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A NOTE ON THE SPACES WHICH ADMIT A WHITNEY MAP

Ivan Lončar

Abstract

Let X be a non-metric continuum, and $C(X)$ be the hyperspace of subcontinua of X. It is known that there is no Whitney map on the hyperspace 2^X for non-metrizable Hausdorff compact spaces X. On the other hand, there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$. In this paper we will study the propertis of non-metric spaces X which admit a Whitney map for $C(X)$.

Key words and phrases: Hyperspace, Whitney map

1. Introduction

All spaces in this paper are Hausdorff and all mappings are continuous. The weight of a space X is denoted by $w(X)$. The cardinality of a set A is denoted by card (A). We shall use the notion of inverse system as in [4, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

A *generalized arc* is a Hausdorff continuum with exactly two nonseparating points. Each separable arc is homeomorphic to the closed interval $I=[0,1]$.

For a space X we denote by 2^X the hyperspace of all nonempty closed subsets of X equipped with the Vietoris topology. $C(X)$ stand for the sets of all compact connected members of 2^X considered as subspace of 2^X.

For a mapping $f: X \to Y$ define $C(f): C(X) \to C(Y)$ by $C(f)(F) = f(F)$ for $F \in C(X)$. By [13, 5.10] $C(f)$ is continuous.

An element $\{x_a\}$ of the Cartesian product $\prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ if $p_{ab}(x_b) = x_a$ for any $a, b \in A$ satisfying $a \leq b$. The subspace of $\prod\{X_a : a \in A\}$ consisting of all threads of $\mathbf{X}$ is called the limit of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by lim $\mathbf{X}$ or by lims $\{X_a, p_{ab}, A\}$ [4, p. 135].

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system with the natural projections $p_a : \lim \mathbf{X} \to X_a$, $a \in A$. Then $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ is an inverse system.

Lemma 1.1. [4, Problem 6.3.22.(f), p. 465]. *Suppose that* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *is an inverse system. Then* $C(\lim \mathbf{X}) = \lim C(\mathbf{X})$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *σ-directed* if for each sequence $a_1, a_2, \ldots, a_k, \ldots$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

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An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *factorizing* [3, p. 17] if for each real-valued continuous function $f : \lim X \to \mathbb{R}$ there exist an $a \in A$ and a mapping $f_a : X_a \to \mathbb{R}$ such that $f = f_a p_a$.

If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is σ-directed inverse system and if lim**X** is a Lindelöf space, then we have the following theorem.

Theorem 1.2. [3, Corollary 1.3.2, p. 18]. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a σ-directed inverse system with surjective projections* $p_a : \lim \mathbf{X} \to X_a$. *If lim***X** *is Lindelöf, then* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *is factorizing.*

If lim**X** is compact, then we have the following corollary.

Corollary 1.3. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a σ-directed inverse system of compact spaces with surjective projections* $p_a : \lim \mathbf{X} \to X_a$. *Then X is factorizing.*

In the sequel we shall use the following result.

Lemma 1.4. [4, Corollary 2.5.11]. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system and B a subset cofinal in A. The mapping consisting in restricting all threads from X* = lim **X** *to B is a homeomorphism of X onto the space* $\lim \{X_b, p_{bc}, B\}$.

We will use the following expanding theorem of non-metric compact spaces into σ-directed inverse systems of compact metric spaces.

Theorem 1.5. *Let X be compact Hausdorff space such that* $w(X) \geq \aleph_1$. *There exists a σ-directed inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *of metric compacta* X_a *such that X is homeomorphic to lim***X**.

Proof. See [10, Theorem 4].

2. Whitney map and hereditarily irreducible mappings

The notion of an irreducible mapping was introduced by Whyburn [19, p. 162]. If X is a continuum, a surjection $f : X \to Y$ is *irreducible* provided no proper subcontinuum of X maps onto all of Y under f.

Some theorems for the case when X is semi-locally-connected are given in [19, p. 163].

A mapping $f : X \to Y$ is said to be *hereditarily irreducible* [14, p. 204, (1.212.3)] provided that for any given subcontinuum Z of X, no proper subcontinuum of Z maps onto $f(Z)$.

A mapping $f : X \to Y$ is *light (zero-dimensional)* if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [4, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subsets of cardinality larger than one ($\dim f^{-1}(y) \leq 0$). Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

Lemma 2.1. *Every hereditarily irreducible mapping is light.*

Lemma 2.2. [14, (1.212.3), p. 204]. *A mapping $f : X \to Y$ of a continuum X into continuum Y is hereditarily irreducible if and only if $C(f): C(X) \to C(Y)$ is light.*

Now we shall prove that the assumption that X is a continuum can be omitted.

Lemma 2.3. *A mapping $f : X \to Y$ of a space X into space Y is hereditarily irreducible if and only if $C(f): C(X) \to C(Y)$ is light.*

Proof. *The "if" part.* Suppose that $C(f): C(X) \to C(Y)$ is light and $f : X \to Y$ is not hereditarily irreducible. This means that there are subcontinua Z and W of X such that $Z \subset W, Z \neq W$, and $f(Z) = f(W)$. Then for every subcontinuum V with $Z \subset V \subset W$ we have $f(Z) = f(V) = f(W)$. The family of all subcontinua V such that $Z \subset V \subset W$ is a continuum L in C ([6, Theorem, p. 1209]). Now, $C(f)(L) = C(f)(\{W\})$. This is impossible since $C(f)$ is light.

The "only if" part. Suppose that f is hereditarily irreducible and the $C(f)$ is not light. This means that there exists $\{K\} \in C(Y)$ such that $(C(f))^{-1}(\{K\})$ contains a subcontinuum L of $C(X)$ such that $C(f)(L) = \{K\}$. Let $\{K_a : a \in A\}$ be a set of points of L. Every K_a is a subcontinuum of X such that $f(K_a) = K$. On the other hand $M = \cup\{K_a : a \in A\}$ is compact subset of X [13, Theorem 2.5.2, p. 157]. Moreover, M is connected [13, Proposition 2.8, p. 158]. Hence, M is a subcontinuum of X. It is clear $f(M) = K$. This is impossible since $f(K_a) = K, K_a \subset K$ and f is hereditarily irreducible.

Lemma 2.4. *If $f : X \to Y$ is monotone and hereditarily irreducible, then f is one-to-one.*

Lemma 2.5. *Every on-to-one mapping $f : X \to Y$ is hereditarily irreducible.*
Lemma 2.6. *If $f : X \to Y$ is a mapping such that*

$$\dim\{x : x \in X, \{x\} \neq f^{-1}f(x)\} \leq 0,$$

then f is hereditarily irreducible.

Proof. Let K, L be a pair of subcontinua of X such that $K \subset L$ and that $K \neq L$. This means there exists a point $x \in L \setminus K$. Let U be a neighborhood of x in $L \setminus K$ such that $Cl_{L\setminus K}(U) \subset L \setminus K$. There exists a component C of $Cl_{L\setminus K}(U)$ containing x and such that $C \cap Bd(U) \neq 0$ [14, Theorem (20.1), p. 625]. We infer that C is a non-degenerate continuum. From $\dim\{x : x \in X, \{x\} \neq f^{-1}f(x)\} \leq 0$ it follows that C is non contained in $\{x : x \in X, \{x\} \neq f^{-1}f(x)\}$. This means there is a point $y \in C$ such that $\{y\} = f^{-1}f(y)$. Hence $f(K) \neq f(L)$.

Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [14, p. 24, (0.50)] we will mean any mapping $g : \Lambda \to [0, +\infty)$ satisfying
a) if $\{A\}, \{B\} \in \Lambda$ such that $A \subset B, A \neq B$, then $g(\{A\}) < g(\{B\})$ and
b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If X is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ ([14, pp. 24-26], [5, p. 106]). On the other hand, if X is non-metrizable, then it admits no any Whitney map for 2^X [2]. It is known that there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$ [2]. Moreover, if X is a non-metrizable locally connected or a rim-metrizable continuum, then X admits no a Whitney map for $C(X)$ [9, Theorem 8, Theorem 11].

Lemma 2.7. *Any metric space X admits a Whitney map for $C(X)$.*

Proof. From [17, Lemma 3.] it follows that there exists a Whitney map μ for $C\,P(X)$, i.e., for the hyperspace of all compact subsets of X. Hence, $\mu|C(X)$ is a Whitney map for $C(X)$.

The proof of the next lemma is obvious.

Lemma 2.8. *If $f : X \to Y$ is a hereditarily irreducible mapping and if Y admits a Whitney map for $C(Y)$, then X admits a Whitney map for $C(X)$.*

Lemma 2.9. *If a Hausdorff space X admits one-to-one continuous mapping $f : X \to Y$ onto a metric space Y, then X admits a Whitney map for $C(X)$.*

Proof. By virtue of Lemma 2.7 we conclude that Y admits a Whitney map $\mu : C(Y) \to [0,+\infty)$. Let us prove that $\mu C(f)$ is a Whitney map for $C(X)$. If $\{x\} \in C(X)$, then $C(f)(\{x\}) = \{f(x)\}$ and $\mu C(f)(\{x\}) = \mu\{f(x)\} = 0$ since μ is a Whitney map for $C(Y)$, Furthermore, let K, L be a pair of subcontinua of X such that $K \subset L, K \neq L$. By virtue of Lemma 2.5 we have $f(K) \subset f(L)$ and $f(K) \neq f(L)$. Moreover, $\{f(K)\}, \{f(L)\} \in C(Y)$. Therefore, $\mu(\{f(K)\}) < \mu(\{f(L)\})$, i.e., $\mu C(f)(\{K\}) < \mu C(f)(\{L\})$.

Theorem 2.10. *If X is a paracompact space and the diagonal Δ is G_δ-set in $X \times X$, then X admits a Whitney map for $C(X)$.*

Proof. By virtue of [4, Problem 5.5.7, p. 421] there exists one-to-one continuous mapping $f : X \to Y$ onto a metric space Y. Apply Lemma 2.9.

Lemma 2.11. *If there exists a mapping $f : X \to Y$ onto a metric space Y such that*

$$\dim\{x : x \in X, \{x\} \neq f^{-1}f(x)\} \leq 0,$$

then X admits a Whitney map for $C(X)$.

Proof. This is a part of theorem (0.51) from [14] for metric continua. By virtue of Lemma 2.6 it follows that f is hereditarily irreducible. By virtue of Lemma 2.7 we conclude that Y admits a Whitney map $\mu : C(Y) \to [0,\infty)$. The proof that $\mu C(f)$ is a Whitney map for $C(X)$ is similar to the proof of Lemma 2.9.

3. An external characterization of non-metric spaces *X* which admit a Whitney map for *C(X)*

In this section we shall give an external characterization of non-metric spaces which admit a Whitney map. We shall use the inverse system method.

Theorem 3.1. *Let* $\mathbf{X}=\{X_a, p_{ab}, A\}$ *be an inverse system of spaces which admit Whitney maps for* $C(X_a)$ *and let* $X = \lim \mathbf{X}$*. If* $C(\mathbf{X})=\{C(X_a), C(p_{ab}), A\}$ *is factorizing, then X admits a Whitney map for C(X) if and only if there exists a cofinal subset* $B \subset A$ *such that for every* $b \in B$ *the projection* $p_b : \lim \mathbf{X} \to X_b$ *is hereditarily irreducible.*

Proof. Necessity. Consider inverse system $C(\mathbf{X})=\{C(X_a), C(p_{ab}), A\}$ whose limit is *C(X)* (Lemma 1.1). If $\mu : C(Y) \to [0,\infty)$ is a Whitney map for *C(X)*, then there exists a cofinal subset *B* of *A* such that for every $b \in B$ there is a mapping $\mu_b : C(X_b) \to [0,\infty)$ with $\mu = \mu_b C(p_b)$ since $C(\mathbf{X})=\{C(X_a), C(p_{ab}), A\}$ is factorizing. Suppose that p_b is not hereditarily irreducible. Then there exists a pair *F*, *G* of subcontinua of *X* with $F \subseteq G$, $F \neq G$, (i.e., *F* is a proper subcontinuum of *G*) such that $p_b(F)=p_b(G)$. It is clear that $C(p_b)(\{F\})=C(p_b)(\{G\})$. This means that $\mu_b C(p_b)(\{F\})=\mu_b C(p_b)(\{G\})$. From $\mu = \mu_b C(p_b)$ it follows that $\mu(\{F\})=\mu(\{G\})$. This is impossible since μ is a Whitney map for *C(X)* and from $F \subseteq G$, $F \neq G$ it follows $\mu(\{F\})<\mu(\{G\})$.

Sufficiency. Suppose that there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $p_b : \lim \mathbf{X} \to X_b$ is hereditarily irreducible. Consider inverse system $C(\mathbf{X})=\{C(X_a), C(p_{ab}), A\}$ whose limit is *C(X)* (Lemma 1.1). Let $\mu_b : C(X_b) \to [0,\infty)$ be a Whitney map for $C(X_b)$, where $b \in B$ is fixed. We shall prove that $\mu = \mu_b C(p_b) : C(X) \to [0,\infty)$ is a Whitney map for *C(X)*. Let *F*,*G* be a pair of subcontinua of *X* with $F \subseteq G$, $F \neq G$. We must prove that $\mu(\{F\})<\mu(\{G\})$. Now, $p_b(F) \subset p_b(G)$ and $p_b(F) \neq p_b(G)$ since p_b is hereditarily irreducible. We infer that $\mu_b(\{p_b(F)\})<\mu_b(\{p_b(G)\})$ since μ_b is a Whitney map for $C(X_b)$. Moreover, $\{p_b(F)\}=C(p_b)(\{F\})$ and $\{p_b(G)\}=C(p_b)(\{G\})$. From $\mu_b(\{p_b(F)\})<\mu_b(\{p_b(G)\})$ we have $\mu_b(C(p_b)(\{F\}))<\mu_b(C(p_b)(\{G\}))$, i.e., $\mu_b C(p_b)(\{F\})<\mu_b C(p_b)(\{G\})$.
Finally, $\mu(\{F\})<\mu(\{G\})$ since $\mu = \mu_b C(p_b)$.

If *C(X)* is a Lindelöf space and $\mathbf{X}=\{X_a, p_{ab}, A\}$ is σ-directed, then $C(\mathbf{X})=\{C(X_a), C(p_{ab}), A\}$ is factorizing by Lemma 1.2. Thus, we have the following corollary.

Corollary 3.2. *Let X be a non-metric compact space. Then X admits a Whitney map for C(X) if and only if for each* σ*-directed inverse system* $\mathbf{X}=\{X_a, p_{ab}, A\}$ *of compact spaces which admit Whitney maps for* $C(X_a)$ *and* $X = \lim \mathbf{X}$ *there exists a cofinal subset* $B \subset A$ *such that for every* $b \in B$ *the projection* $C(p_b) : C(\lim \mathbf{X}) \to C(X_b)$ *is light.*

Proof. Apply Theorem 3.1 and Lemma 2.3.

From Theorem 3.1 we have the following corollaries.

Corollary 3.3. [9, Theorem 8] *Locally connected compact space X admits a Whitney map for C(X) if and only if it is metrizable.*

Corollary 3.4. [9, Theorem 12] *A rim-metrizable continuum X admits a Whitney map for C(X) if and only if it is metrizable.*

Corollary 3.5. *If a non-metric continuum X admits a Whitney map for C(X), then each locally connected compact subspace (or rim-metrizable subcontinuum) of C(X) is metrizable. In particular, each arc in C(X) is metrizable.*

Proof. This follows from Corollary 3.2.

Example 1. *Let X be a non-metric pseudo-arc as constructed in* [15]. *Then X admits no a Whitney map*. It suffices to see that the pseudo-arc X is constructed as a limit of $X = \{X_a, p_{ab}, \omega_1\}$, where each X_a is a metric pseudo-arc and each p_{ab} is monotone. Moreover, X is non-metric.

The following theorem is from [1, Sledstvie 2, p. 392].

Theorem 3.7. *If* $f : X \to Y$ *is a light mapping of a compact space X onto a metric space Y, then* $\dim X = \operatorname{ind} X = \operatorname{Ind} X$.

From Theorem 3.7 we have the following result.

Theorem 3.8. *Let X be a non-metrizable finite-dimensional compact space. If X admits a Whitney map for C(X), then* $\dim X = \operatorname{ind} X = \operatorname{Ind} X$.

Proof. Let X be compact Hausdorff space such that $w(X) \geq \aleph_1$. There exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric compacta X_a such that X is homeomorphic to $\lim \mathbf{X}$. If X admits a Whitney map for $C(X)$, then there exists a light projection $p_a : X \to X_a$ onto a metric space X_a. Apply Theorem 3.7.

Theorem 3.9. *Let X be a non-metrizable finite-dimensional compact space. If X admits a Whitney map for C(X) and C(X) is finite-dimensional, then* $\dim C(X) = \operatorname{ind} C(X) = \operatorname{Ind} C(X)$.

Corollary 3.6. *Let X be a non-metrizable compact space. If either* $\dim X \neq \operatorname{ind} X \neq \operatorname{Ind} X$, *or* $\operatorname{ind} X \neq \operatorname{Ind} X$, *then X admits no a Whitney map for C(X).*

Example 2. *There exists a chainable continuum X such that* $\dim X = 1$ *and indX* > 1 [11, Theorem 6, p. 225]. *This means X does not admits a Whitney map for C(X).*

Question 1. Is true that a chainable continuum admits a Whitney map if and only if it is metrizable?

4. σ-locally connected spaces

We say that a space X is *σ-locally connected*, provided X is the countable union of its closed locally connected subsets [7, p. 53].

If a metric continuum X is σ-locally connected, then so is its hyperspace C(X) [7]. The converse implication does not hold true.

We say that a subspace Y of a space X is *relatively locally connected* (abr. r.l.c) *in X* [7, p.54] provided for any $p \in Y$ and for any neighborhood U of p in X there is a connected set K contained in U (not necessarily in Y) such that $K \cap Y$ is a neighborhood (not necessarily open) of p in Y.

A space X is said to be *σ-relatively locally connected* (abbr. σ-r.l.c.) provided X is the countable union of its closed r.l.c. subsets.

Theorem 4.1. [7, Theorem 2, p. 56]. *For any metric continuum X the following conditions are equivalent:*

(a): *C(X) is σ-locally connected,*

(b): *X is σ-r.l.c.,*

(c): *C(X) is σ-r.l.c.*

Remark 3. Theorem 4.1. was proved in [7, Theorem 2, p. 56] in metric setting. A straightforward modification of the proof shows that Theorem is valid in non-metric setting.

Theorem 4.2. [7, Corollary 8, p. 59]. *If X is a continuum whose hyperspace C(X) is σ-locally connected and if $f: X \to Y$ is a continuous surjection, then C(Y) is σ-locally connected.*

Example 3. *There exists non-locally connected continuum which is σ-locally connected.* Let X be the union of the well-known $\sin \frac{1}{x}$-curve for $0 < x \leq 1$ and the segment $\{(0, y): -1 \leq y \leq 1\}$. Then X is non-locally connected continuum which is the countable union of locally connected subcontinua of X. One can construct a similar continuum C in the Cartesian product $[0, 1] \times L$, where L is any non-metric arc.

Lemma 4.3. *Let X be a σ-locally connected compact space. If $f: X \to Y$ is a continuous surjection, then Y is σ-locally connected.*

Proof. From the definition of σ-local connectedness it follows that there exists a family $\{X_i : i \in \mathbb{N}\}$ of locally connected closed subspaces of X such that $X = \{X_i : i \in \mathbb{N}\}$. Each $f(X_i)$ is locally connected by Lemma 1.5 of [18, p. 70]. It follows that Y is σ-locally connected since $Y = \cup\{f(X_i): i \in \mathbb{N}\}$.

Lemma 4.4. *Let X be a σ-locally connected compact space. If $f: X \to Y$ is a light surjection, then $w(X) = w(Y)$.*

Proof. It is clear that $w(Y) \leq w(X)$ [4, Theorem 3.1.22, p. 171]. Let us prove that $w(X) \leq w(Y)$. Let us prove that $w(X) \leq w(Y)$. Let $w(Y)=m$. There exists a family $\{X_i : i \in \mathbb{N}\}$ of locally connected closed subspaces of X such that $X = \cup \{X_i : i \in \mathbb{N}\}$. Each restriction $f_i = f|X_i$ is light. This means that $w(X_i) = w(f_i(X_i)) \leq m$, $i \in \mathbb{N}$, because of Theorem 1 of [12]. By virtue of Theorem 3.1.20 of [4, p. 171] if $w(X_i) = w(f_i(X_i)) \leq m$, then $w(X) \leq m$. Hence, $w(X) \leq w(Y)$. Finally, we have $w(X) = w(Y)$.

Corollary 4.5. *Let X be a σ-locally connected compact space. If $f : X \to Y$ is a light surjection onto a metric space Y, then X is metrizable.*

The proof of the following theorem is a straightforward modification of the proof of Theorem 9 of [10, p. 205].

Theorem 4.6. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces and surjective bonding mappings p_{ab}. Then:*

1): *There exists an inverse system $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ of compact spaces such that m_{ab} are monotone surjections and lim$\mathbf{X}$ is homeomorphic to lim$M(\mathbf{X})$,*

2): *If $\mathbf{X}$ is σ-directed, then $M(\mathbf{X})$ is σ-directed,*

3): *If every X_a is a metric space and lim$\mathbf{X}$ is σ-locally connected, then every M_a is metrizable.*

Theorem 4.7. *If X is a non-metric σ-locally connected compact space, then there exists a σ-directed system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric σ-locally connected compact spaces X_a and monotone surjective bonding mappings p_{ab} such that X is homeomorphic to lim $\mathbf{X}$.*

Proof. By Theorem 1.5 there exists a σ-directed system $\mathbf{Y} = \{Y_a, p_{ab}, A\}$ of metric compacta Y_a such that X is homeomorphic to lim$\mathbf{Y}$. Applying Theorem 4.6 we obtain a σ-directed system M($\mathbf{Y}$). If we set $\mathbf{X} = M(\mathbf{Y})$, we obtain the desired σ-directed system.

In the light of the fact that the limit of a σ-directed inverse system of locally connected compact spaces is locally connected, the following question arises.

Question 2. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of compact σ-locally connected spaces X_a. Is lim$\mathbf{X}$ σ-locally connected?

Theorem 4.8. *A σ-locally connected compact space admits a Whitney map for $C(X)$ if and only if X is metrizable.*

Proof. If X is metrizable, then X admits a Whitney map for $C(X)$. Suppose that X is non-metric and admits a Whitney map for $C(X)$. By Theorem 1.5 there exists a σ-directed system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua X_a such that X is homeomorphic to lim $\mathbf{X}$. It follows that $C(X)$ is a limit of $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$. From Theorem 3.1 it follows that there exists a cofinal subset $B \subset A$ such that for every

$b \in B$ the projection $p_b : \lim \mathbf{X} \to X_b$ is hereditarily irreducible, and consequently, light (Lemma 2.1). From Corollary 4.5 it follows that X is metrizable.

Lemma 4.9. *Let $f : X \to Y$ be a hereditarily irreducible mapping of σ-r.l.c. continuum X onto a metric space Y. Then X is metrizable.*

Proof. From Lemma 2.3 it follows that a mapping $f : X \to Y$ is hereditarily irreducible if and only if $C(f) : C(X) \to C(Y)$ is light. Now, *C(Y)* a metric space and *C(X)* is σ-locally connected (Theorem 4.1). By virtue of Corollary 4.5 it follows that *C(X)* is metrizable. Hence, X is metrizable.

Theorem 4.10. *A σ-r.l.c. continuum X admits a Whitney map for C(X) if and only if X is metrizable.*

Proof. If X is metrizable, then X admits a Whitney map for *C(X)*. Suppose that X admits a Whitney map for *C(X)* and that X is non-metrizable. By Theorem 1.5 there exists a σ-directed system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua X_a such that X is homeomorphic to lim $\mathbf{X}$. It follows that *C(X)* is a limit of $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$. From Theorem 3.1 it follows that there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $C(p_b) : C(\lim \mathbf{X}) \to C(X_b)$ is light. Moreover, because of Theorem 4.1 *C(X)* is σ-locally connected continuum. From Corollary 4.5 it follows that *C(X)* is metrizable since $C(p_b)(C(\lim \mathbf{X})) \subset C(X_b)$ is metrizable. Hence, X is metrizable since X is homeomorphic to $X(1)$ and $X(1) \subset C(X)$.

At the end of this section we shall use the following Smirnov's theorem from [4, Exercise 5.4A, p. 415].

Theorem 4.11. *If a paracompact X space is locally metrizable (i.e., every point $x \in X$ has a metrizable neighbourhood), then X is metrizable.*

We say that a space X is *locally σ-locally connected* provided each point $x \in X$ has a neighbourhood U such that *Cl(U)* is σ-locally connected and compact.

Theorem 4.12. *A paracompact locally σ-locally connected space X admits a Whitney map for C(X) if and only if X is metrizable.*

Proof. Let $\{U_x : x \in X\}$ be a family of the neighbourhoods of points $x \in X$ such that $Cl(U_x)$ is σ-locally connected and compact. If $\mu : C(X) \to [0, \infty)$ is a Whitney map for *C(X)*, then the restriction $\mu | C(Cl(U_x))$ is a Whitney map for $C(Cl(U_x))$. By Theorem 4.8 we infer that $Cl(U_x)$ is metrizable. Theorem 4.11 completes the proof.

5. σ-rim-merizable continua

We say that a space X is *σ-rim-merizable* provided X is the countable union of its rim-metrizable subcontinua.

Lemma 5.1. *Let $f: X \to Y$ be a light surjection of compact σ-rim-merizable space X onto a space Y. Then $w(X) = w(Y)$.*

Proof. It is clear that $w(X) \leq w(Y)$ [4, Theorem 3.1.22, p. 171]. Let us prove that $w(X) \leq w(Y)$. Let $w(Y) = m$. There exists a family $\{X_i : i \in \mathbb{N}\}$ of rim-metrizable subcontinua of X such that $X = \cup\{X_i : i \in \mathbb{N}\}$. Each restriction $f_i = f|X_i$ is light. This means that $w(X_i) = w(f_i(X_i)) \leq m, i \in \mathbb{N}$, because of Theorem 1.2 of [16]. By virtue of Theorem 3.1.20 of [4, p. 171] if $w(X_i) = w(f_i(X_i)) \leq m$, then $w(X) \leq m$. Hence, $w(X) \leq w(Y)$. Finally, we have $w(X) = w(Y)$.

Corollary 5.2. *Let $f: X \to Y$ be a light surjection of compact σ-rim-merizable space X onto a metric space Y. Then X is metrizable.*

Theorem 5.3. *A compact σ-rim-merizable space X admits a Whitney map for C(X) if and only if X is metrizable.*

Proof. By Theorem 1.5 there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric compacta X_a such that X is homeomorphic to lim$\mathbf{X}$. There exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $p_b : \lim \mathbf{X} \to X_b$ is hereditarily irreducible (theorem 3.1). Because of Lemma 2.1 each p_b is light. Corollary 5.2 completes the proof.

6. σ-fans

An *arboroid* is an hereditarily unicoherent continuum which is arcwise connected by generalized arcs. A metrizable arboroid is a *dendroid*. If X is an arboroid and $x, y \in X$, then there exists a unique arc $[x,y]$ in X with endpoints x and y. If $[x,y]$ is an arc, then $[x,y] \setminus \{x,y\}$ is denoted by (x,y).

A point t of an arboroid X is said to be a *ramification point* of X if t is the only common point of some three arcs such that it is the only common point of any two, and an end point of each of them.

A point e of an arboroid X is said to be *end point* of X if there exists no arc $[a,b]$ in X such that $x \in [a,b] \setminus \{a,b\}$.

If an arboroid X has only one ramification point t, it is called a *generalized fan* with the top t. A metrizable generalized fan is called a *fan*.

Theorem 6.1. [8, Theorem 4.20]. *A generalized fan X admits a Whitney map for C(X) if and only if it is metrizable.*

In connection of Theorem 6.1 the following question is natural.

Question 3. Is it true that an arboroid X admits a Whitney map for *C(X)* if and only if it is metrizable?

We say that a space X is a σ-fan provided X is the countable union of its subcontinua which are the generalized fans.

Theorem 6.2. *If a space X is a σ-fan, then X admits a Whitney map for C(X) if and only if it is metrizable.*

Proof. By the definition of σ-fans $X = \cup\{X_n : n \in \mathbb{N}\}$, where each X_n is a generalized fan. If $\mu : C(X) \to \mathbb{R}$ is a Whitney map for $C(X)$, then the restriction $\mu|X_n$ is a Whitney map for $C(X_n)$, for every $n \in \mathbb{N}$. By Theorem 6.1 each X_n is metrizable. Hence, X is metrizable because [4, Theorem 3.1.20, p. 171].

7. Concluding remark

There are many metrizabtion theorems in the literature with different hypotheses to ensure metrizability. From this point of view some theorems of the present paper may be regarded and reformulated as the metrization theorems. The typical examples are the following theorems.

Theorem 7.1. *Locally connected compact space X is metrizable if and only if it admits a Whitney map for C(X).*

Proof. See Corollary 3.3.

Theorem 7.2. *A rim-metrizable continuum X is metrizable if and only if it admits a Whitney map for C(X).*

Proof. See Corollary 3.4.

Similarly, from Theorem 6.1 it follows the following theorem.

Theorem 7.3. *A generalized fan X is metrizable if and only if it admits a Whitney map for C(X).*

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Bilješka o prostorima koji dozvoljavaju Whitney-ovo preslikavanje

Ivan Lončar

SAŽETAK

Neka je X nemetrički kontinuum, a $C(X)$ hiperprostor podkontinuuma od X. Poznato je da ne postoji Whitney-ovo preslikavanje na hiperprostoru 2^X na nemetričke kompaktne Hausdorf-ove prostore X. S druge strane, postoje nemetrički kontinuumi koji dozvoljavaju i oni koji ne dozvoljavaju Whitney-ovo preslikaanje za $C(X)$. U radu se izučavaju svojstva nemetričkih prostora X koji dozvoljavaju Whitney-ovo preslikavanje za $C(X)$.

Ključne riječi i fraze: Hiperprostor, Whitney-ovo preslikavanje.

Ivan Lončar,
Fakultet organizacije i informatike Sveučilišta u Zagrebu
42000 Varaždin, Pavlinska 2, Croatia
iloncar@foi.hr