here introduced.
Let $X=(x_1, ..., x_n)$ be a sequence, where $x_1, ..., x_n$ are real numbers. Then by $S(X, r)$ will be denoted the polynomial $S(x_1, ..., x_n, r)$.
The sequences $X_1=(x_{i_1},\ldots,x_{i_j})$, ..., $X_k=(x_{i_p},\ldots,x_n)$ are called a partition of the sequence $X = (x_1, ..., x_n)$ if there is a bijection

$$f:\{1, ..., n\}\to\{1, ..., n\}$$

such that $f(u) = v$ implies $x_u = x_v$ for each $u = 1, ..., n$.

Corollary 4.1. *Let m, n, q be positive integers such that mq = n, and let $T_1, \ldots, T_q$ be a partition of the sequence $(t_1, \ldots, t_n)$ such that*

$$\left|T_i\right| = m, i=1,\ldots q \tag{32}$$

where $\left|T_i\right|$ is the number of members in T_i, If

$$S_j(T_i) = S_j(T_1), i = 1,\ldots, q, j = 1, 3, \ldots, \overline{m} \tag{33}$$

then $S(T_1,r) = S(T_i, r)$, $i=1, \ldots, q$ and

$S(t_1, \ldots, t_n, r)$ is divisible by $S(T_1, r)$.

Proof. Because of (32) and (33) the situation is the same as in the case when (31) holds.

Here are some examples.

Example 1. If $n = 6$, $m = 3$, $q = 2$, and $t_1 = 1$, $t_2 = 3$, $t_3 = \frac{16}{5}$, $t_4 = 2$, $t_5 = 4$, $t_6 = \frac{6}{5}$, then

$$t_1 + t_2 + t_3 = t_4 + t_5 + t_6,\ t_1t_2t_3 = t_4t_5t_6 \tag{34}$$

$$(14.4r^4 - 242.4r^2 + 297.6) : (7.2r^2 - 9.6) = 2r^2 - 31.$$

Generally, if (34) hold, then

$$S_1^6r^4 - S_3^6r^2 + S_5^6 = (S_1^3r^2 - S_3^3)(2r^2 - t_1t_2 - t_2t_3 - t_3t_1 - t_4t_5 - t_5t_6 - t_6t_4)$$

where $S_1^3 = t_1 + t_2 + t_3$, $S_3^3 = t_1t_2t_3$.

Especially if $t_1 = t_4$, $t_2 = t_5$, $t_3 = t_6$, then

$$S_1^6r^4 - S_3^6r^2 + S_5^6 = (S_1^3r^2 - S_3^3)(2r^2 - 2S_2^3).$$

Example 2. If $n = 10$, $m = 5$, $t_i = t_{i+5}$, $i=1, \ldots 5$, then

$$S_1^{10}r^8 - S_3^{10}r^6 + S_5^{10}r^4 - S_7^{10}r^2 - S_9^{10} = 2(S_1^5r^4 - S_3^5r^2 + S_5^5)(r^4 - S_5^5r^2 + S_4^5)$$

Analogously holds generally if $n = 2m$.

Corollary 4.2. *Let u, v, z, be elements of the set {0, 1, 2, ...} where $z \geq 3$ is an odd number, and let $T_1, \ldots, T_{u+v}$ be a partition of the sequence $(t_1, \ldots, t_n)$ such that for each $i = 1, \ldots, u+v$ is valid*

$$\left|T_i\right| = z \quad or \quad \left|T_i\right| = z+1, \tag{35}$$

$$S_j(T_i) = S_j(T_1), j = 1, 3, 5, \ldots, z. \tag{36}$$

Then

$$S(T_i, r) = S(T_1, r),\ i = 1, \ldots, u + v \tag{37}$$

$$S(t_1, \ldots, t_n, r) \text{ is divisibile by } S(T_1, r). \tag{38}$$

Proof. Since (36) implies (37) and each root of the equation $S(T_1, r) = 0$ is a root of the equation $S(t_1, \ldots, t_n, r) = 0$, it is clear that (38) must be valid.

The following examples will be discussed in detail.

Example 1. If $n = 7$, $t_1 = 1$, $t_2 = 4$, $t_3 = 7$, $t_4 = t_5 = t_6 = 1$, $t_7 = 9$, then

$$T_1 = (t_1, t_2, t_3),\ T_2 = (t_4, t_5, t_6, t_7)$$

is a partition of the sequence $(t_1, \ldots, t_n)$ such that

$$S_1(T_1) = S_1(T_2) = 12,\ S_3(T_1)=S_3(T_2)=28,$$

where $u = v = 1$, $z = 3$. The root of the equation $S(T_1, r) = 0$, that is of $12r^2 - 28 = 0$, is a root of the equation

$$S_1^7 r^6 - S_3^7 r^4 + S_5^7 r^2 - S_7^7 = 0,$$

where $S_1^7 = 24$, $S_3^7 = 884$, $S_5^7 = 2040$, $S_7^7 = 252$.

The root $r=\sqrt{\dfrac{7}{3}}$ is the radius of the 2-tangential heptagon whose tangents have lengths 1, 4, 7, 1, 1, 1, 9. The radii of 1-tangential and 2-tangential heptagon whose tangents have lengths 1, 4, 7, 1, 1, 1, 9 are the positive solutions of the equation

$$2r^4 - 69r^2 + 9 = 0,$$

where $(24r^6 - 884r^4 + 2\,040r^2 - 252) : (12r^2 - 28) = 2r^4 - 69r^2 + 9$.

Example 2. Again let be $n = 7$, but $t_1 = 1$, $t_2 = -1$, $t_3 = 2$, $t_4 = -1$, $t_5 = t_6 = t_7 = 1$. Then

$$T_1 = (1, -1, 2),\ T_2 = (-1, 1, 1, 1)$$

is a partition of (1, –1, 2, –1, 1, 1, 1) such that

$$S_1(T_1) = S_2(T_2) = 2,\ S_3(T_1) = S_3(T_2) = -2.$$

Hence $S(T_1, r) = S(T_2, r) = 2r^2 + 2$, $S(1, -1, 2, -1, 1, 1, 1) = 4r^6 + 6r^4 - 2$, and

$$(4r^6 + 6r^4 + 0r^2 - 2) = (2r^4 + r^2 - 1)\,(2r^2 + 2).$$

In this case there is only one tangential heptagon whose tangents have lengths 1, –1, 2, –1, 1, 1, 1. The radius of its inscribed circle is the positive root of the equation $2r^4 + r^2 - 1 = 0$, that is $r = \frac{\sqrt{2}}{2}$. The equation $2r^2 + 2 = 0$ has not positive root, i.e. there is not a triangle whose tangents have lengths 1, –1, 2.

Thus, the polynomial $S(t_1, ..., t_n, r)$ may be reducible although there is no a tangential polygon whose tangets have given lengths. In this connection let us remark that in the proof of Theorem 4 the only condition for $\beta_1, ..., \beta_n$ is to be $\beta_i \neq 0$, $i = 1, ..., n$.

Also, let us remark that converse of Corollary 4.1. and Corollary 4.2. need not be valid. For example, if $n = 7$, $t_1 = -1$, $t_2 = ... = t_7 = 1$, then

$$S(-1, 1, 1, 1, 1, 1, 1, \mathrm{r}) = 5r^6 - 5r^4 - 9r^2 + 1$$

$$5r^6 - 5r^4 - 9r^2 + 1 = (5r^4 - 10r^2 + 1)(r^2 + 1).$$

Here the following properties may be interesing.

Property 1. Let $n \geq 3$ be any given integer and let $t_1, ..., t_n$ be such that $t_1 = -1$, $t_2 ... = t_n = 1$. Then

$$S\,(-1, 1, ..., 1, r) \text{ is divisible by } r^2 + 1.$$

Proof. If n is odd, then

$$S_1^n = n-2,\ S_j^n = (-1)\binom{n-1}{j-1} + \binom{n-1}{j},\ j = 3, 5, \dots, n-2,\ S_n^n = -1.$$

From this it can be easily seen that $r = i$ is a root of the polynomial $S_1^n r^{n-1} - S_3^n r^{n-3} + \dots + (-1)^s\ S_n^n$, where $s = (1 + 3 + ... + n) + 1$.

Also it can easily found that

$$(S_1^n r^{n-1} - S_3^n r^{n-3} + \dots + (-1)^s\ S_n^n):(r^2+1) =$$
$$= \binom{n-2}{1} r^{n-3} - \binom{n-2}{3} r^{n-5} + \dots + (-1)^s \binom{n-2}{n-2}. \tag{39}$$

Analogously is when n is an even number.

Thus, if n – 2 is a prim number, then the polynomial on the right side of (39) is irreducible over the field **Q** (rational numbers).

Property 2. In the sam way it can be found that generally holds:

If $t_1 = ...\ t_j = -1$, $t_{j+1} = ... = t_n = 1$, $2j < n$, then for n odd

$$(S_1^n r^{n-1} - S_3^n r^{n-3} + \dots + (-1)^s\ (S_n^n):(r^2+1)^j =$$
$$= \binom{n-2j}{1} r^{n-1-2j} - \binom{n-2j}{3} r^{n-3-2j} + \dots + (-1)^s \binom{n-2j}{n-2j}.$$

Analogously is when n is an even number.
For example, if $n = 8$, $t = 2$, then

$$4r^6 + 4r^4 - 4r^2 - 4 = (4r^2 - 4)(r^4 + 2r^2 + 1).$$

Now we state the following two corollaries of Theorem 4.
Corollary 4.3. If $t_i = -t_{i+m}$, $i = 1, ..., m$, where $n > 2$, then

$$S(t_1, ..., t_n, r) \text{ is divisible by } S(t_{2m+1}, ..., t_n, r)$$

For example, the polyomial $S(1, 2, 3, -1, -2, -3, 1, 1, 1, 2, r)$ can be written as

$$(x^2+1)(x^2+4)(x^2+9)(5r^2-7).$$

Corollary 4.4. If $S_j(t_1, ..., t_m) = -S_j(t_m+1, ..., t_{2m})$, $j = 1, 3, ..., \overline{m}$, then

$$S(t_1, ..., t_n, r) \text{ is divisibile by } S(t_{2m+1}, ..., t_n, r).$$

As we see, the Corollary 4.3. in a way corresponds to the Corollary 4.1, and the Corollary 4.4 to the Corollary 4.2.

Theorem 5. *Let* $\boldsymbol{A} = A_1 ... A_n$ *be a given k-tangential polygon and let* $t_1, ..., t_n$ *be the lengths of its tangents. If* r_k *is the radius of the inscribed circle into* $\boldsymbol{A}$, *then*

$$a \tan(n-2k)\frac{\pi}{2n} \le r_k \le b \tan(n-2k)\frac{\pi}{2n}, \tag{40}$$

where $a = \min\{t_1, ..., t_n\}$, $b = \max\{t_1, ..., t_n\}$.

Proof. According to the Definition 2, each β_i, $i = 1, ..., n$ is either positive or each negative. Let each β_i, $i = 1, ..., n$, be positive. Obviously, if $t_i < t_j$ then $\beta_i > \beta_j$ and if

$$\beta_1 + ... + \beta_n = (n-2k)\frac{\pi}{2},$$

then

$$\min\{\beta_1, ..., \beta_n\} \le (n-2k)\frac{\pi}{2n},$$

$$\max\{\beta_1, ... \beta_n\} \ge (n-2k)\frac{\pi}{2n}.$$

Since $r_k = t_1 \tan b_1 = ... = t_n \tan\beta_n$, it is clear that the inequalities (40) are valid.
In the same way can be seen that inequalities (40) are valid if each β_i, $i=1, ..., n$, is negative.

Corollary 5.1. Let $t_1, \ldots, t_n$ *be any given positive numbers and let* $r_1, \ldots, r_m$ *be all positive roots of the equation*

$$S_1^n r^{\bar{n}-1} - S_3^n r^{\bar{n}-3} + S_5^n r^{\bar{n}-5} - \ldots + (-1)^s S_{\bar{n}}^n = 0.$$

Then for each $i = 1, \ldots, m$ *it is valid*

$$a\tan(n-\bar{n}+1)\frac{\pi}{2n} \le r_i \le b\tan(n-2)\frac{\pi}{2n}.$$

Corollary 5.2. Let $t_1, \ldots, t_n$ be any given positive numbers and let the equation

$$S_1^n x^{\bar{n}-1} - S_3^n x^{\bar{n}-3} + S_5^{\bar{n}-5} - \ldots + (-1)^s S_{\bar{n}}^n = 0$$

be such that $x = y\tan(n-2k)\frac{\pi}{2n}$, $k \in \{1, \ldots, m\}$ *where* $m = \left[\frac{n-1}{2}\right]$. *Then for each positive root* y_j, $j = 1, \ldots, m$, *of the above equation is valid*

$$a \cdot \frac{\tan(n-2j)\frac{\pi}{2n}}{\tan(n-2k)\frac{\pi}{2n}} \le y_j \le b \cdot \frac{\tan(n-2j)\frac{\pi}{2n}}{\tan(n-2k)\frac{\pi}{2n}}.$$

Especially, if $j = k$, then $a \le y_k \le b$.

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O algebarskim jednadžbama u vezi sa tetivnim i tangencijalnim poligonima

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SAŽETAK

Dokazana su važna svojstva jednadžbi (11) i (12) (Theorem 1-5) i utvrđeno da su pozitivna rješenja tih jednadžbi polumjeri jednog niza tetivnih odnosno tangencijalnih poligona s istim stranicama odnosno tangentama.

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