UDK 514.112
514.142
Izvorni znanstveni članak

Prihvaćeno za tisak: 30.3.2004.

ON PENCIL OF QUADRICS IN $I_3^{(2)}$

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Abstract

An affine space A_3 is called a double isotropic space $I_3^{(2)}$, if in A_3 a metric is induced by an absolute $\{\omega, f, F\}$, consisting of the line f in the plane of infinity ω of A_3, and a point $F \in f$.

The pencil of quadrics is a set of ∞^1 2nd order surfaces having common 4th order space curve. Intersecting a pencil of quadrics by a general plane we obtain a pencil of 2nd order curves.

In this paper pencils of quadrics in a double isotropic space $I_3^{(2)}$ are analysed whereby the pencil of surfaces is observed as the pencil associated with the pencil of second order curves (conics) belonging to isotropic absolute plane ω. In this process we use the classification of pencils of conics in the isotropic plane given in [2], the classification of 2nd order surfaces in $I_3^{(2)}$ [4], and the projective properties of the pencils of second order surfaces [9, 16]. In order to obtain a more complete classification, the fundamental curve of the pencil, the curve of the centres, and the focal surface of the pencil of quadrics are analysed.

Key words: quadrics, plane isotropic geometry, geometry of the double isotropic space, pencil of quadrics
MSC 2000: 51N25

1 INTRODUCTION

The first ideas about the pencil of quadrics understood as a set of ∞^1 quadrics having common 4th order space curve originate from the first halt of the 19th century and can be attributed to the French engineer and mathematician Lamè. Ever since many mathematicians, as for example Poncelet (who proved the existence of singular surfaces of the pencil), Monge (who studied the breaking up of the fundamental curve of the pencil), von Staudt, Sturm, Cremona, then Reye and many others*, have been researching the pencils of quadrics.

* Luigi Cremona (1830-1903), Italian mathematician
Gabriel Lamè (1795-1870), French engineer and mathematician
Gaspard Monge (1746-1818), French mathematician
Jean Victor Poncelet (1788-1867), French engineer and mathematician
Theodor Reye (1838-1919), German mathematician
Karl Von Staudt (1798-1867), German mathematician
Jacques C. F. Sturm (1788-1867), French engineer and mathematician

Apart from analysing their properties, the pencils were also classified according to various criteria.

Thus, the classification depending on the reality of the singular surfaces of the pencil, and on the reality, i.e. breaking up of the fundamental curve, has been given simultaneously by Von Staudt, Strum, and Cremona [16]. Considering the characteristic equation of the pencil defined by means of the fundamental quadrics of the pencil, the pencils have been classified in the projective space by Heffter and Köhler [9]. They refer to the earlier made classification by Von Staudt [16]. The projective classification by means of synthetic method has been carried out by Reye [12]. Affine and equiform classification according to the projective classification of the pencil of conics in which the plane of infinity intersects the pencil of surfaces has also been given by Heffter and Köhler.

2 DOUBLE ISOTROPIC SPACE

2.1 Motion Group in $I_3^{(2)}$

Let $P_3(R)$ be a three-dimensional real projective space, quadruples $(x_0 : x_1 : x_2 : x_3) \neq (0 : 0 : 0 : 0)$ the projective coordinates, ω a plane in P_3, and $A_3 = P_3 \backslash \omega$ the derived affine space. A affine space A_3 is called a double isotropic space $I_3^{(2)}$ if in A_3 a metric is induced by an *absolute* $\{\omega, f, F\}$, consisting of the line f in the plane of infmity ω and a point $F \in f$. The geometry of $I_3^{(2)}$ could be seen, for example in Brauner [6]. In the affine model, where

$$x = x_1/x_0, \quad y = x_2/x_0, \quad z = x_3/x_0, \quad (x_0 \neq 0) \tag{2.1}$$

the absolute figure is determined by $\omega \equiv x_0 = 0$, $f \equiv x_0 = x_1 = 0$, and $F(0:0:0:1)$.

All regular projective transformations that keep the absolute figure fixed form a 9-parametric group

$$G_9 \begin{cases} \bar{x}_0 = x_0 \\ \bar{x}_1 = c_1 x_0 + c_2 x_1 \\ \bar{x}_2 = c_3 x_0 + a x_1 + c_4 x_2 \\ \bar{x}_3 = c_5 x_0 + b x_1 + c x_2 + c_6 x_3 \end{cases}, \; c_1, \ldots, c_6, a, b, c, \in R \text{ and } c_2 c_4 c_6 \neq 0. \tag{2.2.}$$

G_9 represents the *group of similarities* of the double isotropic space $I_3^{(2)}$.

The unimodular transformations in G_9 form a 6-parametdc *motion group* of $I_3^{(2)}$, i.e.,

$$G_6 \begin{cases} \bar{x}_0 = x_0 \\ \bar{x}_1 = c_1 x_0 + x_1 \\ \bar{x}_2 = c_3 x_0 + a x_1 + x_2 \\ \bar{x}_3 = c_5 x_0 + b x_1 + c x_2 + x_3 \end{cases}. \tag{2.3.}$$

In the plane of infinity ω, G_6 induces a 3-parametric group

$$G_3 \begin{cases} \overline{x}_1 = x_1 \\ \overline{x}_2 = ax_1 + x_2 \\ \overline{x}_3 = bx_1 + cx_2 + x_3 \end{cases} \tag{2.2.}$$

G_3 is the *motion group* of an isotropic plane, (see [13]).

2.2 Isometric Invariants of Quadrics in $I_3^{(2)}$

Quadric equation is a second-degree equation in three variables that can be written in the form

$$Q(x,y,z) \equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{01}x + 2a_{02}y + 2a_{03}z + a_{00} = 0, \tag{2. 5}$$

where $a_{11}, \ldots\ldots, a_{00} \in R$ and at least one of the coefficients $a_{11}, \ldots, a_{23} \neq 0$ [1]. Using the matrix notation,

$$Q(x,y,z) \equiv \begin{bmatrix} 1 & x & y & z \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = 0. \tag{2.6}$$

In the sequel we will use

$$\Delta = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix}, \quad \Delta_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix}, \quad \Delta_2 = |a_{33}| = a_{33} \tag{2.7}$$

the 3×3 - minors D_{ij} $(i, j=1,2,3)$ of Δ, the 2×2 - minors Δ_{ij} (i, j =1,2,3) of Δ_0, as well as

$$\Gamma_0 = \gamma_{01} + \gamma_{02} + \gamma_{03} = \begin{vmatrix} a_{00} & a_{01} \\ a_{01} & a_{11} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{02} \\ a_{02} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{03} \\ a_{03} & a_{33} \end{vmatrix}$$

$$\Gamma_1 = \Delta_{11} + \Delta_{22} + \Delta_{33}, \quad (\Delta_{11} = \Delta_1), \quad \Gamma = \Gamma_0 + \Gamma_1, \quad \text{and} \tag{2.8}$$

$$\alpha_1 = \alpha_{11} + \alpha_{22} + \alpha_{33}, \quad \alpha_1 = \alpha_0 + \alpha_{00}$$

As it is known (see [5]), the sign of the determinant Δ as well as that of Δ_0 is invariant with respect to the affine (linear, regular) transformations. The main quadric's invariants with respect to the group G_6 of motions in $I_3^{(2)}$ are [4]:

$$\Delta,\ \Delta_0,\ \Delta_1,\ \Delta_2\,. \tag{2.9}$$

2.3 The Absolute Plane ω

The quadric equation (2.5) written in homogenous coordinates $(x_0 : x_1 : x_2, : x_3)$ has the form

$$\begin{aligned} a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 \\ + 2a_{01}x_1x_0 + 2a_{02}x_2x_0 + 2a_{03}x_3x_0 + a_{00}x_0^2 = 0 \end{aligned} \tag{2.10}$$

With $x_0 = 0$ we obtain the section of the quadric with the plane of infinity ω, i.e,

$$k_\omega \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 = 0 \tag{2.11}$$

where $(x_1 : x_2 : x_3)$ are plane projective coordinates. The affine coordinates ξ, η in ω are given by

$$\xi = \frac{x_2}{x_1},\quad \eta = \frac{x_3}{x_1} \tag{2.12}$$

and the absolute figure is determined with

$$F(0 : 0 :1),\quad f \equiv x_1 = 0\,. \tag{2.13}$$

Equation (2.10) obtains the form

$$k_\omega \equiv a_{22}\xi^2 + a_{33}\eta^2 + 2a_{12}\xi + 2a_{13}\eta + 2a_{23}\xi\eta + a_{11} = [1\ \ \xi\ \ \eta]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}\begin{bmatrix} 1 \\ \xi \\ \eta \end{bmatrix} = 0 \tag{2.14}$$

As it has been shown in [3], the basic conic invariants with respect to the group of motions of the isotropic plane ω, in terms of the quadrics invariants, are Δ_0, Δ_1, and Δ_2, given in the relation (2.7).

2.4 Classification of Quadrics

It is known that in the isotropic plane there are 20 different types of conics [see 3, 13]. We use the classification given in [3], made with respect to the conics isometric invariants in the isotropic plane, as well as the results related to k_ω, obtained in the same work.

For the quadrics classification in $I_3^{(2)}$, according [4], the conics division in four families depending on their relation towards the absolute figure given by the relation (2.13) is used. In such a way we have:

k_ω belongs to the 1st family ≡ it doesn't have an isotropic direction ⇒ 1st *family of quadrics*; consisting of two subfamilies defined by

$$\begin{cases} k_\omega \cap f = \emptyset; & \alpha \text{ subfamily} \\ k_\omega \cap f \neq \emptyset; & \beta \text{ subfamily} \end{cases};$$

k_ω belongs to the 2nd family ≡ it has one isotropic direction ⇒ 2nd *family of quadrics*;
k_ω belongs to the 3rd family ≡ it has a double isotropic direction ⇒ 3rd *family of quadrics;*
k_ω belongs to the 4th family ≡ it contains an absolute line f ⇒ 4th *family of quadrics*.

On the other hand, according Brauner [6], in $I_3^{(2)}$ we distinguish six classes of straight lines. Those are:

- Nonisotropic lines ≡ lines l with the property $l \cap f = \emptyset$;
- Isotropic lines ≡ lines l with the property $l \cap f \neq \emptyset,\ l \cap f \neq F$;
- Double isotropic lines ≡ lines l with the property $l \cap f = F$;
- Nonisotropic lines in the plane of infinity ω ≡ lines for which $l \cap f = \emptyset$;
- Isotropic lines in the plane of infinity ω ≡ lines for which $l \cap f = F \ \ l \neq f$;
- Absolute line f.

Hence, quadrics in $I_3^{(2)}$ will also be classified according to the direction of the longitudinal axes, i.e., according to the straight fine class the axes belongs to. We will distinguish:

- Nonisotropic surfaces;
- Isotropic surfaces;
- Double isotropic surfaces.

3 PENCILS OF QUADRICS IN $I_3^{(2)}$

Due to numerous types of pencils of conics in the isotropic plane (in the most general case where the pencil of conics is determined with 4 real mutually different points there are 8 main subtypes, i.e. 20 cases) as well as to second order surfaces in $I_3^{(2)}$, the classification is not carried out completely, but the approach to this task is explained on one example. The pencils of quadrics in $I_3^{(2)}$ can continue to be completely classified following the principle given in the paper.

We will first define some terms and point out some projective properties of the pencils of 2nd order surfaces that are going to be used further on. These general notions can be fouud e.g. in [9, 15, 16].

Fundamental Pencil of Conics

In the absolute isotropic plane ω, one out of 20 different cases of pencils of conics [2], has been chosen. This very pencil is called *fundamental pencil of conics*.

Fundamentul Quadrics

As f*undamental quadrics* any two quadrics of the pencil can be chosen.

Let the fundamental quadrics of the pencil be given in the equations:

$$\mathrm{K}_1(x,y,z) \equiv a_{11}x^2+a_{22}y^2+a_{33}z^2+2a_{12}xy+2a_{13}xz+2a_{23}yz+2a_{01}\,x+2a_{02}y+2a_{03}z+a_{00} = 0\,,$$

$$\mathrm{K}_2(x,y,z) \equiv b_{11}x^2+b_{22}y^2+b_{33}z^2+2b_{12}xy+2b_{13}xz+2b_{23}yz+2b_{01}\,x+2b_{02}y+2b_{03}z+b_{00} = 0\,,$$

where $a_{11}, \ldots, a_{00}, b_{11}, \ldots, b_{00} \in R$ and at least one of the numbers $a_{11}, \ldots, a_{23}$, as well as one of the numbers $b_{11}, \ldots, b_{23}$, is different from zero.

Pencil of Quadrics

The pencil of quadrics, defined by the fundamental quadrics $K_1(x, y, z) = 0$ and $K_2(x, y, z) = 0$, is given in the form

$$H(x,y,z) \equiv K_1(x,y,z)+\lambda K_2(x,y,z)=0, \quad \lambda \in R, \tag{3.2}$$

that is

$$H(x,y,z)=c_{11}x^2+c_{22}y^2+c_{33}z^2+2c_{12}xy+2c_{13}xz+2c_{23}yz+2c_{01}x+2c_{02}y+2c_{03}z+c_{00} =$$

$$=[1 \;\; x \;\; y \;\; z]\begin{bmatrix} c_{00} & c_{01} & c_{02} & c_{03} \\ c_{01} & c_{11} & c_{12} & c_{13} \\ c_{02} & c_{12} & c_{22} & c_{23} \\ c_{03} & c_{13} & c_{23} & c_{33} \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}=$$

$$=[x \;\; y \;\; z]\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}+2[c_{01} \;\; c_{02} \;\; c_{03}]\begin{bmatrix} x \\ y \\ z \end{bmatrix}+c_{00}=0 \tag{3.3}$$

where

$$c_{ik} = a_{ik} + \lambda b_{ik}\,, \qquad i, k = 0,1,2,3\,. \tag{3.4}$$

For each $\lambda\in R$ the set of zeros of the polynomial $H(x, y, z)=K_1(x, y, z)+\lambda K_2(x, y, z)$ presents one of the second order surfaces if at least one of the numbers $c_{11}, \ldots, c_{23}$ is different from zero. $c_{11}= \ldots =c_{23}=0$ implies the degeneration of the surface in two planes, one being an absolute plane ω. Pencils of quadrics with $a_{ik}=\alpha b_{ik}$, $i, k=0,\ldots, 3$, $\alpha \in R$ will not be concerned, whereby the so-called *identical pencils* (*identische*

Büschel) have been excluded. Apart from that, the so-called *degenerated pencils* (*entartete Büschel*), consisting of pairs of planes or cones will not be considered as well.

Let the main invariants of the quadric $H(\lambda)= 0$ with respect to the group G_6 of motions in $I_3^{(2)}$, according (2.9), be denoted by

$$\Delta(\lambda),\ \ \Delta_0(\lambda),\ \ \Delta_1(\lambda),\ \ \Delta_2(\lambda). \tag{3.5}$$

Further on, let

$$\Delta(K_1), \Delta_0(K_1), \Delta_1(K_1), \Delta_2\,(K_1), \text{ and } \Delta(K_2\,), \Delta_0(K_2\,), \Delta_1(K_2\,), \Delta_2(K_2), \tag{3.6}$$

be the main quadric invariants of the pencil surfaces $K_1(x, y, z)=0$ and $K_2(x, y, z)=0$ with respect to the group G_6 of motions in $I_3^{(2)}$.

Fundamental Points

Each point lying on two surfaces of the pencil belongs to all the surfaces of the pencil. Such a point is called *fundamental point* of the pencil.

Fundamental Curve

The geometric locus of all the fundamental points within a pencil is called the *fundamental curve* of the pencil. The fundamental curve can be obtained as an intersection of any two surfaces of the pencil and represents a 4th order space curve.

Characteristic Equation of the Pencil – Singular Surfaces of the Pencil

By the *characteristic equation* of the pencil $K_1(x, y, z)+\lambda\, K_2(x, y, z)=0$ we understand the equation

$$\Delta(\lambda)=0,\ \text{i.e.}\ \ \Delta(\lambda)\equiv\left|c_{ik}(\lambda)\right|\equiv\left|a_{ik}+\lambda b_{ik}\right|=0,\ \ i,k=0,\dots,3. \tag{3.7}$$

$\Delta(\lambda)$ is simply called λ – *determinant* of the pencil. Following the definitions from linear algebra, the *rank* of the determinant $\Delta(\lambda)$, depending on λ, is defined as the rank of the joined matrix. It is obvious that the rank of a λ – deterninant can adopt the following values: 4, 3, 2, or l. Since we have excluded identical and degenerated pencils, the λ – determinant will generally be of the rank 4, and the pencil surfaces will be non-singular ($\Delta(\lambda) \neq 0$). Such a pencil of quadrics is called a *no degenerated pencil* (*nichtentarteten Flächenbüschel*). The only singular surfaces within the no degenerated pencil are obtained by the solutions $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of (3.7). The vertices-centres of the singular surfaces of the pencil are called *main points* of the pencil.

Curve of the Centres

Curve of the centres within the pencil of quadrics is the geometric locus of all centres of the pencil surfaces. It will be shown in the paper that the very curve, denoted by k_S^3, is a 3rd order space curve, and its classification is given as well.

Focal Surface

The notion of the absolute point in the isotropic plane is connected with the notion of the foci of 2nd order curves, as well as with the focal curve as the geometrical locus of all foci of the conics of the pencil.

In $I_3^{(2)}$ all the polar planes of the quadric surfaces of the certain pencil with respect to the absolute point $F(0{:}0{:}0{:}1)$ form a pencil of planes. Each plane of the pencil cuts the affiliated surface into a 2nd order curve. The geometrical locus of all such obtained curves of intersection is called *focal surfac*e of the pencil, and will be denoted by Φ.

4 PENCILS OF QUADRICS IN $I_3^{(2)}$ GENERATED BY PENCILS OF CONICS IN THE ABSOLUTE ISOTROPIC PLANE ω

4.1 On Pencil of Conics of the Subtype I.6

The pencil of conic sections is determined by two arbitrarily chosen conics of the pencil (see [11]). These two conics, called *fundamental conics* of the pencil, have four common points that we *call fundamental points* of the pencil and mark them A, B, C, and D. Depending on the reality, the position towards the absolute figure $\{F, f\}$, and the multiplicity of fundamental points, various types of conic pencils can be distinguished. The most general case is obtained by choosing A, B, C, and D being real and mutually different, known as type I.

For the purpose of this paper, the classification of pencils of type I given in [2] has been used. In the absolute isotropic plane ω a pencil of subtype I.6 has been chosen to represent the fundamental pencil of conics.

Description of the pencils of subtype I.6: Two fundamental points are finite points and two points are placed on the absolute line f. One of the two fundamental points on the absolute line f coincides with the point F. The fundamental points are joined by three singular, in three pairs of straight lines degenerated conics. Two arbitrarily chosen pairs, out of this three pairs of straight lines can be chosen as fundamental conics of the pencil.

Let the coordinates of the fundamental points, in projective coordinates, be

$$A(1{:}0{:}0),\ B(0{:}1{:}0),\ C(1{:}c_1{:}c_2),\ c_1, c_2 \neq 0, D(0{:}0{:}1) \tag{4.1}$$

The straight lines $v : AB$, $p :=AD$, $q :=BC$, $u :=CD$ are given in the equations:

$$\begin{array}{lll} v\ldots\ldots & y=0, & \\ u\ldots\ldots & x+c_u=0, & c_u=-c_1, \\ p\ldots\ldots & x=0, & \\ q\ldots\ldots & y+c_q=0, & c_q=-c_2. \end{array} \tag{4.2}$$

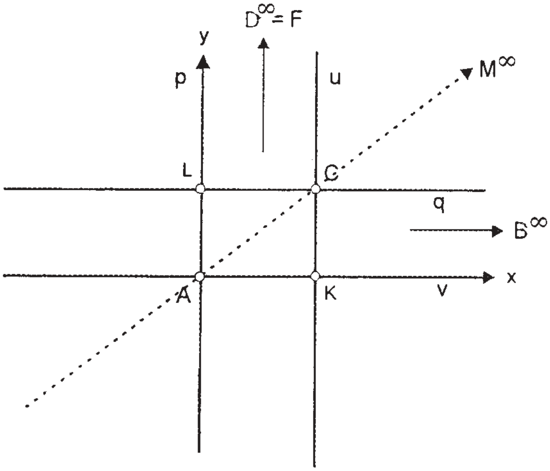

Fig.l: Fundamental elements within the pencil subtype I.6

The fundamental conic section k_1 determined by the lines p and q and its invariants with respect to the group G_3 of motions of an isotropic plane [3] are given by

$$k_1(x, y) \equiv p\,q = x\,(y + c_q) = 0, \qquad \Delta_2 = 0,\ \ \Delta_1 < 0,\ \ \Delta_0 = 0\,. \tag{4.3}$$

For the fundamental conic section k_2 determined by the lines u and v we have

$$k_2(x,y) \equiv u\,v = (x + c_u)y = xy + c_u y = 0, \qquad \Delta_2 = 0,\ \Delta_1 < 0,\ \Delta_0 = 0 \tag{4.4}$$

The conic pencil generated with the curves k_1 and k_2 is given in the equation

$$h(x, y) \equiv k_1\,(x, y) + \lambda\,k_2(x, y) =$$

$$= xy + c_q x + \lambda(xy + c_u y) = (1 + \lambda)xy + c_q x + \lambda c_u y =$$

$$= [1\ x\ y]\begin{bmatrix} 0 & c_q/2 & \lambda c_u/2 \\ c_q/2 & 0 & (1+\lambda)/2 \\ \lambda c_u/2 & (1+\lambda)/2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = 0 \tag{4.5}$$

4.2 Pencil of Quadrics

The equation of the pencil of conic sections given in (4.5), in projective coordinates $(x_1 : x_2 : x_3)$ of ω, has the form

$$h \equiv (1+\lambda)x_2x_3 + c_q x_1x_2 + \lambda c_u x_1x_3 = 0, \tag{4.6}$$

that is, in space affine coordinates (x, y, z):

$$\mathrm{h} \equiv (1+\lambda)yz + c_q xy + \lambda c_u xz = c_q xy + yz + \lambda(c_u xz + yz) = 0. \tag{4.7}$$

Let's now connect the discussed pencil of conics with pencils of quadrics in $I_3^{(2)}$, in a way that (4.7) is understood as an equation of the section of the pencil of quadrics with the absolute isotropic plane ω. All quadrics having (4.7) as an section with the plane of infinity ω are

$$H \equiv c_q xy + yz + a_{01}x + a_{02}y + a_{03}z + a_{00} + \lambda(c_u xz + yz + b_{01}x + b_{02}y + b_{03}z + b_{00}) = \tag{4.8}$$

So, the fundamental quadrics are given in the equations

$$K_1(x, y, z) \equiv c_q xy + yz + 2a_{01} + 2a_{02}y + 2a_{03}z + a_{00} = 0, \text{ and}$$

$$K_2(x, y, z) \equiv c_u xz + yz + 2b_{01}x + 2b_{02}y + 2b_{03}z + b_{00} = 0\ . \tag{4.9}$$

Rewriting (4.8), and using the matrix notation, we get
$H \equiv c_q xy + c_u \lambda xz + (1+\lambda)yz + (a_{01} + \lambda b_{01})x + (a_{02} + \lambda b_{02})y + (a_{03} + \lambda b_{03})z + a_{00} + \lambda b_{00} =$

$$= [1 \quad x \quad y \quad z]\begin{bmatrix} 2(a_{00}+\lambda b_{00}) & a_{01}+\lambda b_{01} & a_{02}+\lambda b_{02} & a_{03}+\lambda b_{03} \\ a_{01}+\lambda b_{01} & 0 & c_q & c_q \lambda \\ a_{02}+\lambda b_{02} & c_q & 0 & 1+\lambda \\ a_{03}+\lambda b_{03} & c_u \lambda & 1+\lambda & 0 \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = 0 \tag{4.10}$$

4.3 Surfaces within the Pencil

We are going to determine by means of analysing the pencil of conics of the subtype I. 6 in the isotropic plane, which 2nd order surfaces are placed within the pencil (4.8). Appropriate 2nd order surfaces of the double isotropic space have been affiliated with individual classes of conics within a fundamental pencil, as it is shown in [4]. The connection among the invariants of conics in the absolute isotropic plane of infinity ω, and of the 2nd order surfaces in the double isotropic space with respect to the group of motions in the plane and the group of motions in the space is given in 2.3.

Within the conic pencils of subtype I.6, apart from the singular conic sections, all of the curves are special hyperbolas. The three siugular conic sections of tlie pencil are:

– two pairs of straight lines, with one element of each pair being an isotropic line,

– pair of straight lines consisting of a no isotropic and an absolute line f.

Further on, the conic invariants with respect to the groupG_3 of motions of an isotropic plane ω, and the possibilities for the associated quadrics in $I_3^{(2)}$, according to [4], are as follows:

(i) For special hyperbolas we have
$\Delta_2 = 0, \quad \Delta_1 < 0, \quad \Delta_0 \neq 0$
and the possibilities for the surfaces are
$\Delta_2 = 0, \Delta_1 < 0, \Delta_0 \neq 0 +$
$\Delta > 0, \Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0 \Rightarrow$ one sheet hyperboloid of the second family (isotropic and nonisotropic);
$\Delta < 0, \Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0 \Rightarrow$ two sheets hyperboloid of the second family (isotropic and nonisotropic);
$\Delta = 0, \Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0 \Rightarrow$ real cone of the second family (isotropic and nonisotropic).

(ii) For pair of lines, one being an isotropic line we have
$\Delta_2 = 0, \Delta_1 < 0, \Delta_0 = 0,$
and the possibilities for the surfaces are
$\Delta_2 = 0, \Delta_1 < 0, \Delta_0 = 0 +$
$\Delta > 0 \Rightarrow$ nonisotropic hyperbolical paraboloid of the second family;
$\Delta = 0, \Delta_{11} \neq 0 \Rightarrow$ nonisotropic hyperbolical cylinder of the second family;
$\Delta = 0, \Delta_{11} = 0 \Rightarrow$ pair of planes with a nonisotropic intersection of the second family.

(iii) For pair of lines, consisting of a nonisotropic line and an absolute line f, we have the invariants
$\Delta_2 = 0, \Delta_1 = 0, \Delta_0 = 0, \Delta_{22} < 0,$
and the possibilities for the associated quadrics are
$\Delta_2 = 0, \ \Delta_1 = 0 \ \ \Delta_0 = 0 \ \ \Delta_{22} < 0 +$
$\Delta > 0 \Rightarrow$ isotropic hyperbolical paraboloid of the fourth family;
$\Delta = 0, \Delta_{22} \neq 0 \Rightarrow$ isotropic hyperbolical cylinder of the fourth family;
$\Delta = 0, \Delta_{22} = 0 \Rightarrow$ pair of planes, one being a double isotropic plane, with an isotropic intersection of the fourth family.

On the other hand, for all the surfaces $H(\lambda)=0$ of the pencil (4.8), the main invariants given in the relation (3.5) can be computed. Thus we obtain

$$\begin{aligned}
&\Delta_2(\lambda)=0,\ \Delta_1(\lambda)=-(1+\lambda)^2 \leq 0,\ \Delta_0(\lambda)=c_u c_q(1+\lambda)\,\lambda \lesseqgtr 0,\\
&\Delta(\lambda) = -(a_{03} + b_{03}\lambda)\,(a01cq-a03cq^2+a01cq\,\lambda+b01\mathrm{cq}\,\lambda-\\
&\qquad -b03cq^2\,\lambda+a02cq\;cu\,\lambda+b01\;cq\,\lambda^2+b02\;cq\;cu\,\lambda^2)-\\
&\qquad -(1+\lambda)\,(-(a01+b01\,\lambda)\,(a01-a03cq+a01\,\lambda+\\
&b01\,\lambda-b03cq\,\lambda+b01\,\lambda^2)-cu\,\lambda\,(-a01\;a02+\\
&2\,a00\;cq-a02\;b01\,\lambda-a01\;b02\,\lambda+2\,b00\;cq\,\lambda-b01\;b02\;\lambda^2))+\\
&cu\,\lambda\,(\,(1+\lambda)\,(-a01\;a02+2\,a00\;cq-a02\;b01\,\lambda-a01\;b02\,\lambda\\
&+2b00\;cq\,\lambda-b01b02\,\lambda^2)+(a02+b02\,\lambda)\,(-a03\;cq-b03\;cq\,\lambda+\\
&a02\;cu\,\lambda+b02\;cu\,\lambda^2)\,)
\end{aligned} \tag{4.11}$$

Proposition 4.3.1

Pencils of quadrics of the double isotropic space $I_3^{(2)}$ generated by the pencils of conic sections of subtype I.6 in the absolute isotropic plane ω, can contain

– isotropic and nonisotropic one and two sheets hyperboloids of the second family,
– two nonisotropic hyperbolical paraboloids of the second family,
– one isotropic hyperbolical paraboloid of the fourth family

being proper (non-singular) surfaces of the pencil, as well as

– isotropic and nonisotropic cones of the second family,
– nonisotropic hyperbolical cylinders of the second family;
– isotropic hyperbolical cylinders of the fourth family
being singular surfaces of the pencil.

Proof.

The affine classification of pencils of quadrics made according the projective classification of pencils of conic sections [8] in which the plane of infinity intersects the observed pencil of quadrics is given in [9]. According to that very classification, all pencils of quadrics in $I_3^{(2)}$ generated by pencils of conics of type I of the absolute isotropic plane ω belong to the group of

- *Nonhomothetic* pencils consisting of hyperboloids or hyperboloids and ellipsoids, i.e. to the subgroup consisting of
  - • hyperboloids and 3 hyperbolical paraboloids or cylinders.

Homothetic pencils of quadrics are pencils having common 2nd order curve in the plane of infinity, which is not our case. The proposition now follows according to the classification of the 2nd order surfaces in $I_3^{(2)}$, as well as the values of the main invariants of the quadrics $H(\lambda)=0$ of the pencil that are given in the relation (4.11).

4.4 Fundamental Quadrics

Let's consider the equation of the pencil given by

$$H \equiv c_q xy + yz + a_{01}x + a_{02}y + a_{03}z + a_{00} + \lambda(c_u xz + yz + b_{01}x + b_{02}y + b_{03}z + b_{00}) = 0 \qquad (4.12\ )$$

After the translation of the coordinate system in y- and z- direction we obtain

$$H \equiv -c_q\overline{xy} + \overline{yz} + a_{02}\overline{y} + a_{03}\overline{z} + a_{00} + \lambda(c_u\overline{xz} + \overline{yz} + b_{02}\overline{y} + b_{03}\overline{z} + b_{00}) = 0. \qquad (4.13)$$

The main invariants of the fundamental quadric

$$K_1(x, y, z) = c_q xy + yz + a_{02}y + a_{03}z + a_{00} = 0 \qquad (4.14)$$

with respect to the group G_6 of motions in $I_3^{(2)}$ are

$$\Delta(K_1) = (a_{03}c_q)^2, \quad \Delta_0(K_1) = \quad \Delta_1(K_1) < 0, \quad \Delta_2(K_1) = 0. \qquad (4.15)$$

One concludes that, providing that $a_{03} \neq 0$, it is a hyperbolical paraboloid of the second family.

The main invariants of the fundamental quadric

$$K_2(x, y, z) = c_u xz + yz + b_{02}y + b_{03}z + b_{00} = 0 \tag{4.16}$$

are

$$\Delta(K_2) = (b_{02}c_u)^2, \quad \Delta_0(K_2) = 0 \quad \Delta_1(K_2) < 0 \quad \Delta_2(K_2) = 0. \tag{4.17}$$

Following the above procedure, with $b_{02} \neq 0$, it is a hyperbolical paraboloid of the second family.

Since the selection of coefficients a_{02}, a_{00}, b_{03}, b_{00} does not affect the class that the fundamental quadrics belong to, we shall further on presume that the following has been fulfilled:

$$a_{03} \neq 0 \ , \ b_{02} \neq 0 \ , \ a_{02} = a_{00} = b_{03} = b_{00} = 0 \ . \tag{4.18}$$

Thus we obtain

$$H \equiv c_q xy + yz + a_{03}z + \lambda(c_u xz + yz + b_{02}y) = 0 \ . \tag{4.19}$$

4.5 Characteristic Equation and Fundamental Curve of the Pencil

The characteristic equation (3.7) of the pencil (4.19) acquires the form:

$$\Delta(\lambda) \equiv (a_{03}cq - b_{02}cu\,\lambda^2)^2 = 0. \tag{4.20}$$

Singular surfaces of the pencil are obtained for the following values of the parameter λ:

$$\lambda_1 = \lambda_2 = \sqrt{\frac{a_{03}c_q}{b_{02}c_u}} \quad \text{and} \quad \lambda_3 = \lambda_4 = -\sqrt{\frac{a_{03}c_q}{b_{02}c_u}}$$

From (4.21) one concludes that both values are either real or complex. Let's point out that with the complex values of the parameter λ equations of the second degree in three variables with imaginary coefficients are obtained that are understood as surfaces as well [9]. Two surfaces having the equation coefficients conjugate imaginaries are called *mutually conjugate imaginaries surfaces* [9].

Proposition 4.5.1

Pencils of quadrics of the double isotropic space $I_3^{(2)}$ generated by the pencils of conic sections of subtype I.6 in the isotropic plane, under the conditions given in (4.18), can contain:

– isotropic and nonisotropic one sheet hyperboloids of the second family,
– two nonisotropic hyperbolical paraboloids of the second family,

– one isotropic hyperbolical paraboloid of the fourth family
being proper (non-singular) surfaces of the pencil, as well as
– two double nonisotropic cones of the second family
being singular surfaces of the pencil.

The fundamental curve of the pencil is the 4th order curve that breaks up into:

– *nonisotropic straight line*, as the connecting line $\overline{S_1 S_2}$ of the centres of the singular surfaces within the pencil, containing the fundamental point A of the pencil I.6,
– *cubic hyperbola*, which is the 3rd order space curve containing three points B, C and D at infinity being fundamental points of the pencil I.6 of conic sections.

Proof.

According the projective classification of pencils of quadrics made with respect to the solutions of the characteristic equation of the pencil, pencils of quadrics in $I_3^{(2)}$ generated by pencils of conics of subtype I.6, under the conditions given in the relation (4.18), belong to the group consisting of

- 2 cones as singular surfaces of the pencil, with the vertices - centres - S_1, S_2 being fundamental points of the pencil. S_1 and S_2 are either real or conjugate imaginaries that imply the connecting line $\overline{S_1 S_2}$ being real. $\overline{S_1 S_2}$ is a part of the fundamental curve consisting of that very line and the 3rd order space curve that is not breaking up further on. The connecting line $\overline{S_1 S_2}$ cuts the 3rd order space curve in the point S_1 and in S_2 ([9], [16]).

The way the fundamental curve breaks up implies that the pencil consists only of ruled surfaces. It is now easy to verify the second part of the proposition using the values from the relation (4.21).

4.6 Curve of the Centres

From the pencil equation

$$H = c_q xy + yz + a_{03} z + \lambda(c_u xz + yz + b_{02} y) = 0. \tag{4.19}$$

computing

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = \frac{\partial H}{\partial z} = 0 \tag{4.22}$$

the curve of the centres is obtained (see [10]), i.e.,

$$\frac{\partial H}{\partial x} = c_q y + c_u \lambda z, \quad \frac{\partial H}{\partial y} = c_q x + (1+\lambda) z + b_{02}\lambda, \quad \frac{\partial H}{\partial z} = c_u \lambda x + (1+\lambda) y + a_{03}. \tag{4.23}$$

In homogenous coordinates $(x_0 : x_1 : x_2 : x_3)$, with $\lambda \to \dfrac{\lambda}{\mu}$, from (4.22) and (4.23), we get the parametric equation of the curve of the centres, that is

$$k_s^3 \begin{cases} x_0 = 2c_u c_q \lambda\mu(\lambda+\mu) \\ x_1 = -(a_{03}c_q\mu^2 + b_{02}c_u\lambda^2)(\lambda+\mu) \\ x_2 = -(a_{03}c_q\mu^2 + b_{02}c_u\lambda^2)c_u\lambda \\ x_3 = -(-a_{03}c_q\mu^2 + b_{02}c_u\lambda^2)c_u\mu \end{cases}$$

k_s^3 is the 3rd order space curve not lying in a plane, known as the *twisted cubic.* In the affine space the twisted cubics are divided in four cases with respect to their points at infinity (classification given in [7]). Those are:

(i) *Cubic parabolas* (Die kubische Parabel) - one triple point at infinity;
(ii) *Cubic hyperbolical parabolas* (Die kubische hyperbolische Parabel) - one double and one single point at infinity;
(iii) *Cubic hyperbolas* (Die kubische Hyperbel) - three real mutually different points at infinity;
(iv) *Cubic ellipses* (Die kubische Ellipse) - one real and two conjugate imaginaries at infinity.

In the double isotropic space the twisted cubics has to be divided according to the power and multiplicity of isotropy as well. We will distinguish

(i) *Single, double, ..., 2-isotropic curve,* if it contains single, double, ..., the absolute point at infinity F;
(ii) *Single, double, ... , 1-isotropic curve,* if it contains single, double, ... , the points belonging to the absolute line $f\setminus\{F\}$;
(iii) *Nonisotropic curve,* if it doesn't contain points belonging to the absolute line f.

Proposition 4.6.1

The curve k_s^3 of the centres of the pencil of quadrics in $I_3^{(2)}$ generated by pencils of conics of subtype I.6, under the conditions given in the relation (4.18) is a single 1-isotropic cubical hyperbola.

Proof. With $x_0=0$ and the equation given in (4.24) we obtain the intersection of k_s^3 with the plane of infinity, that is

(i) $\lambda=0 \Rightarrow (x_0 : x_1 : x_2 : x_3)=(0 : 1 : 0 : -c_q)$,
understood as the centre of the nonisotropic hyperbolical paraboloid of the second family, that is the centre $K(1 : 0 : -c_q)$ of the singular conic $k_1(x, y)=0$ of the pencil I.6;

(ii) $\mu=0 \Rightarrow (x_0 : x_1 : x_2 : x_3)=(0 : 1 : -c_u : 0)$,
understood as the centre of the nonisotropic hyperbolical paraboloid of the second family, that is the centre $L(1 : -c_u : 0)$ of the singular conic $k_2(x, y)=0$ of the pencil I.6;

(iii) $\dfrac{\lambda}{\mu}=-1 \Rightarrow (x_0 : x_1 : x_2 : x_3)=(0 : 0 : c_u : c_q)$,
understood as the centre of the 1-isotropic hyperbolical paraboloid of the fourth family, that is the centre $M(0 : c_u : c_q)$ of the third singular conic of the pencil I.6.

It follows that all the three points are real and mutually different and one of them is incident with the absolute line *f*.

In the sequel we generalise the above proposition.

Proposition 4.6.2

Curves k_s^3 of the centres of all pencils of quadrics in $I_3^{(2)}$ generated by pencils of conics of type I in the isotropic plane are cubic hyperbolas while the power of isotropy is given in the coordinates of the main points of the fundamental pencil of conic sections.

Proof. Main points *K, L* and *M* of the pencil of conic sections are the centres of the singular conics of the pencil. The proposition follows from the properties of pencils of conics of type I in the isotropic plane given in [2].

4.7 Focal Surface

Let's consider the equation of the pencil of quadrics given in the relation (4.19), i.e.,

$$H \equiv c_q xy + yz + a_{03}z + \lambda(c_u xz + yz + b_{02}y) = 0. \tag{4.19}$$

All polar planes of the quadric surfaces of the observed pencil with respect to the absolute point $F(0{:}0{:}0{:}1)$ are obtained by computing

$$\frac{\partial H}{\partial z} \equiv \lambda c_u x + (1+\lambda)y + a_{03} = 0. \tag{4.25}$$

Equation (4.25) represents a pencil of isotropic planes, with a double isotropic straight line

$$\begin{cases} x = \dfrac{a_{03}}{c_u}, \\ x = -a_{03} \end{cases} \tag{4.26}$$

being its axis.

From (4.25) we have

$$\lambda = -\frac{y + a_{03}}{c_u x + y}, \qquad c_u x + y \neq 0. \tag{4.27}$$

Substituting such obtained λ in the pencil equation (4.19) one obtains the geometrical locus of all 2nd order curves as curves of intersection of the surfaces of the pencil with the affiliated polar planes, known as *focal surface:*

$$\Phi = c_u c_q x^2 y + c_q x y^2 - b_{02} y^2 - a_{03} b_{02} y = 0. \tag{4.28}$$

$c_u x + y = 0$ has to be excluded from consideration because it would derive unallowed changes on the fundamental pencil of conic section.

From (4.28) one concludes that Φ is the 3rd order cylindrical surface with double isotropic generating lines. According to [11] it is the 3rd order cone having a vertex in the absolute point F. In our case the cone brakes up as follows:

$$\Phi\begin{cases} y = 0 \\ c_u c_q x^2 + c_q xy - b_{02} y - a_{03} b_{02} = 0 \end{cases} \tag{4.29}$$

It is easy to verify that the absolute isotropic plane at infinity ω cuts the focal surface Φ in the *focal curve* of the fundamental pencil of conic sections of subtype I.6.

As a conclusion, we have

Proposition 4.7.1

The focal surface Φ associated with the pencil of quadrics of the double isotropic space $I_3^{(2)}$ which is generated by the pencil of conics of subtype I.6, under the conditions given in the relation (4.18), is the 3rd order cone having a vertex in the absolute point F. The 3rd order cone breaks up into an isotropic plane and the 2-isotropic hyperbolical cylinder of the 4th family providing that $c_q a_{03} - c_u b_{02} \neq 0$, that is, into an isotropic plane and a pair of planes with a double isotropic intersection of the 4th family for $c_q a_{03} - c_u b_{02} = 0$.

Proof.

According to the classification of the 2nd order surfaces in $I_3^{(2)}$ and the fact that the focal curve k_f^3 of each pencil of conic sections in the isotropic plane belonging to subtype I.6 consists of three straight lines two being isotropic lines and the third being an absolute line f [2].

4.8 Subcases

Let's consider now the more general equation of pencil of quadrics given in the relation (4.13), i.e.,

$$H \equiv c_q xy + yz + a_{02} y + a_{03} z + a_{00} + \lambda (c_u xz + yz + b_{02} y + b_{03} z + b_{00}) = 0.$$

Apart from the conditions on the equation coefficients given by the relation (4.18) one could discuss the following subcases as well:

Ad 1) Let's assume that $a_{03}=0$ and $b_{02} \neq 0$.

$$a_{03}=0 \Rightarrow K_1(x,y,z) = c_q xy + yz + a_{02} y + a_{00} = 0. \tag{4.30}$$

The main invariants are

$$\Delta(K_1)=0, \quad \Delta_0(K_1)=0, \quad \Delta_1(K_1)<0, \quad \Delta_2(K_1)=0. \tag{4.31}$$

Besides, $D_{11} = -2a_{00}$.

Providing that $a_{00}\neq 0$ it is a nonisotropic hyperbolical cylinder of the 2nd family (see[4]).

$$b_{02}\neq 0 \Rightarrow K_2(x, y, z)=c_u xz+yz+b_{02}y+b_{03}z+b_{00}=0. \qquad (4.32)$$

The main invariants are

$$\Delta(K_2)=(b_{02}c_u)^2, \quad \Delta_0(K_2)=0, \quad \Delta 1(K_2)<0, \quad \Delta_2(K_2)=0, \qquad (4.33)$$

wherefrom, with $b_{02}\neq 0$ it is a nonisotropic hyperbolical paraboloid of the 2nd family. Since the selection of the coefficient a_{02} , b_{03} , b_{00} has no affect the class the fundamental quadrics of the pencil belong to, it can be assumed that

$$a_{02}=b_{03}=b_{00}=0, \qquad (4.34)$$

and the pencil equation acquires the form

$$H\equiv c_q xy+yz+a_{00}+\lambda(c_u xz+yz+b_{02}y)=0. \qquad (4.35)$$

Ad 2) Let's assume that $a_{03} \neq 0$ and $b_{02} = 0$.

$$a_{03}\neq 0 \Rightarrow K_1(x,y,z)=c_q xy+yz+a_{02}y+a_{03}z+a_{00}=0. \qquad (4.36)$$

The main invariants are

$$\Delta(K_1)=(a_{03}c_q)^2, \quad \Delta_0(K_1)=0 \quad \Delta_1(K_1)<0, \quad \Delta_2(K_1)=0, \qquad (4.37)$$

wherefrom, with $a_{03}\neq 0$ it is a nonisotropic hyperbolical paraboloid of the 2nd family.

$$b_{02}=0 \Rightarrow K_2(x, y, z)=c_u xz+yz+b_{03}z+b_{00}=0. \qquad (4.38)$$

The main invariants are

$$\Delta(K_2)=0, \quad \Delta_0(K_2)=0, \quad \Delta_1(K_2)<0, \quad \Delta_2(K_2)=0. \qquad (4.39)$$

Besides, $D_{11}=-2b_{00}$ wherefrom, with $b_{00}\neq 0$ it is a nonisotropic hyperbolical cylinder of the 2nd family.

Similarly, the selection of the coefficient a_{02}, a_{00}, b_{03} has no effect on the class the fundamental quadrics of the pencil belong to, it can be assumed that

$$a_{02}=a_{03}=b_{03}=0, \qquad (4.40)$$

and the pencil equation acquires the form

$$H\equiv c_q xy+yz+a_{03}z+l(c_u xz+yz+b_{00})=0. \qquad (4.41)$$

Ad 3) Let's assume that $a_{03}=0$ and $b_{02}=0$.

Ad 4) Let's assume that $a_{03}=0$, $a_{00}=0$ and $b_{02}=0$.

Ad 5) Let's assume that ......

We could repeat the procedure given in the chapters 4.5, 4.6, 4.7 for any of the above and further cases with no difficulties.

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O pramenovima ploha 2. reda u dvostruko izotropnom prostoru $I_3^{(2)}$

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SAŽETAK

Dvostruko izotropni prostor $I_3^{(2)}$ je trodimenzionalni afini prostor s apsolutnom figurom $\{\omega, f, F\}$, gdje je f pravac beskonačno daleke ravnine ω , a F točka pravca f.

Pramen kvadrika je skup od ∞^1 kvadrika koje imaju zajedničku prostornu krivulju 4. reda. Presjek pramena kvadrika bilo kojom ravninom je pramen krivulje 2. reda.

U radu se, u prostoru $I_3^{(2)}$, pramenovi kvadrika promatraju kao pramenovi pridruženi pramenovima konika izotropne apsolutne ravnine ω. Analiza pramenova kvadrika vrši se s obzirom na izotropnu klasifikaciju konika, izotropnu klasifikaciju pramenova konika, izotropnu klasifikaciju kvadrika dvostruko izotropnog prostora i projektivna svojstva pramenova kvadrika. Promatra se temeljna jednadžba i temeljna krivulja pramena, krivulja središta i žarišna ploha pramena kvadrika. Dan je pristup po kojem se u nastavku mogu obraditi proizvoljni pramenovi kvadrika u $I_3^{(2)}$ kao i provesti njihova potpuna klasifikacija.

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