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$E_{25} \cdot Z_3$ AS AN AUTOMORPHISM GROUP OF A SYMMETRIC (101,25,6) DESIGN

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Abstract

We prove that there is exactly one possible orbit structure for the action of the group $E_{25} \cdot Z_3$ as an automorphism group on a symmetric (101,25,6) design.

Key words and phrases: Symmetric design, automorphism group, orbit structure

We assume that the reader is familiar with the construction of a symmetric design on wich acts a given automorphism group (see [3]). Terminology and notation are mostly the same as in [2] and [4].

Symmetric (101,25,6) designs belong to the family of symmetric designs with parameters $v = 4t^2 + 1, k = t^2, \lambda = \frac{t^2 - 1}{4}$ (t an odd positive integer, v a prime). For t=5, the only design known so far admits an automorphism of order 101 (see [1]). Assuming that a group isomorphic to $E_{25} \cdot Z_3$ (semidirect product of the elementary abelian group $E_{25} = Z_5 \times Z_5$ and Z_3, a cyclic group of order 3) acts on a symmetric (101,25,6) design the following result was obtained:

Theorem 1 *If there exists a symmetric (101,25,6) design having an automorphism group $G \cong E_{25} \cdot Z_3$, then G acts in three orbits of lengths 1,25 and 75 (on points and blocks) and the orbit structure for that group is*

$$\begin{pmatrix} 0 & 25 & 0 \\ 1 & 6 & 18 \\ 0 & 6 & 19 \end{pmatrix}.$$

In terms of generators and relations the group $G \cong E_{25} \cdot Z_3$ is given by

$$G = \left\langle a, b, c \,\middle|\, a^5 = b^5 = c^3 = 1, a^b = a, a^c = b, b^c = a^4 b^4 \right\rangle.$$

The set of all different G-orbit lengths is $T = \{1,3,15,25,75\}$. In order to reduce the number of possible cases for types of orbit lengths we prove

Lemma 1 *If a cyclic group Z_5 acts on a symmetric design (101,25,6) than it has exactly one fixed point.*

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Proof: Let *f(g)* be the number of fixed points of an authomorphism $g \in G$ acting on a symmetric design (v, k, λ). The well known result that $f(g) \le k + \sqrt{k-\lambda}$ gives the upper bound for *f(g)*. So, together with $f(g) \equiv 101 (mod\, 5)$ for $|g| = 5$, we get $f(g) \in \{1,6,11,16,21,26\}$. If N_i denotes the number of fixed blocks containing exactly i points fixed by $g \in Z_5$, then obviously $i \equiv k\, (mod\, 5)$. Moreover, $N_0=0$ for $f(g) > 1$, because the intersection of two blocks fixed by Z_5 contains whole Z_5-orbits and $\lambda = 6$. Thus, $i \in I = \{5,10,15,20,25\}$. The number of fixed blocks equals the number of fixed points so it holds $\sum_{i \in I} N_i = f(g)$. The rest of the proof consists in testing this equation for each $f(g) \in \{6,11,16,21,26\}$. Considering that two blocks fixed by Z_5 have in common either six fixed points or one fixed point and one whole Z_5-orbit one can verify that in each case it holds $\sum_{i \in I} N_i < f(g)$.

Let t_m be the number of orbits of length m. An inspection of permutation representations of the group G shows that in the representations of degree 3 and 15 an element of the group of order 5 fixes 3 and 5 points respectively. So, as a consequence of Lemma 1, $t_3=t_{15}=0$ and also $t_1=1$. Therefore the equation $\sum_{t_m \in T} mt_m = v$ is reduced to $t_{25} + 3t_{75} = 4$. Obviously, there are only two solutions: $t_{25} = 4, t_{75} = 0$ and $t_{25} = t_{75} = 1$ which means that the group G acts on a symmetric (101,25,6) design either in five orbits 1,25,25,25,25 when Z_3 fixes five points, or in three orbits 1,25,75 when Z_3 fixes two points.

It is easily verified that in each case of orbit distribution there exists only one orbit structure:

$$\begin{bmatrix} 0 & 25 & 0 & 0 & 0 \\ 1 & 6 & 6 & 6 & 6 \\ 0 & 6 & 9 & 6 & 4 \\ 0 & 6 & 6 & 4 & 9 \\ 0 & 6 & 4 & 9 & 6 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 25 & 0 \\ 1 & 6 & 18 \\ 0 & 6 & 19 \end{bmatrix} \text{ respectively.}$$

Now we have to specify, for each of these two orbit structures, which points from particular point orbit lie in particular block. This procedure is often called indexing, and usually makes the most demanding phase of design construction, because of a huge number of possibilities of point selection.

The attempt at indexing of the first orbit structure was carried out in full. In what follows, we give a brief account of the procedure.

A representative of each of four block orbits of length 25 is Z_3-invariant. Z_3 fixes exactly one point from 25 points. A part of each representative consists of 4,6 or 9 Z_3-invariant points from a particular point orbit of length 25. Four points will be Z_3-invariant if we take one fixed point and one Z_3-orbit and for that there are 8 different possibilities. We can denote these 8 sets of four points by numbers 1,2,…,8 and consider them as sets of indices of points from n-th point orbit of length 25,

n=1,2,3,4. In the same way there are $\binom{8}{2} = 28$ different index sets containing six Z_3-invariant points which can be denoted by numbers 9,10,…,36, and $\binom{8}{3} = 56$ different index sets containing nine Z_3-invariant points denoted by 37,38,…,92. Now a Z_3-invariant block can be represented as an ordered quadruple of numbers (i, j, k, l) where $i, j, k, l = 9,10,\ldots,36$ for a block from the first block orbit of length 25; $i, k = 9,10,\ldots,36$, $j = 37,38,\ldots,92$, $l = 1,2,\ldots,8$ for the second block orbit etc. The number of possibilities for a Z_3-invariant block is 28^4 in case of the first block orbit and $28^2 \cdot 56 \cdot 8$ in case of the other three block orbits.

After all of these blocks were constructed we firstly examine how many of them can be representative of a particular block orbit, that means, we check each block and its E_{25}-images for pairwise intersection. It turns out that there are 13464 representatives of the first and 6720 representatives for each of the other three orbits. Next, a test on pairwise intersections among representatives gave 1558 solutions for the first and the second block orbit. Checking these representatives of first block orbit with all representatives of the third block orbit we got 127 solutions, but among these 127 representatives of the first block orbit and all of the representatives of fourth block orbit there is no pair of blocks with exactly 6 common points.

This completes the proof of the Theorem 1.

Remark. The indexing of the second orbit structure is a huge computational problem because the third block orbit is of length 75 which is the order of the group G so the number of possible cases which has to be checked for the third block orbit representative is of size 10^{22}.

References

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$E_{25} \cdot Z_3$ kao grupa automorfizama simetričnog (101, 25, 6) dizajna

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SAŽETAK

Dokazuje se da postoji točno jedna orbitna struktura za djelovanje grupe $E_{25} \cdot Z_3$ kao grupe automorfizama simetričnih (101, 25, 6) dizajna.

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