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Mathematical Meaning and Importance of the Topological Index *Z*

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The role of the topological index, Z_G, proposed by the present author in 1971, in various problems and topics in elementary mathematics is introduced, namely, (i) Pascal's and asymmetrical Pascal's triangle, (ii) Fibonacci, Lucas, and Pell numbers, (iii) Pell equation, (iv) Pythagorean, Heronian, and Eisenstein triangles. It is shown that all the algebras in these problems can be easily obtained, graph-theoretically interpreted, and systematically related with each other by introducing certain series of graphs whose Z_G values represent the series of numbers involved therein. Finally, an ambitious conjecture is proposed: for any recursive relation of the type of Fibonacci numbers, there always exist a series of graphs whose *Z*-indices obey the same recursive relation. Important role of Z_G in algebraic number theory is also discussed.

INTRODUCTION

My first graph-theoretical paper "Topological Index. A Newly Proposed Quantity Characterizing the Topological Nature of Saturated Hydrocarbons" was published in 1971.[1] Since then I have been working in mathematical chemistry for more than thirty years. If the essence of all my papers is to be abstracted, one may say that many chemical problems and phenomena can be described and analyzed mathematically, and then interpreted chemically by using proper counting polynomials.[2] Although the "topological index" was coined by myself for the *Z*-index, this term has now become the general name for topological descriptors.[3]

Good correlation is observed between the boiling point of alkanes and *Z*, which is the sum of the non-adjacent numbers, $p(\mathrm{G},k)$s, of graph G representing their carbon atom skeleton.[1,4–7] For tree graphs those $p(\mathrm{G},k)$ values are identical to the absolute values of the coefficients of the characteristic polynomial of G, while for non-tree graphs some correction terms become necessary. Then one can systematically analyze the graph-theoretical meaning of *Z* and then the chemical meaning of the observed correlation. Later the *Z* index was found to be utilized for analyzing the graph theoretical meaning of the aromatic stability of conjugated hydrocarbons,[8,9] and also for the classification and coding of hydrocarbons.[10]

Possibly due to its mathematically naive definition, the $p(\mathrm{G},k)$ and *Z* values of several series of typical graphs were found to be closely related to several series of numbers and the related mathematical objects. For example, the *Z*'s of path and monocycle graphs are nothing else but the Fibonacci and Lucas numbers, respectively, which have the same recursion formula, as $f_n = f_{n-1} + f_{n-2}$.[1] Moreover, the whole families of $p(\mathrm{G},k)$ numbers of these two series of graphs form Pascal's and asymmetrical Pascal's triangles, respectively.[11,12] This means that by using the *Z*-index one can obtain new graph-theoretical meaning of the well known algebraic concepts and theorems. Quite recently various series of graphs were discovered whose *Z*-indices are closely related to fundamental concepts and theorems in elementary mathematics as follows.

1) Pell numbers, 1, 2, 5, 12, 29, *etc*., which recur as $f_n = 2 f_{n-1} + f_{n-2}$.
2) Solutions of Pell equation, $x^2 - D y^2 = N$, for special values of D and N.
3) Pythagorean triangles (a, b, c) composed of all integers with $a^2 + b^2 = c^2$.
4) Heronian triangles (a, b, c) composed of all integers with integral area.

A very important role of the Z index in elementary mathematics has thus been verified. The purpose of this paper is to expose and relate the important mathematical roles of the Z-index, and finally to propose an ambitious conjecture connecting algebra and geometry through Z.

PRELIMINARIES

Topological Index and Related Polynomials

In this section various numbers and polynomials relevant to the discussion on the topological index, Z, will briefly be introduced. Graph-theoretical concepts used here are those which have conventionally been approved.[13]

Non-adjacent Number, p(G,k).[1] – The number of ways for choosing k disjoint edges from a given graph G is defined as the non-adjacent number, $p(\mathrm{G},k)$. Here $p(\mathrm{G},0)$ is defined as unity for any G including vacant graph, and $p(\mathrm{G},1)$ is equal to the number of edges of G.

Z-*Counting Polynomial*, Q_G(x).[1] – By using the set of $p(\mathrm{G},k)$'s for G the Z-counting polynomial, $\mathrm{Q_G}(x)$, is defined as

$$\mathrm{Q_G}(x) = \sum_{k=0}^{m} p(\mathrm{G},k)\,x^k \tag{1.1}$$

where m is the maximum number of $k = \lfloor N/2 \rfloor$ with N being the number of vertices of G.

Matching Polynomial, M_G(x).[14–16] – By using $p(\mathrm{G},k)$'s for a given G three groups of researchers independently proposed the same polynomial in the following form, which is now called matching polynomial,

$$\mathrm{M_G}(x) = \sum_{k=0}^{m} (-1)^k\, p(\mathrm{G},k)\,x^{N-2k} \;. \tag{1.2}$$

Mathematically it is essentially the same as $\mathrm{Q_G}(x)$.

Topological Index, Z_G.[1] – Total sum of $p(\mathrm{G},k)$'s for a given G is defined as the topological index, Z_G, as

$$Z_\mathrm{G} = \sum_{k=0}^{m} p(\mathrm{G},k) = \mathrm{Q_G}(1)\,. \tag{1.3}$$

Characteristic Polynomial, P_G(x). – By using the adjacency matrix, $\boldsymbol{A}$, and unit matrix, $\boldsymbol{E}$, of the same size the characteristic polynomial for G with N vertices is defined as

$$\mathrm{P_G}(x) = (-1)^N \det(\boldsymbol{A} - x\,\boldsymbol{E})\;. \tag{1.4}$$

For tree graphs the coefficients of $\mathrm{P_G}(x)$ exactly coincide with the set of $p(\mathrm{G},k)$'s as[1]

$$\mathrm{P_T}(x) = \sum_{k=0}^{N} a_k x^{N-k} = \sum_{k=0}^{m} (-1)^k\, p(\mathrm{T},k)\,x^{N-2k} \quad (\mathrm{T:\ tree}) \tag{1.5}$$

and then we have

$$Z_\mathrm{T} = \sum_{k=0}^{m} |\,a_k| \quad (\mathrm{T:\ tree})\;. \tag{1.6}$$

For non-tree graphs $\mathrm{P_G}(x)$ can be expressed by the set of $p(\mathrm{G},k)$'s for G and its subgraphs obtained by deleting the component rings. Details have already been discussed elsewhere.[17,18]

Chebyshev Polynomial, T_n(x) *and U_n*(x).[19,20] – Although trigonometric definitions are more common, Chebyshev polynomials of the first and second kinds, $\mathrm{T}_n(x)$, $\mathrm{U}_n(x)$, are respectively defined here as

$$\mathrm{T}_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{2k} x^{n-2k}\,(x^2-1)^k \tag{1.7}$$

$$\mathrm{U}_n(x) = \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} x^{n-2k}\,(x^2-1)^k \;. \tag{1.8}$$

Modified Chebyshev Polynomials, S_n(x) *and C_n*(x).[19–21] – By using the following formulas modified Chebyshev polynomials, $\mathrm{S}_n(x)$ and $\mathrm{C}_n(x)$, are defined as

$$\mathrm{C}_n(x) = 2\,\mathrm{T}_n(x/2) \tag{1.9}$$

$$\mathrm{S}_n(x) = \mathrm{U}_n(x/2), \tag{1.10}$$

or explicitly expressed by

$$\mathrm{C}_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k} x^{n-2k} \tag{1.11}$$

$$\mathrm{S}_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} x^{n-2k} \;. \tag{1.12}$$

These two polynomials have been found to be identical to the matching polynomials of monocyclic graph, C_n, and path graph, S_n, respectively.[21] It is interesting to note that the notations of C and S for these modified Chebyshev polynomials come from cosine and sine, respectively, and the coincidence with the conventional notations for cycle and path graphs is quite accidental.

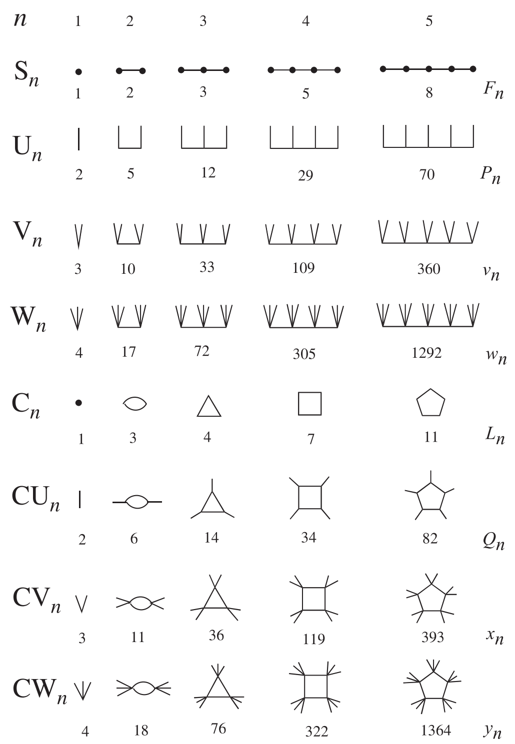


Figure 1. Various series of graphs derived from path (S_n) and monocycle (C_n) graphs together with their topological indices, Z_G.

Various Series of Numbers

Fibonacci Number, F_n.[22] – Contrary to the conventional choice of the initial values, the Fibonacci numbers, F_n, are defined here as

$$F_n = F_{n-1} + F_{n-2} \tag{1.13}$$

with $F_0 = F_1 = 1$, and one can see that F_n represents the Z-index of path graph S_n as shown in Figure 1.

Lucas number, L_n.[22] – The definition of the Lucas Numbers, L_n, is the same as the conventional one as

$$L_n = L_{n-1} + L_{n-2} \tag{1.14}$$

with $L_0 = 2$ and $L_1 = 1$, and L_n is shown to represent the Z-index of cycle graph C_n as shown in Figure 1.

Pell Number, P_n.[23,24] – Contrary to the conventional choice of the initial values, the Pell numbers, P_n, are defined here as

$$P_n = 2P_{n-1} + P_{n-2} \tag{1.15}$$

with $P_0 = 1$ and $P_1 = 2$. It is shown in Figure 1 that P_n represents the Z-index of comb graph, U_n.[25] The change in the initial values of F_n and P_n from the conventional definition is linked with each other.

Pell-Lucas Number or Companion Pell Number, Q_n.[23,24,26] – The definition of the Lucas numbers, L_n, is the same as the conventional one as

$$Q_n = 2Q_{n-1} + Q_{n-2} \tag{1.16}$$

with $Q_0 = 2$ and $Q_1 = 2$. It is shown that Q_n represents the Z-index of gear graph, CU_n, as seen in Figure 1.

Examples of all these characteristic quantities explained above are given below for path graph S_5 which is composed of five vertices and four consecutive edges as shown in Figure 1, where several typical series of graphs and their Z_G values are given:

for $G = S_5$:

$p(G, 0) = 1, \quad p(G, 1) = 4, \quad p(G, 2) = 3$

and $\quad p(G, k) = 0 \quad (k \geq 3)$

$Q_G(x) = 1 + 4x + 3x^2 \qquad Z_G = 1 + 4 + 3 = 8 = F_5$

$M_G(x) = x^5 - 4x^3 + 3x = S_5(x)$

$P_G(x) = x^5 - 4x^3 + 3x.$

Similarly,

for $G = C_5$:

$p(G, 0) = 1, \quad p(G, 1) = 5, \quad p(G, 2) = 5$

and $\quad p(G, k) = 0 \quad (k \geq 3)$

$Q_G(x) = 1 + 5x + 5x^2 \qquad Z_G = 1 + 5 + 5 = 11 = L_5$.

$M_G(x) = x^5 - 5x^3 + 5x = C_5(x)$

$P_G(x) = x^5 - 5x^3 + 5x - 2$.

See also the Pascal and asymmetrical Pascal triangles[12] given in Figures 2a and 2b, where the coefficients

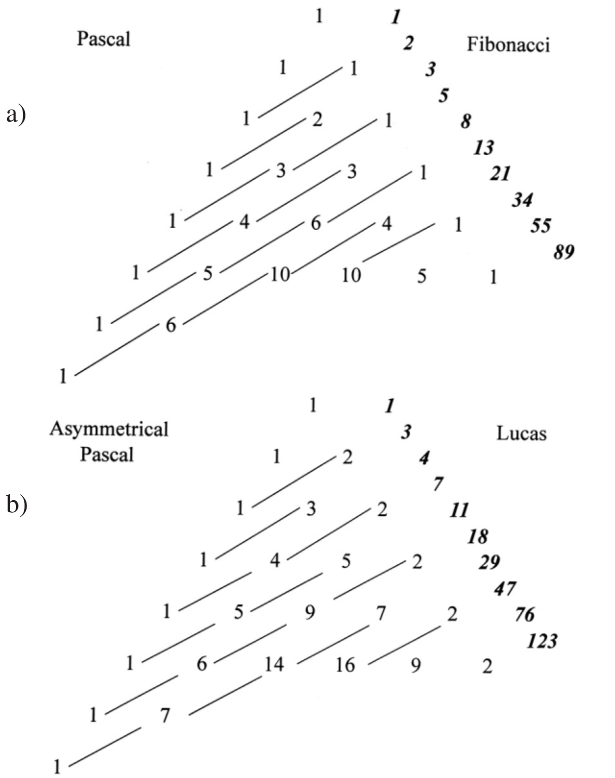


Figure 2. Pascal's (a) and asymmetrical Pascal's (b) triangles. Along the slant lines from the left end to the upper right $p(G,k)$ numbers for path and monocycle graphs can be read, and their sums give the Fibonacci and Lucas numbers, respectively.

of the $Q_G(x)$ for path graphs and monocycle graphs can be read along the slant lines from left to upper right. Note also that the absolute values of the coefficients of the modified Chebyshev polynomials of the second and first kinds, S_n, (1.12), and C_n, (1.11), are respectively, identical to the above two polynomials.

Recursive Relations[1,2,11,21]

Useful recursive relations for $p(G,k)$, $Q_G(x)$, and Z_G have been known, respectively, as

$$p(G,k) = p(G - l,k) + p(G\Theta l,k - 1) \qquad (1.17)$$

$$Q_G(x) = Q_{G-l}(x) + x\, Q_{G\Theta l}(x) \qquad (1.18)$$

$$Z_G = Z_{G-l} + Z_{G\Theta l} \qquad (1.19)$$

where $G\Theta l$ is the subgraph of G obtained by deleting edge l from G together with all the edges incident to l, while $G–l$ is the subgraph of G obtained by deleting only edge l. These recursive relations are derived from the inclusion-exclusion principle.[27,28] As many examples have already been illustrated elsewhere, no further explanation will be given here except for the following case.

Namely, as shown in Figure 1, Lucas numbers can be obtained from Fibonacci numbers as in the following way,

$$L_n = F_n + F_{n-2}, \qquad (1.20)$$

This is also the case with the pair of Pell and Pell--Lucas numbers as

$$Q_n = P_n + P_{n-2}, \qquad (1.21)$$

and also for other pairs of the series of numbers as

$$x_n = v_n + v_{n-2}, \qquad (1.22)$$

and

$$y_n = w_n + w_{n-2}. \qquad (1.23)$$

The first terms of the right hand sides of (1.20)–(1.23) are the contribution of $G–l$, while the second terms count the $G\Theta l$ contribution.

Each of the four recursive relations, (1.20)–(1.23), respectively, connects a pair of graphs or numbers whose recursive relations are the same. In some sense these relations describe the ring closure property along a »latitude.« On the other hand, another group of recursive relations can be obtained along a "longitude." Namely, the fundamental skeletons of the three tree graphs (See Figure 1), U_n, V_n, and W_n, with the same n are in common with that of S_n, but they differ only in the number (a) of unit edges branching from each vertex of the common skeleton of S_n leading to a different recursive relation for each stage (latitude) as

$$f_n = (a+1)\, f_{n-1} + f_{n-2}\ . \qquad (1.24)$$

Similar relations also hold for another group of non--tree graphs, C_n, CU_n, CV_n, and CW_n.

Let us here consider the process of growth of path graph, S_n, into the corresponding comb graph, U_n. By taking into account the meaning and function of the counting polynomial U_5 can be obtained from that of S_5, with $Q_G(x) = 1 + 4x + 3x^2$, in the following way:

$$U_5 = (1 + x)^5 + 4x\,(1 + x)^3 + 3x^2\,(1 + x) = \\ 1 + 9\,x + 25\,x^2 + 25\,x^3 + 9\,x^4 + x^5\ .$$

Similarly $Q_G(x)$ for V_5 and W_5 can be obtained from that of S_5 as

$$V_5 = (1 + 2x)^5 + 4x\,(1 + 2x)^3 + 3x^2\,(1 + 2x) = \\ 1 + 14\,x + 67\,x^2 + 134\,x^3 + 112\,x^4 + 32\,x^5\ ,$$

and

$$W_5 = (1 + 3x)^5 + 4x\,(1 + 3x)^3 + 3x^2\,(1 + 3x) = \\ 1 + 19\,x + 129\,x^2 + 387\,x^3 + 513\,x^4 + 243\,x^5.$$

By putting $x = 1$ into these two polynomials one gets $v_5 = 360$ and $w_5 = 1292$, respectively, for their Z-indices (See Figure 1).

The results obtained here assert that all the numbers from F_n to w_n and from L_n to y_n with the same n in Figure 1, respectively, represent the Z-indices of those graphs whose fundamental skeletons are tree S_n and non-tree C_n. The shift of the initial conditions for the definition of F_n and P_n are thus justified.

Various Series of Graphs

Although several important series of graphs have been introduced in Figure 1, we will add several more others and their Z-indices which are closely related to the problems discussed in this paper.

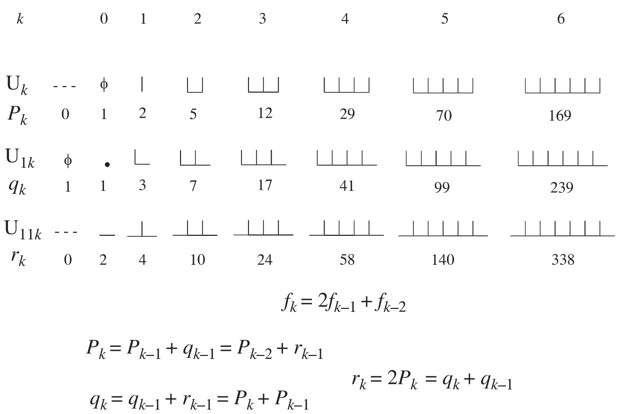
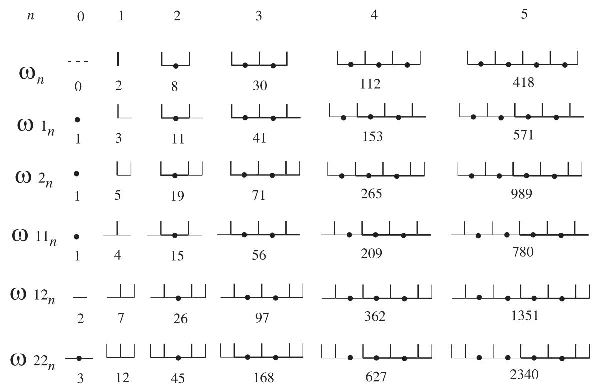


Figure 3. Comb graph and the related graphs, and their Z-indices.



$$f_n = 4f_{n-1} - f_{n-2}$$

Figure 4. ω-shaped graph and the related graphs, and their *Z*-indices.

Comb-type graphs, $U1_n$ and $U11_n$, with their Z-indices, q_n and r_n (See Figure 3).

Wide Comb graph or ω-shaped graph ω_n and ω-related graphs, $\omega 1_n$, $\omega 2_n$, $\omega 11_n$, $\omega 12_n$ and $\omega 22_n$ (See Figure 4).

ROLE OF Z-INDEX

Fibonacci, Lucas and Pell Numbers, and Pascal's Triangle

It has already been pointed out in my earlier paper[11] that the set of $p(G,k)$'s for the series of path graphs exactly correspond to the Pascal's triangle as illustrated in Figure 2a. That the set of $p(G,k)$'s for the series of monocycle graphs correspond to the asymmetridal Pascal's triangle was rather later pointed out in my paper dealing with 4-dimensional atomic orbitals[12] as shown in Figure 2b. Since both these numbers obey the recursive relation of the following type as (1.13) and (1.14),

$$f_n = f_{n-1} + f_{n-2}, \tag{2.1}$$

the well-known recursive relation applied to the Pascal's triangle can be explained diagrammatically.

On the other hand, as the Pell and Pell-Lucas numbers situate in different stage or latitude, they obey a different recursive relation as

$$f_n = 2\, f_{n-1} + f_{n-2}. \tag{2.2}$$

See Table I, where the set of $p(G,k)$ numbers, or the coefficients of $Q_G(x)$, of the two series of graphs, comb graph, U_n, and gear graph, or a cyclic comb graph, CU_n, (See Figure 1) are arranged to form a big triangle, respectively. Notice that the pattern of the numbers is symmetrical with respect to the bisecting line of each equilateral triangle. The reason for this symmetrical nature of these polynomials can be explained by the process of generating U_5 polynomial from S_5 as already shown in the preceding section.

TABLE I. Coefficients of the *Z*-counting polynomials of graphs, U_n and CU_n, generating Pell and Pell-Lucas numbers

n		P_n
0	1	1
1	1 1	2
2	1 3 1	5
3	1 5 5 1	12
4	1 7 13 7 1	29
5	1 9 25 25 9 1	70
6	1 11 41 63 41 11 1	169
7	1 13 61 129 129 61 13 1	408
8	1 15 85 231 321 231 85 15 1	985

n		Q_n
0	2	2
1	1 1	2
2	1 4 1	6
3	1 6 6 1	14
4	1 8 16 8 1	34
5	1 10 30 30 10 1	82
6	1 12 48 76 48 12 1	198
7	1 14 70 154 154 70 14 1	478
8	1 16 96 272 384 272 96 16 1	1154

Another feature of these two triangles in Table I is in their recursive construction rule. That is an element in each triangle is the sum of the triplet forming a small triangle just above it. This is an outcome of the recursive relation of $Q_G(x)$ as

$$Q_{U_n}(x) = (1+x)\, Q_{U_{n-1}}(x) + Q_{U_{n-2}}(x). \tag{2.3}$$

One can extend this discussion to other pairs of graphs, (V, CV) and (W, CW), in Figure 1, which situate on higher stages, but this problem will be left for the interested readers.

Aside from these interesting algebraic properties the most important message emanating from Table I and Figure 1 is that graph-theoretical meaning of the Pell and Pell-Lucas numbers was clarified and their close relationship with Fibonacci and Lucas numbers was found.

Pell Equation

Irrespective of the long history of the Pell equation, let us here call (2.4) and (2.5), respectively, Pell-*N* and Llep-*N*,[29]

$$x^2 - D\, y^2 = N \qquad \text{(Pell-}N\text{)} \tag{2.4}$$

$$x^2 - D\, y^2 = -N, \qquad \text{(Lepp-}N\text{)} \tag{2.5}$$

where positive integer solutions (x, y) are to be sought for positive integers N and square-free D.[23,29,30] When both Pell-*N* and Llep-*N* are to be discussed simultaneously, one may use another new terminology, Pellep-*N*. For any value

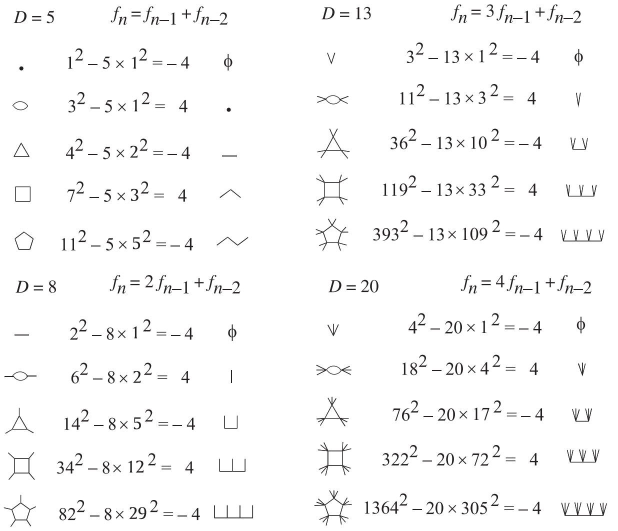


Figure 5. The solutions of Pellep-4 with D=5, 8, 13, and 20. The series of graphs whose Z-indices correspond to these solutions are shown.

of D Pell-1 has an infinite number of solutions besides the trivial solution (x_0 = 1, y_0 = 0), whereas Llep-1 has solutions only for special values of D.[23,29,30] This is also the case with Pellep-N, if N is a square number. Although a number of interesting features have been known, it is to be remarked here that Pellep-4 has a key role in the whole Pellep-N problem, and the solutions of Pellep-4 for special series of D values have important graph-theoretical meaning especially related to the problems discussed in this paper.

See Figure 5 where the solutions (x, y) of Pellep-4 with four different D values are given together with the graphs whose Z-indices correspond to these solutions. Note that all the series of graphs given in Figure 1 are found here. As mentioned before all the tree and non-tree graphs in Figure 1 were constructed, respectively, from S_n and C_n graphs by joining the same number ($a \geq 0$) of unit edges to each vertex. As shown in Figure 5 the solutions (x, y) of Pellep-4 with $D = a^2 + 2a + 5$ correspond, respectively, to the Z-indices of these graphs.

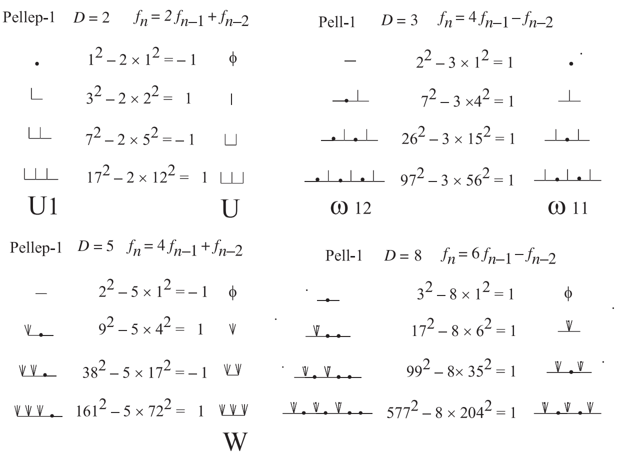


Figure 6. The solutions of Pellep-1 and Pell-1 with selected values of D. The series of graphs whose Z-indices correspond to these solutions are shown.

Similarly one can find graph-theoretical interpretation of the solutions of Pellep-1 through the Z-indices of other series of graphs introduced in Figure 3 and 4. Namely, in Figure 6 the Z-indices of two pairs of series of graphs, U and U1 in Figure 3 and ω11 and ω12 in Figure 4 are found to be the solutions of Pellep-1 with D = 2 and Pell-1 with D = 3. The series of graphs, W, in Figure 1 are also found to be the solutions of Pellep-1 with D = 5 as in Figure 6, where other interesting series of graphs are also shown.

In this way the whole mathematical structure of the solutions of Pellep-N is beginning to be clarified by assigning proper series of graphs whose Z-indices exactly represent these solutions. Study along this line is being in progress.

Pythagorean Triangle[31]

A Pythagorean triangle is a triangle whose edges (a, b, c) with c as the hypotenuse are all integers, and these triplet edges are sometimes called Pythagorean triplet. If (a, b, c) are prime with each other, the triangle is called primitive. It has long been known that any Pythagorean triplet can be expressed by a pair of positive integers (m>n) as

$$m^2 - n^2, \ \ 2\,m\,n\ , \ \ m^2 + n^2, \qquad (*)$$

which satisfy the following identity,

$$(m^2 - n^2)^2 + (2\,m\,n\,)^2 = (m^2 + n^2)^2. \qquad (2.6)$$

By using this relation various series of Pythagorean triangles have been known for more than two thousand years. Although Pythagorean triangle is a geometrical object, its discussion has been performed mainly in algebraic manner, such as on the series of numbers and recursive relations among them. In this paper it will be shown that by using the topological index, Z_G, a positive integer representing the topological structure of a simple graph, G, all these algebras can be easily obtained, graph-theoretically interpreted, and systematically related with each other. Further, algebras of Heronian and Eisenstein triangles can comprehensively be discussed and understood through Z_G.

Relation of Pythagorean triplet and the famous series of numbers, *e.g.*, Fibonacci, Lucas, and Pell numbers, have been discussed but only sporadically. These relations will be shown to be systematically related with each other through Z_G.

(a, a±*1*, c*) or Root-2-Triangle.* – First consider a series of triangles, (a, a±1, c) converging to the equilateral right triangle. The problem is reduced to solve the Pell

TABLE II. Pythagorean triangles approaching euilateral right triangle

k	1	2	3	4	5	6
$m = P_k$	2	5	12	29	70	169
$n = P_{k-1}$	1	2	5	12	29	70
$a = m^2 - n^2$	3	21	119	697	4059	23661
$b = 2\ m\ n$	4	20	120	696	4060	23660
$c = m^2 + n^2$	5	29	169	985	5741	33461
$2\ c\ /\ (a + b)$	1.42	1.4146	1.41422	1.4142139	1.41421357	1.414213563

$a = P_k^2 - P_{k-1}^2 = q_k q_{k-1}$ $\quad b = 2P_k\, P_{k-1}$ $\quad c = P_k^2 + P_{k-1}^2 = P_{2k}$

equation, $x^2 - 2y^2 = -1$, and it is straightforwardly to get the values of (m, n) and (a, b, c) as in Table II.[23]

The arguments m and n are nothing else but the Pell numbers, P_k and P_{k-1}, respectively, while c takes every other Pell number, P_{2k}.[23] Further, b is the product of the pair of consecutive Pell number. That is $b = 2P_kP_{k-1}$. Although the value of a is obtained from the difference of the squares of consecutive pair of P_k's, it can also be expressed by the product $a = q_k\, q_{k-1}$ of the consecutive pair of another series of numbers, q_k, representing the Z-index of graph U1$_k$, which has already been introduced in Figure 3. By noticing the relation,

$$q_k = P_k + P_{k-1}, \tag{2.7}$$

one gets

$$q_{k-1} = P_{k-1} + P_{k-2} = P_k - P_{k-1} \tag{2.8}$$

from the recursive relation (1.15). Then a can also be expressed by the difference of the squares of the pair of consecutive P_k's.

Anyway it was shown here that the Pythagorean triplets converging to equilateral triangle can be constructed from the Z-indices of the comb and related-graphs introduced in Figure 3, where another series of comb-related graphs and numbers, r_k, are also shown. P_k, q_k, and r_k, are closely interrelated with each other through their recursive relations as given in Figure 3, but have the same recursive relation (2.2) within each series. The rapid convergence of the ratio $2c/(a+b)$ with such small digits of numbers is also to be remarked here.

However, by trial-and-error method interesting triplet series of graphs were found as shown in Figure 7. First consider the three tree graphs whose Z-indices represent the smallest Pythagorean triangle (3, 4, 5), and then sandwich it with a pair of L-shaped graphs as the (5, 2) graph in Figure 7. The Z-indices of the resultant triplet graphs will be the next larger member (21, 20, 29) of this family of triplets. Although the recursive relation for a and b is different from that of c as given in Figure 7, they are closely related with each other through the following identity,

$$(x^2 - 6\,x + 1)\,(x + 1) = x^3 - 5\,x^2 - 5\,x + 1\ . \tag{2.9}$$

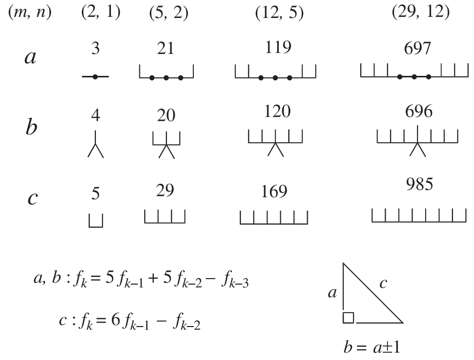


Figure 7. The series of graphs whose Z-indices correspond to the edge lengths of Pythagorean triangles of $(a, a\pm1, c)$ type

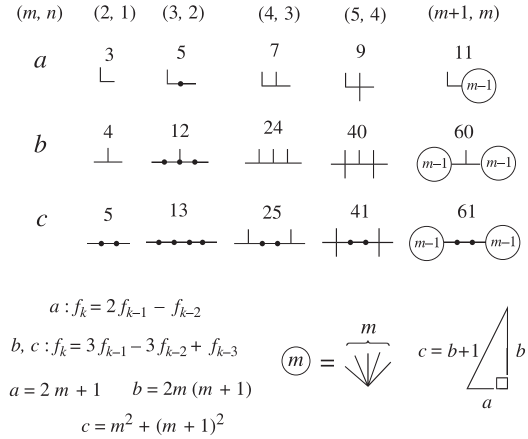


Figure 8. The series of graphs whose Z-indices correspond to the edge lengths of Pythagorean triangles of $(a, b, b+1)$ type.

Although one can derive algebraic closed forms representing the three edges of this type of Pythagorean triangle, manipulation of graphs with simple recursive relations as worked out in Figure 7 is mathematically more enjoyable and productive for suggesting many hints for the extension of the theory.

*(*a, b, b + *1) Triangle.* – The series of triangles of the form $(a, b, b + 1)$ converge to another equilateral but flattened triangle. From this condition it is easy to derive the following series of numbers forming long acute Pythagorean triangles as shown in Figure 8,[23] where three series of graphs whose Z-indices corresponding to these triangle are given. In this case the maximum length of the graphs is fixed and the number of branches are increasing to infinity.

*(*a, 2a ± *1,* c*) or Root-5 Triangle.* – From the condition of

$$a^2 + (2a \pm 1)^2 = c^2 \tag{2.10}$$

the problem is reduced to solve the following type of Pell equation,

$$x^2 - 5\,y^2 = -1. \tag{2.11}$$

After a little manipulation one gets the following formula

$$(F_{k-1}\,F_{k+2})^2 + (2\,F_k\,F_{k+1})^2 = F_{2k+2}, \tag{2.12}$$[22,32]

which is expressed by only Fibonacci numbers.

A similar result is obtained by using only Lucas numbers,[22,33]

$$(L_{k-1}\,L_{k+2})^2 + (2\,L_k\,L_{k+1})^2 = (L_{2k} + L_{2k+2})^2. \tag{2.13}$$

Since F_n and L_n stand path and monocycle graphs, respectively, both (2.12) and (2.13) represent some graph-theoretical meaning of Pythagorean triangle whose edge ratio c/a converges to square root of 5. However, we are not going into more detail here, because the results are not new and the rate of convergence is rather low.

*(*a, b*, 2*a *± 1) or Root-3 Triangle*. – From the condition of

$$a^2 + b^2 = (2a \pm 1)^2 \tag{2.14}$$

the problem is reduced to solve the following type of Pell equation,

$$x^2 - 3\,y^2 = 1. \tag{2.15}$$

Then after a little manipulation the series of Pythagorean triangle whose edge ratio b_k/c_k converges to square root of 3 are obtained as in Table III. However, it was found that these triangles are alternately grouped into two series with even and odd k. Namely, the length of longer leg b can be expressed by one general formula, whereas the shorter leg a and hypotenuse c are, respectively, are found to be a union of two series of numbers as given in Table III. Further, all these series of numbers can be expressed by the Z-indices of several series of »ω-shaped graphs« introduced in Figure 4. Although the recursive relation for a and c is different from that of b, they are related through the following identity,

$$x^3 - 3\,x^2 - 3\,x + 1 = (x^2 - 4\,x + 1)\,(x + 1)\,. \tag{2.16}$$

One can find and discuss as many types of Z-index interpretation of Pythagorean triangles as one may wish, but here only one example will be added without further explanation as shown in Figure 9, where Pythagorean triangles with $c = a + 2$ are given. In Figure 10 it is shown how the Z-indices of several typical graphs introduced in this paper have a key role in relating and understanding the algebras of Pythagorean triangles of various types. One can extend this diagram as large as one may wish.

TABLE III. Pythagorean triangles whose edge ratio converges to square root of 3

k	1	2	3	4	5	6	7	8
a_k	3	8	33	120	451	1680	6273	23408
b_k	4	15	56	209	780	2911	10864	40545
c_k	5	17	65	241	901	3361	12545	46817
b_k/c_k n	1.3	1.8	1.69	1.741	1.729	1.73273	1.73186	1.73210

$a_{2k-1} = \omega_{1k}\,\omega_{1k-1}$ $b_k = \omega_{11k}$ $c_{2k-1} = \omega_{11k-1^2} + \omega_{12k-1^2}$

$a_{2k} = \omega_{0k}\,\omega_{11k}$ $c_{2k} = \omega_{11k^2} + \omega_{11k-1^2}$ (k≥1)

$a\,,\,c : f_k = 3\,f_{k-1} + 3\,f_{k-2} - f_{k-3}$ $b : f_k = 4\,f_{k-1} - f_{k-2}$

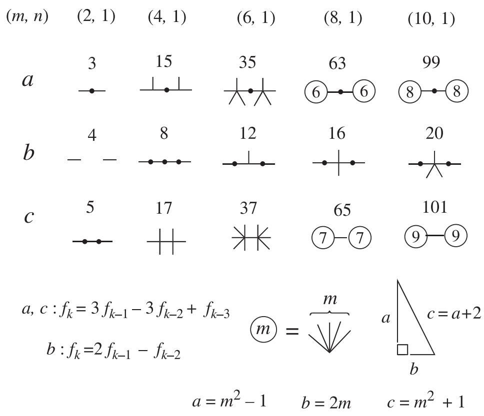


Figure 9. The series of graphs whose Z-indices correspond to the edge lengths of Pythagorean triangles of (a, b, a+2) type.

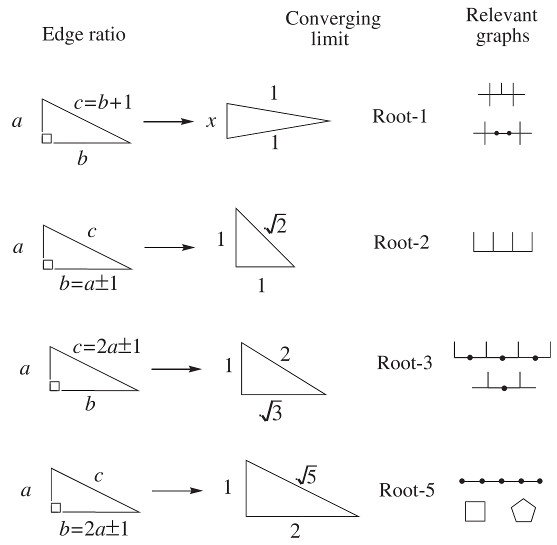


Figure 10. Mathematically interesting series of Pythagorean triangles and their relevant graphs.

Heronian Triangle

A Heronian triangle is a triangle whose edges (a, b, c) and area are all integers. Any Pythagorean triangle is Heronian, and one can construct a Heronian triangle from a pair of Pythagorean triangles which have a leg of common length. Then search for Heronian triangles is usually performed by excluding those cases. Contrary to the case with Pythagorean triangles no universal condition for Heronian triangles is known.

By putting

$$s = (a + b + c)/2 \tag{2.17}$$

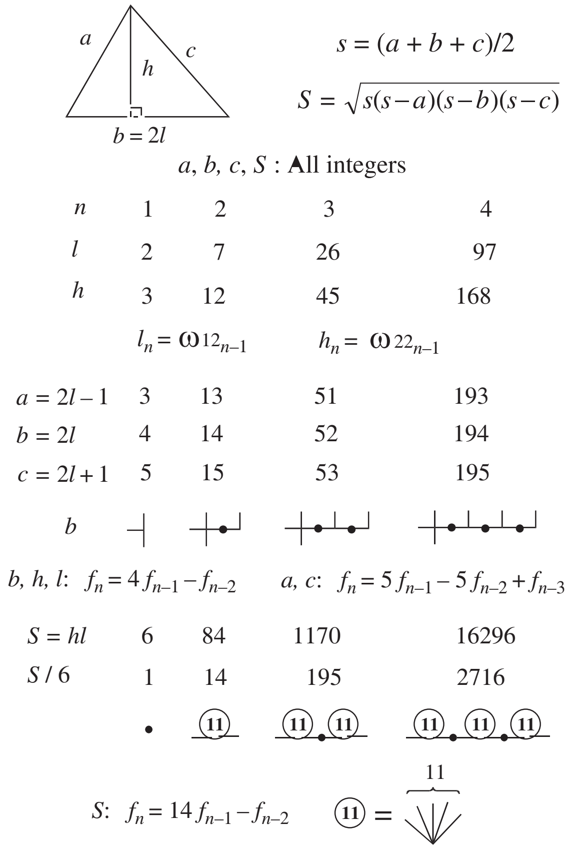


Figure 11. The series of graphs whose Z-indices correspond to the edge lengths of Heronian triangles of (a–1, a, a+1) type.

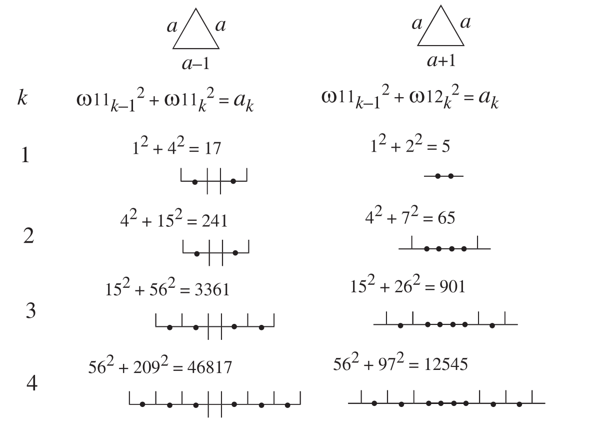


Figure 12. The series of graphs whose Z-indices correspond to the edge lengths of Heronian triangles of (a, a±1, a) type.

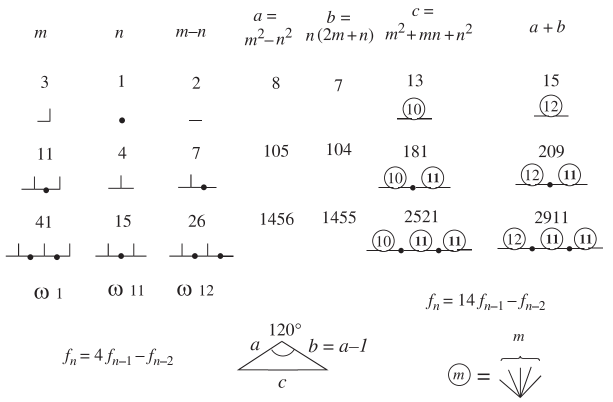


Figure 13. The series of graphs whose Z-indices correspond to the edge lengths of Eisenstein triangles of (a, a–1, c) type.

the area S of the triangle is given by the Heronian formula,

$$S = \sqrt{s(s-a)(s-b)(s-c)}\,. \tag{2.18}$$

The most famous case is for those Heronian triangles whose edges are consecutive integers, (b – 1, b, b + 1). This problem is reduced to solve the Pell equation of the form,

$$x^2 - 3\,y^2 = 4, \tag{2.19}$$

and the results with the height h and area S are obtained as in Figure 11.

Although these numerical results have been known,[23] it is to be noted that $l = b/2$ and h are, respectively, our ω12 and ω22 introduced in Figure 4. Further, one can find two series of graphs whose Z-indices are, respectively, $b = 2l$ and $S/6$ as given in Figure 11.

These two series of graphs, ω12 and ω22, are found to be involved in another series of equilateral Heronian triangles also converging to regular triangle. Namely, the condition for the type of triangles (a, a, a±1) also to solving (2.19), and one gets the results as summarized in Figure 12, where two series of a_k are found to give this type of triangle as

$$a_k = \omega 11_{k-1}^{\,2} + \omega 11_k^{\,2} \quad \text{for } (a,\ a,\ a-1) \tag{2.20}$$

and

$$a_k = \omega 11_{k-1}^{\,2} + \omega 12_k^{\,2} \quad \text{for } (a,\ a,\ a+1) \tag{2.21}$$

two series of graphs whose Z-indices represent the edge lengths of these Heronian triangles also belong to the ω-related graphs with the same recursive relation,

$$f_n = 4\,f_{n-1} - f_{n-2}. \tag{2.22}$$

After a number of trials for finding graph-theoretical features of Heronian triangles of various types, one can conclude that all these algebras can be represented and interpreted by the aid of Z-indices for proper series of relevant graphs.

Eisenstein Triangle[34]

This terminology does not seem to be widely approved in mathematical community, but one can discuss interesting properties of integral triangles with one angle being

60 or 120 degree. The smallest examples are (5, 8, 7) and (3, 5, 7), respectively, for the former and latter ones.

Here we will seek those Eisenstein triangles (a, a – 1, c) which converge to equilateral 120° triangle. The problem is reduced to solve the Pell equation,

$$x^2 - 3\ y^2 = 1. \tag{2.23}$$

Details of the analysis will not be given here, but Figure 13 illustrates and shows all the recipes and graph-theoretical aspects of the relevant numbers. Although the values are not given in Figure 13, the ratio $2c/(a + b)$ is converging to square root of 3.

AMBITIOUS CONJECTURE

$Z(G_m) = Z(G_{m-1}) + Z(G_{m-2})$

In this paper a number of examples have been shown to demonstrate how the Z-index is not only helpful for understanding various concepts and theorems in elementary mathematics and number theory but also indispensable for connecting algebra and geometry (*via* graph theory) visually. The essence of this statement stems from numerous observations that when a simple recursive relation is obtained in any set of numbers, there are found corresponding series of graphs whose Z-indices obey the same or closely related recursive rule.

Then the present author dare to propose an ambitious conjecture as follows.

Conjecture. – Given a pair of positive integers, $n_1 < n_2$, which are prime with each other, there exist a series of graphs, $\{G_n\}$, so that their Z-indices have such a property that

$$Z_{G1} = n_1,\ Z_{G2} = n_2, \quad \text{and}$$

$$Z_{Gm} = Z_{Gm-1} + Z_{Gm-2} \quad (m \geq 3). \tag{3.1}$$

However, there should be added a supplementary remark. Namely, a series of star graphs fulfilling (3.1) always exist for any pair of positive integers. A star graph $K_{1,n}$ is composed of a center vertex and n edges of a unit length (each with a terminal vertex) emanating from the center. Since its Z-value is $n + 1$, for any positive integer one can prepare a star graph whose Z-value is equal to it. However, they are a kind of trivial solutions from which no new mathematically meaningful information nor practically useful consequence comes out, and thus they will be excluded from our discussion.

Instead of trying to find the general proof of the above conjecture a few examples will be given here. Namely, we can show that for the case with $n_1 = 5$ a series of graphs can be prepared for any n_2 larger than and prime to 5. See Figure 14, where the series of graphs for n_2 = 6~9, 11~14, and 16~19 are given. By following this systematic algorithm one can conclude that at least for the case with $n_1 = 5$ the conjecture is true. No violation has been found for this type of checking at least up to $n_1 = 65$.

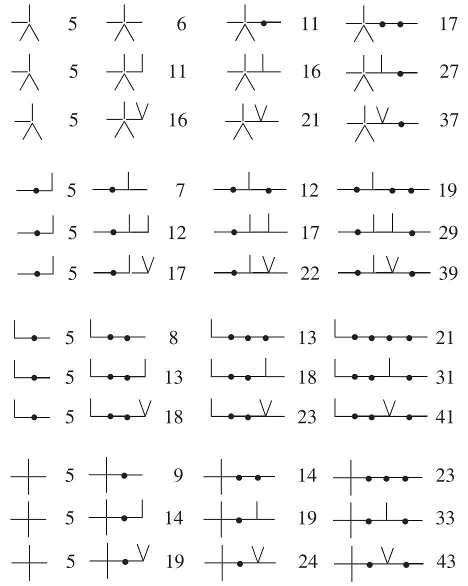


Figure 14. Diagram supporting the Conjecture for the case with n_1=5.

Although in this paper several series of non-tree graphs are shown to support this conjecture, all of them were found to be substituted by another series of tree graphs. Then the term "graphs" in the above conjecture may be substituted by "tree graphs."

If the above conjecture is true, all the algebraic theories derived from the Fibonacci-type recursive relation (1.13) can be interpreted graph-theoretically or geometrically.

Perspective

Although at present the general proof has not yet been obtained, supported by numerous calculations the present author believes that the above conjecture can further be extended to the case with

$$F_n = a\ F_{n-1} + F_{n-2} \quad (a \in \text{positive integer}). \tag{3.2}$$

This means that not only in mathematics but also in sophisticated algebraic number theory, new interpretation or global understanding will be gained by the use of topological index, Z_G. Study for paving the way to this goal is being in progress.

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SAŽETAK

Matematički smisao i značaj topološkog *Z* indeksa

Haruo Hosoya

Autor je 1971. god. predložio topološki indeks Z_G za koji se ovdje predlaže njegova primjena na razne probleme i teme iz elementarne matematike: (i) Pascalov i asimetrični Pascalov trokut, (ii) Fibonaccijevi, Lucasovi i Pellovi brojevi, (iii) Pellova jednadžba, (iv) Pitagorin, Heronov i Eisensteinov trokut. Pokazano je kako se algebra u ovim problemima može lako provesti i graf-teorijski interpretirati, te problemi međusobno povezati ako se promatraju određene klase grafova čije Z_G vrijednosti odgovaraju brojevima u problemima (i)-(iv). Konačno se predlaže smiona hipoteza da za svaku rekurzivnu relaciju Fibonaccijevog tipa uvijek postoji klasa grafova čiji indeksi Z zadovoljavaju iste rekurzivne relacije. Raspravlja se i o važnosti indeksa Z_G u algebarskoj teoriji brojeva.