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IMPROVEMENT OF A GRÜSS TYPE INEQUALITY OF VECTORS IN NORMED LINEAR SPACES AND APPLICATIONS

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Abstract

In this paper, improvement of a Grüss type inequality of vectors in normed linear spaces was proved. In the proofs we used Abel's inequality. That results were applied to Fourier transform, Melin transform and polynomials.

Key words: Grüss type inequality, normed linear spaces, Fourier transform, Melin transform, polynomials.

1. INTRODUCTION

In 1950., M. Biernacky, H. Pidek and C. Ryll-Nardjewski (see [1]) established the following discrete version of Grüss inequality:

Theorem A. Let $a=(a_1,..., a_n)$, $b=(b_1,...,b_n)$ be two n-tuples of real numbers such that $r\le a_i\le R$ and $s\le b_i\le S$ for $i=1,...,n$. Then one has

$$\left|\frac{1}{n}\sum_{i=1}^{n}a_ib_i-\frac{1}{n}\sum_{i=1}^{n}a_i\cdot\frac{1}{n}\sum_{i=1}^{n}b_i\right|\le\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s)$$

where $[x]$ denotes the integer part of $x\in R$.

A weight version of the discrete Grüss inequality was proved by J. Pečarić in 1979 in [2]:

Theorem B. Let a and b be two monotonic n-tuples and p a positive one. Then one has

$$\left|\frac{1}{P_n}\sum_{i=1}^{n}p_ia_ib_i-\frac{1}{P_n}\sum_{i=1}^{n}p_ia_i\cdot\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i\right|\le\left|a_n-a_1\right|\cdot\left|b_n-b_1\right|\cdot\max_{1\le k\le n-1}\frac{P_k\overline{P}_{k+1}}{P_n^2}$$

where $P_k=\sum_{i=1}^{k}p_i$, $k=1,...n$, and $\overline{P}_{k+1}=P_n-P_k$, $k=1,..., n-1$.

Recently, S.S. Dragomir in [3] established the following discrete version of Grüss inequality:

Theorem C. Let $(X,\|\cdot\|)$ be a normed linear space over $K=\boldsymbol{R}, \boldsymbol{C}$, let $x_i\in X, \alpha_i\in K$ and $p_i\ge 0$ $(i=1,...,n)(n\ge 2)$, such that $\sum_{i=1}^{n}p_i=1$. That we have the inequality

$$\left\| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right\| \le \tfrac{1}{2} \sum_{i=1}^{n} p_i \left(1 - p_i\right) \sum_{i=1}^{n-1} \left|\Delta\alpha_1\right| \sum_{i=1}^{n-1} \left\|\Delta x_i\right\|$$

where $\Delta\alpha_i = \alpha_{i+1} - \alpha_i$ and $\Delta x_i = x_{i+1} - x_i$.

In this paper we point out another improvement of inequality from Theorem C and apply it to the discrete Fourier transform, the discrete Mellin transform and to polynomials with coefficients in normed linear spaces.

2. RESULTS

First, we shall give an analogue of an identity from [4] in a vector space. Preposition 1 below is useful consequence of repeated use of Abel´s identity from [1]:

$$\sum_{i=1}^{n} p_i c_i = P_n c_n - \sum_{i=1}^{n-1} P_i \Delta c_i \tag{1}$$

where $P_i = \sum_{j=i}^{i} p_j$, $i=1,\ldots n$, and $\Delta c_i = c_{i+1} - c_i$, $i=1,\ldots n-1$.

Lemma 1. An extension of Abel´s identity (1) is:

$$\sum_{i=1}^{n} p_i c_i = P_n c_j - \sum_{i=1}^{j-1} P_i \Delta c_i + \sum_{i=1}^{n-1} \overline{P}_{i+1} \Delta c_i \tag{2}$$

where $P_i = \sum_{j=i}^{i} p_j$, $i=1,\ldots n$, $\overline{P}_{i+1} = P_n - P_i$, $\Delta c_i = c_{i+j} - c_i$, $i=1,\ldots n-1$.

Proof: From Abel´s identity (1), we have:

$$\sum_{i=1}^{n} p_i c_i = P_n c_n - \sum_{i=1}^{n-1} P_i \Delta c_i = P_n c_n - \left(\sum_{i=1}^{j-1} P_i \Delta c_i + \sum_{i=j}^{n-1} P_i \Delta c_i \right)$$

$$= P_n c_n - \sum_{i=1}^{j-1} P_i \Delta c_i - \sum_{i=j}^{n-1} P_i \Delta c_i = P_n c_n - \sum_{i=1}^{j-1} P_i \Delta c_i - \sum_{i=j}^{n-1} \left(P_n - \overline{P}_{i+1}\right) \Delta c_i$$

$$= P_n c_n - \sum_{i=1}^{j-1} P_i \Delta c_i - \sum_{i=j}^{n-1} P_n \Delta c_i + \sum_{i=j}^{n-1} \overline{P}_{i+1} \Delta c_i$$

$$= P_n c_n - \sum_{i=1}^{j-1} P_i \Delta c_i - P_n c_n + P_n c_j + \sum_{i=j}^{n-1} \overline{P}_{i+1} \Delta c_i = P_n c_j - \sum_{i=1}^{j-1} P_i \Delta c_i + \sum_{i=j}^{n-1} \overline{P}_{i+1} \Delta c_i$$

Proposition 1. Let X be a vector space over $K=\boldsymbol{R},\boldsymbol{C}$, let $x_i \in X$, $\alpha_i \in K$ and $p_i \ge 0$ ($i=1,\ldots n$) ($n \ge 2$). That we have the identity

$$\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \alpha_j x_j - \sum_{i=1}^{n} p_i \alpha_i \sum_{j=1}^{n} p_j x_j = \sum_{i=1}^{n-1} \Delta\alpha_i \left(\overline{P}_{i+1} \sum_{j=1}^{i-1} P_j \Delta x_j + P_i \sum_{k=i}^{n-1} \overline{P}_{j+1} \Delta x_j \right) \quad (3)$$

Proof. From its definition, we have

$$\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \alpha_j x_j - \sum_{i=1}^{n} p_i \alpha_i \sum_{j=1}^{n} p_j x_j$$

$$= \sum_{i=1}^{n} p_i \alpha_i \sum_{j=1}^{n} p_j x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{j=1}^{n} p_j x_j = \sum_{i=1}^{n} p_i \alpha_i \left(\sum_{j=1}^{n} p_j x_i - \sum_{j=1}^{n} p_j x_j \right) \quad (4)$$

$$= \sum_{i=1}^{n} p_i \alpha_i \sum_{j=1}^{n} p_j (x_i - x_j) = \sum_{i=1}^{n} \alpha_i \left(p_i \sum_{j=1}^{n} p_j (x_i - x_j) \right)$$

Accordningly, by Abel´s identity (1), we have

$$\sum_{i=1}^{n} \alpha_i \left(p_i \sum_{j=1}^{n} p_j (x_i - x_j) \right) = \quad (5)$$

$$= \alpha_n \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j (x_i - x_j) - \sum_{i=1}^{n-1} \Delta\alpha_i \sum_{j=1}^{i} p_j \sum_{k=1}^{n} p_k (x_j - x_k)$$

where $\Delta\alpha_i = \alpha_{i+1} - \alpha_i$, $i = 1,\ldots,n-1$
Since

$$\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j (x_i - x_j) = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j x_i - \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j x_j = \sum_{i=1}^{n} p_i x_i \sum_{j=1}^{n} p_j - \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j x_j = 0$$

we have

$$\alpha_n \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j (x_i - x_j) - \sum_{i=1}^{n-1} \Delta\alpha_i \sum_{j=1}^{i} p_j \sum_{k=1}^{n} p_k (x_j - x_k)$$

$$= -\sum_{i=1}^{n-1} \Delta\alpha_i \left(\sum_{j=1}^{i} p_j \left(\sum_{k=1}^{n} p_k (x_j - x_k) \right) \right) \quad (6)$$

Again, by Abel´s identity (1), we have

$$-\sum_{i=1}^{n-1}\Delta\alpha_i\left(\sum_{j=1}^{i}p_j\left(\sum_{k=1}^{n}p_k(x_j-x_k)\right)\right)$$

$$=-\sum_{i=1}^{n-1}\Delta\alpha_i\left(\sum_{j=1}^{i}p_j\sum_{k=1}^{n}p_k(x_i-x_k)-\sum_{j=1}^{i-1}\sum_{i=1}^{j}p_i\left(\sum_{k=1}^{n}p_k(x_{j+1}-x_k)-\sum_{k=1}^{n}p_k(x_j-x_k)\right)\right) \quad (7)$$

$$=-\sum_{i=1}^{n-1}\Delta\alpha_i\left(\sum_{j=1}^{i}p_j\sum_{k=1}^{n}p_k(x_i-x_k)-\sum_{j=1}^{i-1}\sum_{i=1}^{j}p_i\sum_{k=1}^{n}p_k(x_{j+1}-x_j)\right)$$

$$=-\sum_{i=1}^{n-1}\Delta\alpha_i\left(P_i\sum_{k=1}^{n}p_k(x_i-x_k)-\sum_{j=1}^{i-1}P_jP_n\Delta x_j\right)$$

where $\Delta x_i = x_{i+1}-x_i$, $i=1,\dots,n-1$.

Accordningly, by an extensions Abel´s identity (2), we have

$$-\sum_{i=1}^{n-1}\Delta\alpha_i\left(P_i\sum_{k=1}^{n}p_k(x_i-x_k)-\sum_{j=1}^{i-1}P_jP_n\Delta x_j\right)$$

$$=-\sum_{i=1}^{n-1}\Delta\alpha_i\left(P_iP_n(x_i-x_i)-P_i\sum_{k=1}^{i-1}p_k\Delta(x_i-x_k)+P_i\sum_{k=1}^{n-1}\overline{P}_k\Delta(x_i-x_k)-\sum_{j=1}^{i-1}P_jP_n\Delta x_j\right)$$

$$=-\sum_{i=1}^{n-1}\Delta\alpha_i\left(P_i\sum_{k=1}^{i-1}P_k\Delta x_k-P_i\sum_{k=1}^{n-1}\overline{P}_k\Delta x_k-\sum_{j=1}^{i-1}P_jP_n\Delta x_j\right) \quad (8)$$

$$=-\sum_{i=1}^{n-1}\Delta\alpha_i\left(\left(P_i-P_n\right)\sum_{k=1}^{i-1}P_k\Delta x_k-P_i\sum_{k=i}^{n-1}\overline{P}_k\Delta x_k\right)$$

$$=\sum_{i=1}^{n-1}\Delta\alpha_i\left(\overline{P}_{i+1}\sum_{j=1}^{i-1}P_j\Delta x_j+P_i\sum_{j=i}^{n-1}\overline{P}_j\Delta x_j\right)$$

From relations (4), (5), (6), (7) and (8) we proved our identity (3).

Theorem 1. Let $(X,\|\cdot\|)$ be normed linear space over $K = \boldsymbol{R}, \boldsymbol{C}$, let $x_i \in X$, $\alpha_i \in K$ and $p_i \geq 0$ ($i=1,\dots n$) ($n \geq 2$) such that $\sum_{i=1}^{n}p_i=1$. That we have the inequality

$$\left\|\sum_{i=1}^{n}p_i\alpha_i x_i-\sum_{i=1}^{n}p_i\alpha_i\sum_{i=1}^{n}p_i x_i\right\|\leq \max_{i=1,\dots n-1}\left\{P_i\overline{P}_{i+1}\right\}\sum_{i=1}^{n-1}|\Delta\alpha_i|\sum_{i=1}^{n-1}\|\Delta x_i\|$$

Proof. Let start with identity in Proposition 1. Using the generalized triangle inequality, we have successively:

$$\left\|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right\| = \left\|\sum_{i=1}^{n-1}\Delta\alpha_i\left(\overline{P}_{i+1}\sum_{j=1}^{i-1}P_j\Delta x_j + P_i\sum_{j=i}^{n-1}\overline{P}_j\Delta x_j\right)\right\|$$

$$\leq \left\|\sum_{i=1}^{n-1}\sum_{j=1}^{i-1}\Delta\alpha_i\overline{P}_{i+1}P_j\Delta x_j\right\| + \left\|\sum_{i=1}^{n-1}\sum_{j=i}^{n-1}\Delta\alpha_i P_i\overline{P}_{j+1}\Delta x_j\right\|$$

$$\leq \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\overline{P}_{i+1}P_j|\Delta\alpha_i|\|\Delta x_j\| + \sum_{i=1}^{n-1}\sum_{j=i}^{n-1}P_i\overline{P}_{j+1}|\Delta\alpha_i|\|\Delta x_j\|$$

$$\leq \max_{\substack{i=1,\dots,n-1\\ j=1,\dots i-1}}\left\{\overline{P}_{i+1}P_j\right\}\sum_{i=1}^{n-1}|\Delta\alpha_i|\sum_{j=1}^{i-1}\|\Delta x_j\| + \max_{\substack{i=1,\dots,n-1\\ j=1,\dots i-1}}\left\{\overline{P}_i P_{j+1}\right\}\sum_{i=1}^{n-1}|\Delta\alpha_i|\sum_{j=i}^{n-1}\|\Delta x_j\|$$

$$\leq \max_{i=1,\dots,n-1}\left\{\overline{P}_{i+1}P_i\right\}\sum_{i=1}^{n-1}|\Delta\alpha_i|\sum_{j=1}^{i-1}\|\Delta x_j\| + \max_{i=1,\dots,n-1}\left\{P_i\overline{P}_{i+1}\right\}\sum_{i=1}^{n-1}|\Delta\alpha_i|\sum_{j=i}^{n-1}\|\Delta x_j\|$$

$$\leq \max_{i=1,\dots,n-1}\left\{\overline{P}_{i+1}P_i\right\}\left(\sum_{i=1}^{n-1}|\Delta\alpha_i|\sum_{j=1}^{i-1}\|\Delta x_j\| + \sum_{i=1}^{n-1}|\Delta\alpha_i|\sum_{j=i}^{n-1}\|\Delta x_j\|\right) =$$

$$\leq \max_{i=1,\dots,n-1}\left\{P_i\overline{P}_{i+1}\right\}\sum_{i=1}^{n}|\Delta\alpha_i|\sum_{i=1}^{n}\|\Delta x_i\|$$

Remark. Theorem 1. is really an improvement of Theorem C since we have:

$$\tfrac{1}{2}\sum_{i=1}^{n}p_i\left(1-p_i\right) = \tfrac{1}{2}\left(\sum_{i=1}^{n}p_i - \sum_{i=1}^{n}p_i^2\right)$$

$$\tfrac{1}{2}\left(\left(\sum_{i=1}^{n}p_i\right)^2 - \sum_{i=1}^{n}p_i^2\right) = \tfrac{1}{2}\left(\left(P_i+\overline{P}_{i+1}\right)^2 - \sum_{i=1}^{n}p_i^2\right) = \tfrac{1}{2}\left(2P_i\overline{P}_{i+1} + P_i^2 + \overline{P}_{i+1}{}^2 - \sum_{i=1}^{n}p_i^2\right)$$

$$= \tfrac{1}{2}\left(2P_i\overline{P}_{i+1} + \left(p_1+\dots+p_i\right)^2 + \left(p_{i+1}+\dots+p_n\right)^2 - \sum_{i=1}^{n}p_i^2\right)$$

$$= \tfrac{1}{2}\left(2P_i\overline{P}_{i+1} + \sum_{j=1}^{i}p_j^2 + \sum_{j=1}^{i}\sum_{\substack{k=1\\ k\neq j}}^{i}p_jp_k + \sum_{j=i+1}^{n}p_j^2 + \sum_{j=i+1}^{n}\sum_{\substack{k=i+1\\ k\neq j}}^{n}p_jp_k - \sum_{i=1}^{n}p_i^2\right)$$

$$= \tfrac{1}{2}\left(2P_i\overline{P}_{i+1} + \sum_{j=1}^{i}\sum_{\substack{k=1\\ k\neq j}}^{i}p_jp_k + \sum_{j=i+1}^{n}\sum_{\substack{k=i+1\\ k\neq j}}^{n}p_jp_k\right) \geq P_i\overline{P}_{i+1}$$

The following corollary holds.

Corollary 1. Under the assumptions of Theorem 1. we have the inequality

$$\left\|\tfrac{1}{n}\sum_{i=1}^{n}\alpha_i x_i - \tfrac{1}{n}\sum_{i=1}^{n}\alpha_i\cdot\tfrac{1}{n}\sum_{i=1}^{n}x_i\right\| \leq \tfrac{1}{n^2}\left[\tfrac{n^2}{4}\right]\cdot\sum_{i=1}^{n-1}|\Delta\alpha_i|\cdot\sum_{i=1}^{n-1}\|\Delta x_i\|$$

Proof. The proof holds by putting $p_i = \frac{1}{n}$ in Theorem 1.

Remark. This is an improvement of Corollary 1 from [3].

In the next part of this paper we shall give improvements of some other results from [3] by using ideas given in this paper and our improvements of Grüss type inequality.

Especially, the case of real or complex numbers is impottant in practical applications.

Corollary 2. Let $\alpha_i, \beta_i \in K,\ K = \mathbf{R}, \mathbf{C},\ p_i \geq 0\ (i=1,\ldots n)$ with $\sum_{i=1}^{n} p_i = 1$. Then we have inequality

$$\left|\sum_{i=1}^{n} p_i \alpha_i \beta_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i \beta_i\right| \leq \max_{i=1,\ldots,n-1}\left\{P_i \bar{P}_{i+1}\right\} \sum_{i=1}^{n-1} \left|\Delta\alpha_i\right| \sum_{j=1}^{n-1}\left|\Delta\beta_j\right|$$

Remark 1. If in the above inequality we chose $\beta_i = \bar{\alpha}_i$, $(i=1,\ldots,n)$, the we have

$$0 \leq \sum_{i=1}^{n} p_i \left|\alpha_i\right|^2 - \left|\sum_{i=1}^{n} p_i \alpha_i\right|^2 \leq \max_{i=1,\ldots,n-1}\left\{P_i \bar{P}_{i+1}\right\} \left(\sum_{i=1}^{n-1} \left|\Delta\alpha_i\right|\right)^2 .$$

3. APPLICATIONS

We can apply our results to the discrete Fourier transform.

Let $(X, \|\cdot\|)$ be normed space over $K = \mathbf{R}, \mathbf{C}$, and $(x_1,\ldots x_n)$ be an n–tuple of vectors in X. Define discrete Fourier transformation as

$$F(x_1, \ldots, x_n, m) = \sum_{k=1}^{n} \exp\left(\tfrac{2\pi}{n}\, ikm\right) x_k,\ m=1,\ldots,n$$

where $i = \sqrt{-1}$

Theorem 2. Let $(X, \|\cdot\|)$ and $(x_1,\ldots x_n)$ be as above. Then we have the inequality

$$\left\| F(x_1, \ldots, x_n, m) - \sum_{k=1}^{n} \exp\left(\tfrac{2\pi}{n}\, ikm\right) \cdot \tfrac{1}{n} \sum_{k=1}^{n} x_k \right\| \leq 2\, \tfrac{n-1}{n} \left[\tfrac{n^2}{4}\right] \cdot \left|\sin\left(\frac{\pi m}{n}\right)\right| \cdot \sum_{k=1}^{n-1} \left\|\Delta x_k\right\|$$

Proof. Multiplying the inequality from Corollary 1 by $\frac{1}{n}$ and substituting $\alpha k = \exp\left(\frac{2\pi}{n}\, ikm\right)$, $k = 1, \ldots$, n, we obtain

$$\left\| F(x_1, \ldots, x_n, m) - \sum_{k=1}^{n} \exp\left(\tfrac{2\pi}{n}\, ikm\right) \cdot \tfrac{1}{n} \sum_{k=1}^{n} x_k \right\| \leq \tfrac{1}{n} \left[\tfrac{n^2}{4}\right] \cdot \sum_{k=1}^{n-1} \left|\Delta \exp\left(\tfrac{2\pi}{n}\, ikm\right)\right| \cdot \sum_{k=1}^{n-1} \left\|\Delta x_k\right\|$$

Observing that

$$\Delta \exp\left(\tfrac{2\pi}{n} ikm\right) = 2i \sin\left(\tfrac{\pi}{n} m\right) \exp\left(\tfrac{2\pi}{n} i(2k+1)m\right)$$

we have

$$\sum_{k=1}^{n-1}\left|\Delta \exp\left(\tfrac{2\pi}{n} ikm\right)\right| = 2(n-1)\left|\sin\left(\tfrac{\pi m}{n}\right)\right|,$$

so the proof is complete.

We can also apply our results to the discrete Merlline transform.

Let $(X, \|\cdot\|)$ be normed space over $K = \boldsymbol{R}, \boldsymbol{C}$, and $(x_1,...x_n)$ be a n–tuple of vectors in X. Define discrete Fourier transformation as

$$M(x_1, \ldots, x_n, m) = \sum_{k=1}^{n} k^{m-1} x_k, \ (m=1,...n)$$

Then the following results holds.

Theorem 3. Let $(X, \|\cdot\|)$ and $(x_1,...x_n)$ be a above. Then we have the inequality

$$\left\| M(x_1, \ldots, x_n, m) - \sum_{k=1}^{n} k^{m-1} \tfrac{1}{n} \sum_{k=1}^{n} x_k \right\| \leq \tfrac{n^{m-1}-1}{n}\left[\tfrac{n^2}{4}\right] \sum_{k=1}^{n-1} \|\Delta x_k\|$$

Proof. As in the proof of Theorem 2, choosing $\alpha_k = k^{m-1}$ we obtain

$$\left\| M(x_1, \ldots, x_n, m) - \sum_{k=1}^{n} k^{m-1} \tfrac{1}{n} \sum_{k=1}^{n} x_k \right\| \leq \tfrac{1}{n}\left[\tfrac{n^2}{4}\right] \sum_{k=1}^{n-1} \left|\Delta k^{m-1}\right| \sum_{k=1}^{n-1} \|\Delta x_k\|$$

$$= \tfrac{1}{n}\left[\tfrac{n^2}{4}\right] \sum_{k=1}^{n-1} \left((k+1)^{m-1} - k^{m-1}\right) \sum_{k=1}^{n-1} \|\Delta x_k\| = \tfrac{n^{m-1}-1}{n}\left[\tfrac{n^2}{4}\right] \sum_{k=1}^{n-1} \|\Delta x_k\|.$$

Considering the following particular values of Mellin transform:

$$M(x_1, \ldots, x_n, 2) = \sum_{k=1}^{n} k x_k$$

and

$$M(x_1, \ldots, x_n, 3) = \sum_{k=1}^{n} k^2 x_k,$$

the following corollary holds.

Corollary 3. Let $(X, \|\cdot\|)$ and $(x_1,...x_n)$ be as above. Then the inequalites

$$\left\|\sum_{k=1}^{n} kx_k - \frac{n+1}{2}\sum_{k=1}^{n} x_k\right\| \le \frac{n-1}{n}\left[\frac{n^2}{4}\right]\sum_{k=1}^{n-1}\|\Delta x_k\|$$

and

$$\left\|\sum_{k=1}^{n} k^2 x_k - \frac{(n+1)(2n+1)}{6}\sum_{k=1}^{n} x_k\right\| \le \frac{n^2-1}{n}\left[\frac{n^2}{4}\right]\sum_{k=1}^{n-1}\|\Delta x_k\|$$

hold.

Remark. If we assume $(x_1, ..., x_n)=(p_1,....p_n)$ is a probability distribution, that is, pk ≥ 0 (k = 1, ..., n) and $\sum_{k=1}^{n} p_k = 1$, the by Corollary 3, we get the inequalities

$$\left|\sum_{k=1}^{n} kp_k - \frac{n+1}{2}\right| \le \frac{n-1}{n}\left[\frac{n^2}{4}\right]\sum_{k=1}^{n-1}|p_{k+1} - p_k|$$

$$\left|\sum_{k=1}^{n} k^2 p_k - \frac{(n+1)(2n+1)}{6}\right| \le \frac{n^2-1}{n}\left[\frac{n^2}{4}\right]\sum_{k=1}^{n-1}|p_{k+1} - p_k|$$

We can apply our results also to the polynomials.

Let $(X,\|\cdot\|)$ be normed space over K = ***R, C***, and $(c_0, c_1, ..., c_n)$ be $(n+1)$ be n - tuple of vectros in X such that $c_n \neq 0$. Define the polynomial $P : C \to X$ with the coefficients $(c_1,..., c_n)$ by

$$P(z) = c_0 + zc_1 + ... + z^n c_n \quad z \in C.$$

Then the following results for polynomial holds.

Theorem 4. Let $(X,\|\cdot\|)$ and $(c_1, ..., c_n)$ be as above. Then we have the inequality

$$\left\|P(z) - \sum_{k=0}^{n} z^k \cdot \frac{1}{n+1}\sum_{k=0}^{n} c_k\right\| \le \frac{|z-1|(|z|^n-1)}{(n+1)(|z|-1)}\left[\frac{(n+1)^2}{4}\right]\cdot\sum_{k=0}^{n-1}\|\Delta c_k\|, z \in C, z \neq 1.$$

Proof. Chosing $\alpha_k = z^k$ and $x_k = c_k$ n Corollarz 1 we obtain

$$\left\|P(z) - \sum_{k=0}^{n} z^k \cdot \frac{1}{n+1}\sum_{k=0}^{n} c_k\right\| \le \frac{1}{(n+1)}\left[\frac{(n+1)^2}{4}\right]\cdot\sum_{k=0}^{n-1}|z^{k+1} - z^k|\cdot\sum_{k=0}^{n-1}\|\Delta c_k\|$$

$$= \frac{1}{n+1}\left[\frac{(n+1)^2}{4}\right]\cdot\sum_{k=0}^{n-1}|z^k||z-1|\cdot\sum_{k=0}^{n-1}\|\Delta c_k\|$$

$$= \frac{|z-1|(|z|^n - 1)}{(n+1)(|z|-1)} \left[\frac{(n+1)^2}{4} \right] \sum_{k=0}^{n-1} \|\Delta c_k\|$$

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Poboljšanje nejednakosti Grüssovog tipa za vektore u normiranim linearnim prostorima i primjene

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SAŽETAK

U radu je dokazano poboljšanje nejednakosti Grüssovog tipa za vektore u normiranim linearnim prostorima. U dokazima koristili smo Abelovu nejednakost. Dobiveni rezultat je primijenjen na Fourierovu transformaciju, Melinovu transformaciju i polinome.

Ključne riječi: Nejednakost Grüssovog tipa, normirani linearni prostori, Fourierova transformacija, Melinova transformacija, polinomi.

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