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K-dominating Sets on the Associative and Commutative Products of Two Paths*

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This paper first presents some earlier results on *k*-dominating sets of Cartesian products and cardinal products of two paths, and then new results on *k*-dominating sets of a strong product and an equivalent product of two paths (a *k*-domination number of a strong and an equivalent product of $P_m \times P_n$ for arbitrary m and n).

INTRODUCTION

For any graph G we denote the vertex-set and the edge-set of G by V(G) and E(G), respectively. A subset $D \subset V(G)$ is called a *k*-dominating set, $k \geq 2$, if for every vertex y not in D there exists at least one vertex $x \in D$, such that $d(x, y) \leq k$. For convenience, we also say that D *k*-dominates G. The *k*-domination number $\gamma_k(G)$ is the cardinality of the smallest *k*-dominating set. A 1-domination number is also called a domination number.

On the Cartesian product of the vertices of two graphs we can define many products. First systematical researches of such products of graphs were done by H. Izbicki and W. Imrich.[1] They found out that there are 256 such products, 20 of them being associative. Out of these 20, 10 can be defined if we know the structure of both factors. Futhermore, 8 of them are commutative. These eight products can be split in four pairs of mutual dual products.[2]

Graph X' is a complement of graph X if they have the same set of vertices, and two vertices are adjacent to X' if and only if they are not adjacent to X. Then for some product * we can define dual product $\underline{*}$ such that:

$$X \,\underline{*}\, Y = (X' * Y')'.$$

We can also say that $\underline{*}$ is a complementary product of *. From the aforementioned it follows that there are 4 associative and commutative products and their complementary products.

Since it is often possible to deduce properties of the complementary product from the properties of the original product,[1] we will consider only four associative and commutative products of simple graphs. These 4 products are cardinal, Cartesian, strong and equivalent.[3]

We will denote the cardinal product by $\times$. On the cardinal product $G \times H$ of two graphs G and H, $\{(g_1, h_1), (g_2, h_2)\} \in E(G \times H)$ if and only if $\{g_1, g_2\} \in E(G)$ and $\{h_1, h_2\} \in E(H)$. (see Figure 1. ($P_3 \times P_4$))

* Dedicated to Professor Haruo Hosoya in happy celebration of his 70th birthday.

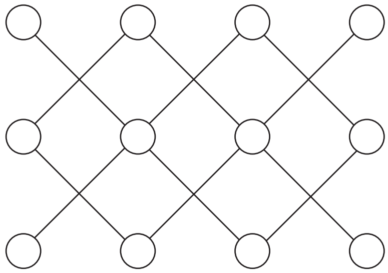

Figure 1.

We will denote the Cartesian product by $\square$. On the Cartesian product G $\square$ H of two graphs G and H, $\{(g_1, h_1), (g_2, h_2)\} \in$ E(G $\square$ H) if and only if $g_1 = g_2$ and $\{h_1, h_2\} \in$ E(H), or $\{g_1, g_2\} \in$ E(G) and $h_1 = h_2$. (see Figure 2. ($P_3 \square P_4$))

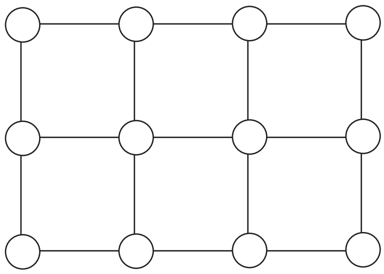

Figure 2.

The strong product will be denoted by $\otimes$. On the strong product G $\otimes$ H of two graphs G and H, $\{(g_1, h_1), (g_2, h_2)\} \in$ E(G $\otimes$ H) if and only if:

1) $\{g_1, g_2\} \in$ E(G) and $\{h_1, h_2\} \in$ E(H), or
2) $g_1 = g_2$ and $\{h_1, h_2\} \in$ E(H), or
3) $\{g_1, g_2\} \in$ E(G) and $h_1 = h_2$. (see Figure 3. ($P_3 \otimes P_4$))

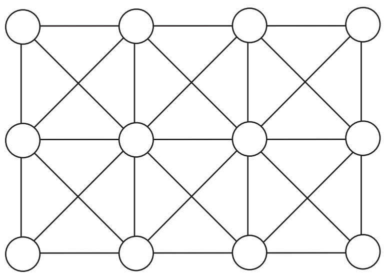

Figure 3.

The equivalent product will be denoted by $\circ$. On the equivalent product G $\circ$ H of two graphs G and H, $\{(g_1, h_1), (g_2, h_2)\} \in$ E(G $\circ$ H) if and only if:

1) $\{g_1, g_2\} \in$ E(G) and $\{h_1, h_2\} \in$ E(H), or
2) $g_1 = g_2$ and $\{h_1, h_2\} \in$ E(H), or
3) $\{g_1, g_2\} \in$ E(G) and $h_1 = h_2$, or
4) $\{g_1, g_2\} \in$ E(G') and $\{h_1, h_2\} \in$ E(H'). (see Figure 4. ($P_3 \circ P_4$))

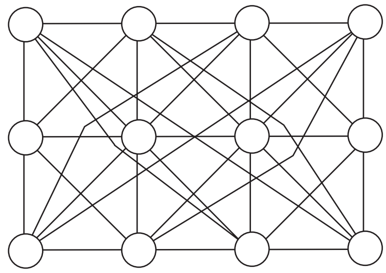

Figure 4.

Notions of a polygraph[4–6] were introduced in chemical graph theory as a formalization of chemical notions of a polymer. A path is a special case of a polygraph called a fascia graph and the Cartesian product of two paths is a fascia-fascia graph, which is of interest in mathematical chemistry.

Also, let maximal matching of graph G be β_1. Then if G is a graph without isolated vertices, with n vertices, it holds:

$$\gamma(G) \leq \min\{\beta_1(G), n - \beta_1(G)\}\ .$$

In the next chapter, we will give some earlier results about k-dominating sets on the cardinal and Cartesian products of two paths.

K-DOMINATING SETS ON THE CARDINAL AND CARTESIAN PRODUCTS OF TWO PATHS

We have shown the following results:[7–8] For the cardinal product of two paths it holds:

Theorem 1. – For $n \geq 3$, $k \geq 2$

$$\gamma_k(P_2 \times P_n) = \ldots = \gamma_k(P_{2k} \times P_n) = 2 \cdot \left\lceil \frac{n}{2k+1} \right\rceil .$$

Let $k \geq 2$, $n \geq 3$. Then

$$\gamma_k(P_{2k+1} \times P_n) = \begin{cases} 2 \cdot \left\lfloor \dfrac{n}{2k} \right\rfloor + 1, & \text{if } n \equiv 1 (\text{mod}\, 2k) \\ 2 \cdot \left\lceil \dfrac{n}{2k} \right\rceil, & \text{otherwise} \end{cases}$$

For $k \geq 1$ and $n \geq 3$,

$$\gamma_k(P_{2k+2} \times P_n) = \begin{cases} 4\cdot\left\lfloor \frac{n}{2k+2} \right\rfloor + 2, & \text{if } n \equiv 1(\text{mod}(2k+2)) \\ 4\cdot\left\lceil \frac{n}{2k+2} \right\rceil, & \text{otherwise} \end{cases}$$

Theorem 2. – For any two paths P_m, P_n,

$$\lim_{m,n\to\infty} \frac{\gamma_k(P_m \times P_n)}{mn} = \frac{1}{\left\lceil \frac{4k^2+4k+1}{2} \right\rceil}.$$

For the Cartesian product the following holds:[9]

Theorem 3. – Let $k \geq 1$. Then

$$\gamma_k(P_2 \,\square\, P_n) = \begin{cases} \frac{n}{2k} + 1, & \text{if } n \equiv 0(\text{mod}\, 2k), \\ \left\lceil \frac{n}{2k} \right\rceil, & \text{otherwise} \end{cases}$$

For every path P_n $n \geq 2$, and $k \geq 2$

$$\gamma_k(P_3 \,\square\, P_n) = \left\lceil \frac{n}{2k-1} \right\rceil.$$

Theorem 4. – For m odd and $k \geq m - 1$

$$\gamma_k(P_m \,\square\, P_n) \leq \left\lceil \frac{n}{2k-m+2} \right\rceil.$$

For m even and $k \geq m - 1$

$$\gamma_k(P_m \,\square\, P_n) \leq \begin{cases} \frac{n}{2k-m+2} + 1, & \text{if } n \equiv 0(\text{mod}(2k-m+2)) \\ \left\lceil \frac{n}{2k-m+2} \right\rceil, & \text{otherwise} \end{cases}$$

Theorem 5. – For any two paths P_m, P_n, m, $n \geq 2$,

$$\lim_{m,n\to\infty} \frac{\gamma_k(P_m \,\square\, P_n)}{mn} = \frac{1}{2k^2+2k+1}.$$

Now it is interesting to consider this problem on the strong and equivalent products of two paths.

K-DOMINATING SETS ON THE STRONG PRODUCT OF TWO PATHS

Observation 1. – For each path P_n, $n \geq 1$ $\gamma_k(P_n) = \left\lceil \frac{n}{2k+1} \right\rceil$.

By $\otimes$ we denote the strong product.

Lemma 1. –

$\gamma_k(P_2 \otimes P_n) = \gamma_k(P_3 \otimes P_n) = \ldots = \gamma_k(P_{2k+1} \otimes P_n) = \left\lceil \frac{n}{2k+1} \right\rceil = \gamma_k(P_n)$

Proof: We will consider the case $P_{2k+1} \otimes P_n$, and for other cases the proof is the same. Let $D = \{(k+1,\, k+1+(2k+1)r \mid r = 0, 1, \ldots, \left\lceil \frac{n}{2k+1} \right\rceil - 1\}$. D is a k-dominating set for $n \equiv (k+1), (k+2), (k+3), \ldots, 0(\text{mod}(2k+1))$.

If $n \equiv 1, 2, \ldots, k(\text{mod}(2k+1))$, then $D = \{(k+1,\, k+1+(2k+1)r \mid r) = 0, 1, \ldots, \left\lceil \frac{n}{2k+1} \right\rceil - 2\} \cup (k+1, n)$. $|D| = \left\lceil \frac{n}{2k+1} \right\rceil$. Minimality follows from the fact that:

$$\left\lceil \frac{n}{2k+1} \right\rceil \geq \gamma_k(P_{2k+1} \otimes P_n) \geq \gamma_k(P_{2k} \otimes P_n) \geq \ldots \geq \gamma_k(P_2 \otimes P_n) \geq \gamma_k(P_n) = \left\lceil \frac{n}{2k+1} \right\rceil. \blacksquare$$

Theorem 6. – Let m, $n \geq 2k + 1$. Then,

$$\gamma_k(P_m \otimes P_n) = \left\lceil \frac{m}{2k+1} \right\rceil \left\lceil \frac{n}{2k+1} \right\rceil = \gamma_k(P_m) \cdot \gamma_k(P_n).$$

Proof: Let $D = \{(k+1+(2k+1)l,\, k+1+(2k+1)r) \mid l = 0, 1, \ldots, \left\lceil \frac{m}{2k+1} \right\rceil - 1,\ r = 0, 1, \ldots, \left\lceil \frac{n}{2k+1} \right\rceil - 1\}$. D is a k-dominating set for $m \equiv (k+1), (k+2), \ldots, 2k, 0(\text{mod}(2k+1))$ and $n \equiv (k+1), (k+2), \ldots, 2k, 0(\text{mod}(2k+1))$.

If $m \equiv (k+1), (k+2), \ldots, 2k, 0(\text{mod}(2k+1))$ and $n \equiv 1, \ldots, k(\text{mod}(2k+1))$,

then $D = \{(k+1+(2k+1)l,\, k+1+(2k+1)r) \mid l = 0, 1, \ldots, \left\lceil \frac{m}{2k+1} \right\rceil - 1,\ r = 0, 1, \ldots, \left\lceil \frac{n}{2k+1} \right\rceil - 2\} \cup \{(k+1+(2k+1)l,\, n) \mid l = 0, 1, \ldots, \left\lceil \frac{m}{2k+1} \right\rceil - 1\}$.

If $m \equiv 1, \ldots, k(\text{mod}(2k+1))$ and $n \equiv (k+1), (k+2), \ldots, 2k, 0(\text{mod}(2k+1))$, then

$D = \{(k+1+(2k+1)l,\ k+1+(2k+1)r \mid l = 0, 1, \ldots, \left\lceil \frac{m}{2k+1} \right\rceil - 2,\ r = 0, 1, \ldots, \left\lceil \frac{n}{2k+1} \right\rceil - 1\} \cup \{(m, (k+1) + (2k+1)r) \mid r = 0, 1, \ldots, \left\lceil \frac{n}{2k+1} \right\rceil - 1\}$.

If $m \equiv 1, \ldots, k(\mathrm{mod}(2k+1))$ and $n \equiv 1, \ldots, k(\mathrm{mod}(2k+1))$, $D = \{(k+1+(2k+1)l,\ k+1+(2k+1)r \mid l = 0, 1, \ldots, \left\lceil \frac{m}{2k+1} \right\rceil - 2,\ r = 0, 1, \ldots, \left\lceil \frac{n}{2k+1} \right\rceil - 2\} \cup \{(m, (k+1) + (2k+1)r) \mid r = 0, 1, \ldots, \left\lceil \frac{n}{2k+1} \right\rceil - 1\} \cup \{(k+1+(2k+1)l, n) \mid l = 0, 1, \ldots, \left\lceil \frac{m}{2k+1} \right\rceil - 2\}$.

D is a k-dominating set on $P_m \otimes P_n$ and it holds $|D| = \left\lceil \frac{m}{2k+1} \right\rceil \left\lceil \frac{n}{2k+1} \right\rceil$.

Proof of minimality:

a) If $m, n \equiv 0(\mathrm{mod}(2k+1))$, it is obvious, because each vertex is k-dominated by exactly one vertex and each dominating vertex dominates $(2k+1) \times (2k+1)$ vertices which are maximal.

b) If $m \equiv 0(\mathrm{mod}(2k+1))$ but $n \equiv h(\mathrm{mod}(2k+1))$, $h \neq 0$, then on $P_m \otimes P_{n-h}$ each vertex is k-dominated by only one vertex, and there are $\left\lceil \frac{m}{2k+1} \right\rceil \left\lfloor \frac{n}{2k+1} \right\rfloor$ vertices.

From the construction of the set D it follows that no vertex on the rest of the graph can be k-dominated by one of the previous k-dominating vertices.

To k-dominate the rest of the graph, which is $P_m \otimes P_h$ $(1 \leq h \leq 2k)$, we need $\left\lceil \frac{m}{2k+1} \right\rceil$ vertices (Lemma 1). Then, on the whole graph we have at least

$$\left\lceil \frac{m}{2k+1} \right\rceil \left(\left\lfloor \frac{n}{2k+1} \right\rfloor + 1 \right) = \left\lceil \frac{m}{2k+1} \right\rceil \left\lceil \frac{n}{2k+1} \right\rceil$$

vertices.

c) If $n \equiv 0(\mathrm{mod}\ 2k+1)$ but $m \equiv h(\mathrm{mod}\ 2k+1)$, $h \neq 0$, the proof is the same as in b).

d) If $m \equiv h(\mathrm{mod}\ 2k+1)$, $h \neq 0$ and $n \equiv z(\mathrm{mod}\ 2k+1)$, $z \neq 0$, then on a $P_{m-h} \otimes P_{n-z}$ we have a perfect k-dominating set (each vertex is k-dominated by exactly one vertex), and it has $\left\lfloor \frac{m}{2k+1} \right\rfloor \left\lfloor \frac{n}{2k+1} \right\rfloor$ vertices. To k-dominate the remaining vertices (on the blocks $P_h \otimes P_n$ and $P_{m-h} \otimes P_z$) we need at least $\left\lceil \frac{n}{2k+1} \right\rceil + \left\lceil \frac{m-h}{2k+1} \right\rceil = \left\lceil \frac{n}{2k+1} \right\rceil + \left\lfloor \frac{m}{2k+1} \right\rfloor$ vertices and then the result follows. ■

K-DOMINATING SETS ON THE EQUIVALENT PRODUCT OF TWO PATHS

Observation 2. – By $\circ$ we denote the equivalent product.

From $P_1 \circ P_n = P_n$, it follows $\gamma_k(P_1 \circ P_n) = \gamma_k(P_n)$.

From $P_2 \circ P_n = P_2 \otimes P_n$, it follows $\gamma_k(P_2 \circ P_n) = \gamma_k(P_2 \otimes P_n) = \gamma_k(P_n)$.

Definition 1. – For a fixed l, $1 \leq l \leq n$, the set $(P_m)_l = \{(i, l) \mid i = 1, \ldots, m\}$ is called a column of $P_m \circ P_n$. For a fixed z, $1 \leq z \leq m$, the set $(P_n)_z = \{(z, j) \mid j = 1, \ldots, n\}$ is called a row of $P_m \circ P_n$.

Theorem 7. –

$$\gamma(P_3 \circ P_n) = \gamma(P_n)\ .$$

Proof: Since on $P_3 \circ P_n$ we have all edges as on $P_3 \otimes P_n$ (and some more), it follows that:

$$\gamma(P_3 \circ P_n) \leq \gamma(P_3 \otimes P_n) = \left\lceil \frac{n}{3} \right\rceil = \gamma(P_n)\ .$$

Let us prove that $\left\lceil \frac{n}{3} \right\rceil \leq \gamma(P_3 \circ P_n)$. Let D be some dominating set and vertex $(i, j) \in D$, $i \in \{1, 2, 3\}$, $j \in \{1, 2, \ldots, n\}$.

If $i = 2$, this vertex dominates only all vertices on the $(j-1)$-th, the j-th and the $(j+1)$-th column on $P_3 \circ P_n$, and we have the same situation as on the strong product.

If we take that if $(i, j) \in D$, then $i = 2$, it follows (from the strong product) that D has at least $\left\lceil \frac{n}{3} \right\rceil$ vertices.

Let $i = 1$ (it is the same for $i = 3$). Vertex $(1, j)$ dominates $(1, j-1)$, $(1, j)$, $(1, j+1)$, $(2, j-1)$, $(2, j)$, $(2, j+1) \cup (3, 1), \ldots (3, j-2), (3, j+2), \ldots, (3, n)$. (If $j = 1$, then there do not exist vertices $(1,j-1),(2,j-1),(3,j-1),\ldots$ but everything else is the same.)

To dominate the remaining vertices from the 3-rd row, we can take some vertices from the the 2-nd row (but these vertices can dominate only 9 vertices and we again have the situation as on the strong product), or we can take some vertices from the 1-st and the 3-rd row.

The best case is if $(3, j) \in D$, because then all vertices from the 1-st and the 3-rd row and 3 vertices from the 2-nd row are dominated.

To dominate $n-3$ vertices from the 2-nd row (on which we have the same situation as on the strong product), we need at least $\left\lceil \frac{n-3}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil - 1$ vertices.

All together, D has then at least $\left\lceil \frac{n}{3} \right\rceil + 1$ vertices, and is not minimal. ■

Theorem 8. –

$\gamma(P_m \circ P_n) = \min\{\gamma(P_m), \gamma(P_n)\}$; $m, n \geq 4$.

Proof: Without loss of generality let $\gamma(P_m) < \gamma(P_n)$.

If $m \equiv 2, 0 (\mathrm{mod}(3))$, then we can take $D = \{(2 + 3l, 2) \mid l = 0, 1, ..., \left\lceil \frac{m}{3} \right\rceil - 1\}$. If $m \equiv 1 (\mathrm{mod}(3))$, then, $D = \{(2 + 3l, 2) \mid l = 0, 1, ..., \left\lceil \frac{m}{3} \right\rceil - 2\} \cup (m, 2)$.

If $m = 4$, then, $D = \{(1, 2), (4, 2)\}$. It holds:

$$|D| = \left\lceil \frac{m}{3} \right\rceil = \gamma(P_m)$$

and D is a dominating set on $P_m \circ P_n$.

Proof of minimality: Let $(i, j) \in D$. Then it dominates vertices $(i-1, j-1)$, $(i-1, j)$, $(i-1, j+1)$, $(i, j-1)$, (i, j), $(i, j+1)$, $(i+1, j-1)$, $(i+1, j)$, $(i+1, j+1)$, and $\{(k, l) : |k - i| > 1, |l - j| > 1\}$. To dominate the remaining vertices on the $(i - 1)$-th, the i-th and the $(i+1)$-th row (like $P_3 \circ P_n$) and the $(j-1)$-th, the j-th and the $(j+1)$-th column (like $P_3 \circ P_m$), we need $\min\{\gamma(P_m), \gamma(P_n)\} - 1$ vertices. ■

Theorem 9. –

$\gamma_2(P_m \circ P_n) = 1$; $m, n \geq 3$.

Proof: Let D be a 2-dominating set and $(i, j) \in D$. As in the previous theorem, only vertices from the $(i-1)$-th, the i-th and the $(i+1)$-th row and the $(j-1)$-th, the j-th and the $(j+1)$-th column, which are at a distance >1 from (i,j), are not 1-dominated by (i,j). All other vertices are already dominated. But now we have 2-domination.

Let $3 \leq i \leq m - 2$ and $3 \leq j \leq n - 2$. It is easy to see that vertices from the $(i-1)$-th, and the $(i+1)$-th row and the $(j-1)$-th and the $(j+1)$-th column are 2-dominated.

Also, for all vertices from the i-th row and the j-th column there exists at least one vertex (k,l) from the first or the last row which is adjacent to them, and which is at a distance one from (i,j). ($|k - i| > 1$, $|l - j| > 1$). Then (i,j) 2-dominate all vertices on $P_m \circ P_n$ for $m,n \geq 3$. For $i \in \{1, 2, m - 1, m\}$ and $j \in \{1, 2, n - 1, n\}$, the proof is the same. ■

Theorem 10. – Let m, $n \geq 3$ and $k \geq 2$. Then it holds:

$$\gamma_k(P_m \circ P_n) = 1 .$$

Proof: From the previous theorem. ■

CONCLUSIONS

K-dominating sets on the Cartesian product, cardinal product, strong product and equivalent product of two paths are given. A path is a special case of a polygraph called a fascia graph and the Cartesian product of two paths is a fascia-fascia graph. Hence, the methods of this paper could be generalized to polygraphs.

Also, from the domination number we can say something about the lower bound on maximal matching. Products of more than two paths can be considered, but this was not done. There is only one article about the dominating Cartesian product of cycles[10] where the product of more than two graphs is considered.

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SAŽETAK

K-dominacijski skupovi na asocijativnim i komutativnim produktima dvaju putova

Antoaneta Klobučar

U članku su prvo dani raniji rezultati o *k*-dominacijskim skupovima na kartezijevom i kardinalnom produktu dvaju putova, a potom novi rezultati o *k*-dominacijskim skupovima na jakom i ekvivalentnom produktu dvaju putova (za slucaj kada su putovi proizvoljno veliki).