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Approximation of Euclidean k-size cycle cover problem

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Abstract. For a fixed natural number k, a problem of k collaborating salesmen servicing the same set of cities (nodes of a given graph) is studied. We call this problem the Minimum-weight k-size cycle cover problem (or Min-k-SCCP) due to the fact that the problem has the following mathematical statement. Let a complete weighted digraph (with loops) be given; it is required to find a minimum-weight cover of the graph by k vertex-disjoint cycles. The problem is a simple generalization of the well-known Traveling Salesman Problem (TSP). We show that Min-k-SCCP is strongly NP-hard in the general case. Metric and Euclidean special cases of the problem are intractable as well. We also prove that the Metric Min-k-SCCP belongs to the APX class and has a 2-approximation polynomial-time algorithm. For the Euclidean Min-2-SCCP in the plane, we present a polynomial-time approximation scheme extending the famous result obtained by S. Arora for the Euclidean TSP. Actually, for any fixed $c > 1$, the scheme finds a $(1 + 1/c)$-approximate solution of the Euclidean Min-2-SCCP in $O(n^3(\log n)^{O(c)})$ time.

Key words: vertex-disjoint cycle cover, Traveling Salesman Problem (TSP), NP-hard problem, polynomial-time approximation scheme (PTAS).

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1. Introduction

We consider the following combinatorial optimization problem, which is closely related to the well-known Traveling Salesman (TSP) and Vehicle Routing (VRP) Problems. For a fixed natural number k and a given complete weighted digraph (with loops) $G = (V, E, w)$, it is required to find a minimum-weight cover of the set V by k vertex-disjoint cycles.

Let C be an arbitrary directed cycle in the digraph G. We will denote sets of its nodes and arcs by $V(C)$ and $E(C)$, respectively.

Arcs of G are weighted by some weighting (cost) function $w : E \to \mathbb{R}$. Since the set E is finite, the weight function w is defined by the matrix

$$W = (w_{ij}), \quad (1 \leqslant i, j \leqslant n)$$

uniquely, so that the weight $w(e)$ of any arc $e = (i, j)$ is determined by the formula $w(e) = w_{i,j}$. The weight (cost) of a cycle C is defined by equation $W(C) = \sum_{e\in E(C)} w(e)$.

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Definition 1. *Let $C_1, \ldots, C_k$ be vertex-disjoint simple directed cycles in the graph G such that $V(C_1) \cup \ldots \cup V(C_k) = V$. The family $\mathcal{C} = \{C_1, ..., C_k\}$ is called a k-size cycle cover (k-SCC) of the graph G. The weight $\mathcal{W}(\mathcal{C})$ of the cover $\mathcal{C}$ is defined by equality $\mathcal{W}(\mathcal{C}) = \sum_{i=1}^{k} W(C_i)$.*

Minimum-weight k-size cycle cover problem (Min-k-SCCP). For a given complete weighted digraph $G = (V, E, w)$ with loops, it is required to find the minimum-weight k-size cycle cover (of the graph G).

The Min-k-SCCP can be stated in the optimization form

$$\begin{aligned} \mathcal{W}(\mathcal{C}) = \min\ & \textstyle\sum_{i=1}^{k} \sum_{e \in E(C_i)} w(e) \\ \text{s.t.}\ & \\ & \bigcup_{i=1}^{k} V(C_i) = V, \\ & V(C_i) \cap V(C_j) = \varnothing, \quad (\{i, j\} \subset \{1, \ldots, k\} = \mathbb{N}_k). \end{aligned}$$

We consider two special cases of the Min-k-SCCP, which are called Metric and Euclidean minimum-weight k-size cycle cover problems, respectively.

In the Metric Min-k-SCCP, a weight function w satisfies the following constraints: (i) $w_{ij} \geq 0$, (ii) $w_{ij} = w_{ji}$, (iii) $w_{ii} = 0$, and (iv) $w_{il} + w_{lj} \geq w_{ij}$ for any $1 \leq i, j, l \leq n$. Further, we denote by $\bar{G}$ the undirected graph induced by G. We will use $\bar{G}$ since the weight of an arbitrary cycle cover does not depend on orientation of the constituent cycles.

The Euclidean Min-k-SCCP is just a subclass of the Metric Min-k-SCCP, in which nodes of the given graph G are points in d-dimensional space (for some $d > 1$), and edge weights are Euclidean distances between the incident nodes.

2. Related work

Min-k-SCCP is a natural generalization of the well-known Traveling Salesman Problem (TSP) [1] which deals with finding the minimum-cost Hamiltonian cycle (salesman tour) in a given complete weighted graph.

It is known [2] that the TSP is NP-hard even in the Euclidean case, i.e., the optimal solution can not be found in polynomial time, unless P = NP. Although the TSP is hardly approximable [3] in the general case, for some special cases polynomial-time approximation algorithms are developed. For instance, the Metric TSP [4] can be approximated in polynomial time with a ratio of 3/2, and, for Euclidean TSP, a polynomial-time approximation scheme [5] and an asymptotically correct [6] algorithm are developed.

Other well-known generalizations of the TSP are Min-L-UCC and m-PSP. Instances of these problems are given by undirected complete graphs with non-negative edge weights. In the Min-L-UCC problem, edge weights satisfy the triangle inequality. It is required to find the cycle cover in which the length of every cycle belongs to the set $L \subseteq \mathcal{U} = \{3, 4, 5, ...\}$. Here the length of a cycle is the number of its edges. The Min-L-UCC problem is NP-hard and APX-hard for almost all sets L [7]. In paper [9], it is shown that the Min-L-UCC problem can be approximated within a factor of 4 for $L = \{k, k+1, ...\}$ and $L = \{k\}$. In m-PSP, the goal is to find m edge-disjoint Hamiltonian cycles $H_1, ..., H_m$, so as to minimize or maximize the total weight of the cycles. For Max-2-PSP, there exists the asymptotically optimal algorithm [6].

We show that the Min-k-SCCP problem is strongly NP-hard in the general case and remains intractable in metric and Euclidean special cases. For Metric Min-k-SCCP we propose a 2-approximation polynomial-time algorithm. For Euclidean Min-2-SCCP in the plane we present a polynomial-time approximation scheme, extending the result of S. Arora [5] obtained for Euclidean TSP.

3. Computational complexity of the Min-k-SCCP

This section contains the intractability results for both the general case of Min-k-SCCP and the metric and Euclidean special cases of the problem in question.

Theorem 1. *Min-k-SCCP, Metric Min-k-SCCP and Euclidean Min-k-SCCP (in d-dimensional space for some $d > 1$) are strongly NP-hard for any fixed $k \geqslant 1$.*

Theorem 1 is proved in [8]. The main idea of the proof is very popular in the computation complexity theory. The TSP is reduced to the Min-k-SCCP by cloning the TSP instance and spreading the clones apart. Further, it is shown that any optimal solution of the obtained instance of the Min-k-SCCP consists of optimal solutions (Hamiltonian cycles of the minimum weight) of the initial TSP instance.

Adapting the technique developed in [3] to the case of k-size cycle covers, it is easy to show that the Min-k-SCCP has no polynomial approximation algorithms with any ratio of $O(2^n)$. In subsequent sections we show that the Metric Min-k-SCCP and the Euclidean Min-k-SCCP can be approximated with much higher accuracy.

4. 2-approximation algorithm for the Metric Min-k-SCCP

We propose the approximation algorithm (Algorithm 1), which extends the scheme of the 2-approximation algorithm [4] for the Metric TSP based on preliminary construction of a minimum spanning tree (for the given graph). Hereinafter we use standard notations APP and OPT for the weight of k-size cycle cover found by the approximation algorithm and the optimal k-size cycle cover, respectively. Also, we assume w.l.o.g. that $n > k$.

We summarize properties of the proposed Algorithm 1 in Assertion 1.

Assertion 1. *Algorithm 1 has a running-time of $O(n^2 \log n)$ and an approximation ratio APP/OPT satisfying (in the worst case) the following inequality*

$$\frac{APP}{OPT} \leq 2(1 - 1/n). \tag{1}$$

Proof. In Algorithm 1, the running time of Step 1 is upper-bounded by the running time of the Kruskal's algorithm, which is $O(|E| \log |E|)$ [11]. The running time of

Algorithm 1 Polynomial-time 2-approximation algorithm

1: For the given graph G, construct a k-trees minimum spanning forest F applying the simple modification of the Kruscal's minimum spanning tree construction algorithm [10];
2: Take all edges in F twice to transform all nonempty trees of F into Eulerian subgraphs;
3: For any obtained Eulerian subgraph, find an Eulerian cycle; after that, transform all of them into Hamiltonian cycles (using the standard procedure);
4: Output the constructed set of cycles augmented by the necessary number of one-node routes.

Steps 2 and 3 is $O(|E|)$. Hence, the running time of Algorithm 1 is upper-bounded by $O(n^2 \log n)$ for a given complete graph G.

To prove inequality (1), we consider an arbitrary minimum-weight k-size cycle cover $\mathcal{C}$ of the graph G. Since $k < n$, $\mathcal{C}$ contains at least one nonempty cycle[‡]. We transform $\mathcal{C}$ into k-spanning forest F by removing the most heavy edge from any nonempty cycle. Further, we denote by SF and by MSF the weights of F and the k-minimum spanning forest, respectively. Then,

$$MSF \leqslant SF \leqslant OPT(1 - 1/n),$$

consequently,

$$APP \leqslant 2MSF \leqslant 2(1 - 1/n)OPT,$$

and

$$\frac{APP}{OPT} \leq 2(1 - 1/n).$$

□

It should be noted that approximation ratio bounds (1) of Algorithm 1 are independent of k. Therefore, Algorithm 1 can be considered as a 2-approximation algorithm for the case of the Min-k-SCCP for which parameter k is a part of an instance.

5. Polynomial-time approximation scheme

It is generally believed that a combinatorial optimization problem has a polynomial-time approximation scheme (PTAS) if, for any fixed $c > 1$, there exists an algorithm, finding a $(1 + 1/c)$-approximate solution of the problem in time bounded by some polynomial of the instance length. Generally speaking, the order and coefficients of this polynomial can be dependent on c.

Further, we present a polynomial-time approximation scheme for the Euclidean Min-2-SCCP in the plane.

[‡]We call a cycle C empty if C is a loop.

5.1. Preliminary preprocessing of the problem instance

We start from the claim that, for any instance of the Euclidean Min-2-SCCP in the plane, one of the following alternatives is valid; their verification can be conducted in polynomial time: (i) the instance in question is decomposable into a pair of independent smaller Euclidean TSP instances; (ii) maximum inter-node distance (diameter) of the instance has an upper bound, which depends linearly on OPT.

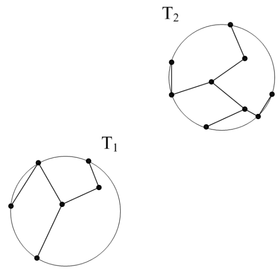

Figure 1: *Minimum spanning forest* $\{T_1, T_2\}$ *and circumscribed circles*

Our considerations are based on the well-known geometric Jung's inequality [12, 13], establishing the dependence between the diameter D of a bounded set in d-dimensional space and the radius R of its circumscribed circle:

$$\frac{1}{2}D \leqslant R \leqslant \left(\frac{d}{2d+2}\right)^{\frac{1}{2}} D.$$

Consequently, in the plane we have:

$$\frac{1}{2}D \leqslant R \leqslant \frac{\sqrt{3}}{3}D. \tag{2}$$

For the instance given by the graph $\bar{G}$ (Figure 1), we construct a 2-minimum spanning forest (2-MSF) $F = \{T_1, T_2\}$. We denote by MSF the weight of the forest F, by D_1 and D_2 the diameters of the trees T_1 and T_2. Also, we denote by R_1, R_2 and by $\rho(T_1, T_2)$ radii of circumscribed circles and the distance between their centers, respectively. Let, further, $D = \max\{D_1, D_2\}$ and $R = \max\{R_1, R_2\}$.

Assertion 2. *Suppose for the given graph* $\bar{G}$ *the following lower bound*

$$\rho(T_1, T_2) > 5R \tag{3}$$

is valid. Then, an arbitrary minimum-weight 2*-size cycle cover of the graph consists of minimum-weight Hamiltonian cycles for subgraphs* $G(T_1)$ *and* $G(T_2)$ *induced by trees* T_1 *and* T_2*. Otherwise, internode distances of the given instance can be upper-bounded by*

$$\frac{7\sqrt{3}}{3}MSF \leqslant \frac{7\sqrt{3}}{3}OPT. \tag{4}$$

Proof. Let inequality (3) be valid for the given instance. Suppose $\mathcal{C} = \{C_1, C_2\}$ is a minimum-weight 2-size cycle cover of the graph G. We assume, by contradiction, that some cycle contains nodes from both trees T_1 and T_2. W.l.o.g., we suppose that the following equations

$$C_1 \cap T_1 \neq \varnothing,\ C_1 \cap T_2 \neq \varnothing,\ C_2 \cap T_2 \neq \varnothing,\ \text{and } C_2 \cap T_1 = \varnothing$$

are valid[§].

By assumption, the cycle C_1 contains at least two edges e_1 and e_2, spanning T_1 and T_2. By condition, the weights $w(e_1)$ and $w(e_2)$ are greater than $3R$, simultaneously. Let us remove these edges and close the cycles inside the circumscribed circles (for T_1 and T_2). To make such a transformation, we should add three new edges and remove one. Summarizing, the total weight of the removed edges is greater than $6R$ and the total weight of the added edges is at most $6R$. Therefore, we construct a lighter 2-size cycle cover than $\mathcal{C}$ that contradicts the optimality of $\mathcal{C}$.

Suppose the given instance violates (3), then the distance between any two nodes in the graph G is at most $7R$. due to the triangle inequality. Applying the right-hand side of (2) and taking into account the straightforward bound $D \leqslant MSF \leqslant OPT$, we have

$$R \leqslant \frac{\sqrt{3}}{3} D \leqslant \frac{\sqrt{3}}{3} OPT.$$

Therefore, the maximum internode distance of the graph G is upper-bounded by (4). □

As it follows from Assertion 2, for any instance of the Euclidean Min-2-SCCP satisfying condition (3), PTAS can be composed of PTASs for two smaller instances of the Euclidean TSP defined by subgraphs $G(T_1)$ and $G(T_2)$.

Hereinafter we consider the special case of the Euclidean Min-2-SCCP, which violates (3).

5.2. Well-rounded problem

To approximate the Euclidean Min-2-SCCP in the plane, it is sufficient to have an efficient approximation algorithm for the special case of the problem in question, which is called a well-rounded Min-2-SCCP. In particular, PTAS for the well-rounded Min-2-SCCP induces PTAS for the general case with the same bound on the running-time.

Definition 2. *An instance of the Euclidean Min-2-SCCP in the plane is called well-rounded if the following conditions are valid: (i) any node i of the input graph $\bar{G}$ has integral coordinates $x_i, y_i \in \mathbb{N}^0_{O(n)}$; (ii) for each edge $e \in E$, $w(e) \geqslant 4$.*

We use the following Lemma 1, which is proved in [8].

Lemma 1. *A PTAS for a well-rounded Min-2-SCCP induces PTAS for the Euclidean Min-2-SCCP.*

§Other cases can be considered similarly.

Further, we construct a geometric partition of the Min-2-SCCP, following the general approach from [5]. For an instance of the Euclidean Min-2-SCCP given by a graph $\bar{G}$, the smallest axis-aligned square $\mathcal{S}$ containing all nodes of $\bar{G}$, such that side-length L of this square is some power of two is called *bounding box*.

We construct a dissection of $\mathcal{S}$ into smaller squares using vertical and horizontal lines. These lines cross the coordinate axes in integer-coordinate points with Step 1. By construction, every smallest-size square contains at most one node of the given instance (Figure 2).

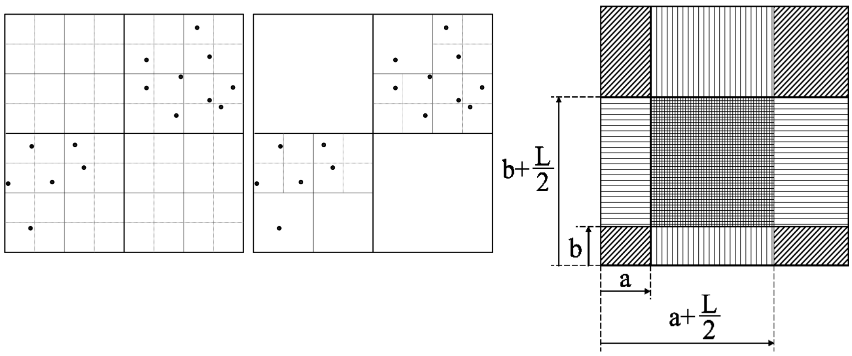

Figure 2: *Dissection of a bounding box, a quadtree, and a shifted quadtree*

Further, we proceed with using a special kind of a 4-regular tree known as *a quadtree* [14]. In our case, the root of the tree is the bounding box $\mathcal{S}$. Each non-leaf square in the tree is partitioned into four equal child sub-squares. This recursive partitioning stops on a square, containing at most one node.

We define levels of the squares of the constructed quadtree as follows. The bounding box $\mathcal{S}$ is the unique square of the first level, its four children belong to the second level, and so on. By construction, the quadtree contains $O(n)$ leaves, $O(\log L) = O(\log n)$ levels and thus $O(n \log n)$ squares in all.

We refer to the point $(L/2, L/2)$ as the centre point. $(L/2, L/2)$ is the point of crossing of inner edges of the first level squares. We consider a quadtree whose centre point is picked randomly in $\mathcal{S}$.

Definition 3. *Let $a, b \in \mathbb{N}_L$ be constants. The quadtree, whose centre point has coordinates $((L/2 + a) \mod L, (L/2 + b) \mod L)$, is called shifted and designated by $T(a, b)$.*

The squares of $T(a, b)$ which have level ≥ 1 are considered *modulo L* and are called *wrapped-around* (Figure 2). Our goal is to show that, if the stochastic variables a, b are distributed uniformly in $\mathbb{N}_L$, in the quadtree $T(a, b)$ there is $(1 + 1/c)$-approximation of the Euclidean Min-2-SCCP on the plane with a probability of at least 1/2.

5.3. Structure theorem

For some parameter values $m, r \in \mathbb{N}$ and any square S (a node in quadtree $T(a, b)$), we assign a regular partition of the border ∂S, consisting of $4(m+1)$ points including all corners of S. We call this partition *m-regular*, and its points are referred to as *portals.*

Definition 4. *The union of m-regular partitions of borders for all nodes of the quadtree $T(a,b)$ (except the root) is called the m-regular portal set and denoted by $P(a,b,m)$.*

Definition 5. *Let C be an arbitrary simple cycle in the graph $\bar{G}$ in the plane. The closed continuous piecewise linear route $l(C)$ such as (i) $l(C)$ bends only the points of $V(C) \cup P(a,b,m)$; (ii) nodes of $V(C)$ are visited by $l(C)$ in the same order as by C; (iii) for any side of any node of $T(a,b)$, the route $l(C)$ crosses this side in the points of $P(a,b,m)$ and no more than r times is called (m,r)-approximation of the cycle C.*

Following the Arora's idea of an approximate Hamiltonian tour, we define the similar construction for our problem.

Definition 6. *Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be an arbitrary k-size cycle cover in the graph $\bar{G}$, and let $l(C_i)$ be some (m,r)-approximation of the cycle C_i. Then the family $\mathcal{L}(\mathcal{C}) = \{l(C_1), \ldots, l(C_k)\}$ is called a cycle (m,r,k)-cover in the graph $\bar{G}$.*

Obviously, an arbitrary cycle $(m,r,1)$-cover consists of the only Hamiltonian cycle. We are interested in $(m,r,2)$-covers (Figure 3).

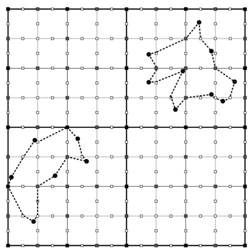

Figure 3: *Example of a cycle $(m,r,2)$-cover*

Theorem 2. *Let the constant $c > 1$ be fixed, let L be the size of the bounding box $\mathcal{S}$ for the given instance of the well-rounded Min-2-SCCP in the plane, and let discrete stochastic variables a and b be distributed uniformly in $\mathbb{N}_L$. Then, for $m = O(c \log L)$ and $r = O(c)$ with a probability of at least $1/2$ there exists a cycle $(m,r,2)$-cover in the graph $\bar{G}$ whose weight is not exceeding $(1 + 1/c)OPT$.*

Theorem 2 is proved in [8].

5.4. Dynamic programming

For parameter values $m = O(c \log n)$ and $r = O(c)$, we describe the dynamic programming procedure for searching for the cycle $(m,r,2)$-cover $\mathcal{L} = \{l_1, l_2\}$ of the minimum weight with the running time of $O(n(\log n)^{O(c)})$.

Definition 7. *A part of the cycle $(m,r,2)$-cover which belongs to some square S (being a node of the quadtree $T(a,b)$) and visits all nodes of the given graph $\bar{G}$ which are located inside the square S, is called an $(m,r,2,S)$-segment (of this cover).*

To describe an adaptation of the well-known Bellman equation (see explanations, e.g., in [15]) for the problem in question, we define the inner task, which is solved recursively for each entry of the dynamic programming lookup table.

Inner Task (S, R_1, R_2, κ).
Input. A square S being some node in quadtree $T(a,b)$. The cortege $R_i : \mathbb{N}_{q_i} \to (P(a,b,m) \cap \partial S)^2$ defines a sequence of the ordered portal pairs (s^i_j, t^i_j); i.e., crossing-points of (m,r)-approximation l_i and the border ∂S of the square S. The number κ defines the number of parts of the cycle $(m,r,2)$-cover which intersect the square S. If $q_1 q_2 > 0$, $\kappa = 2$; otherwise, it can take an arbitrary value from $\{1,2\}$.
Output. An $(m,r,2,S)$-segment defined by the values of input parameters, having the minimum weight $W(S,R_1,R_2,\kappa)$.

Depending on values of the input parameters, there are several cases of this Inner task:

(i) case $q_1 q_2 > 0$ seems to be regular. In this case, the resulting segment consists of parts of both (m,r)-approximations l_1 and l_2;

(ii) case $q_1 \neq 0, q_2 = 0$ (or $q_1 = 0, q_2 \neq 0$) and $\kappa = 1$ is similar to the previous one, apart from the fact that all subroutes in the square S belong to the only (m,r)-approximation (either l_1, or l_2);

(iii) in the case $q_1 \neq 0, q_2 = 0$ (or $q_1 = 0, q_2 \neq 0$) and $\kappa = 2$, one of the building (m,r)-approximations is supposed to belong entirely to the square S; in this case the resulting segment is constructed by augmenting the output of the case b) by closed (m,r)-approximation located inside the square S;

(iv) if $q_1 = q_2 = 0$ and $\kappa = 2$, l_1 and l_2 are supposed to belong to the square S, and the output is the cycle $(m,r,2)$-cover which is located in the square S;

(v) finally, if $q_1 = q_2 = 0$ and $\kappa = 1$, it is required to find one (m,r)-approximation of the minimum weight, which belongs entirely to the square S. This case is the same as the TSP.

Bellman equation. Similarly to PTAS for the Euclidean TSP, we start the dynamic programming procedure from leaves of the quadtree $T(a,b)$. Let S be an arbitrary leaf in the tree $T(a,b)$. By construction, S contains no more than one node of the graph $\bar{G}$, and for any input values of R_1, R_2 and the number κ, the Inner Task (S,R_1,R_2,κ) can be solved by brute force in time of $O(r)$.

Any other node (not a leaf) of the quadtree $T(a,b)$ has four child nodes, let us denote them by $S^I, \ldots, S^{IV}$. According to the recursive assumption, for any entry of the lookup table, which is related to the square S, to the moment of filling this entry, all possible instances of the Inner task for squares S^I – S^{IV} should be solved (and the appropriate entries are filled).

Let us explain a recursive solution of the inner task (S,R_1,R_2,κ) for fixed values of corteges R_1 and R_2 and the parameter κ. Let $\mathfrak{P}$ be a family of multisets P, consisting of no more than $4r$ portals (with their multiplicities), that are located on the inner sides of the child squares $S^I, \ldots, S^{IV}$.

By the choice of m and r, any such side has $m+2$ portals, in which the side can be crossed no more than r times. Therefore, $|\mathfrak{P}| = O((m+2)^{4r})$. For any multiset $P \in \mathfrak{P}$ we assign the set Σ_P of maps $\sigma : P \to \mathbb{N}_{q_1+q_2}$, for each inner portal $p \in P$ each of them assigns the ordered pair

$$\begin{aligned} (s^1_{\sigma(p)}, t^1_{\sigma(p)}),&\ \text{ if } \sigma(p) \leqslant q_1,\\ (s^2_{\sigma(p)-q_1}, t^2_{\sigma(p)-q_1}),&\ \text{ otherwise,} \end{aligned}$$

and, consequently, the route-segment $l_1(s^1_\sigma(p), t^1_\sigma(p))$ or $l_2(s^2_{\sigma(p)-q_1}, t^2_{\sigma(p)-q_1})$, crossing one of inner sides (of child squares) in the portal p. In this case, we call such a portal p *matched* to this route-segment. Since $|P| \leqslant 4r$ and $q_1 + q_2 \leqslant 4r$, the following bound $|\Sigma_P| = O((2r)^{4r})$ is valid.

For any route-segment $l_i(s_{ij}, t_{ij})$, we assign the set $\Lambda_{ij}(\sigma)$ consisting of permutations α of the preimage multiset $\sigma^{-1}(j + (i-1)q_1) \subset P$ for the map σ. It is easy to show that $|\Lambda_{ij}(\sigma)| = O((4r)!)$.

Each triple $\tau = (P, \sigma, \alpha)$ induces the instance quadruple of the inner tasks

$$\big((S^I, R^I_1(\tau), R^I_2(\tau), \kappa^I(\tau)), \ldots, (S^{IV}, R^{IV}_1(\tau), R^{IV}_2(\tau), \kappa^{IV}(\tau))\big)$$

in such a way that

$$W(S, R_1, R_2, \kappa) = \min_\tau \sum_{i=I}^{IV} W(S^i, R^i_1(\tau), R^i_2(\tau), \kappa^i(\tau)),$$

and the solving time-complexity for the task (S, R_1, R_2, κ) has the upper bound $O((m+2)^{4r}(2r)^{4r}(4r)!)$.

Finally, to find a minimum-weight cycle $(m, r, 2)$-cover, we should find a solution for the task $(\mathcal{S}, R^0_1, R^0_2, 2)$, for empty corteges R^0_1 and R^0_2.

To estimate the total running time of the constructed dynamic programming procedure, we need an upper bound for the number of entries in the lookup table. It can be easily verified, that any node S of the quadtree $T(a,b)$ is related to $O(m+2)^{4r}$ ways to choose a multiset of portals located on ∂S; any such multiset can be partitioned into pairs by at most $O((4r)!)$ times, and any such partition can be assigned to routes l_1 and l_2 by at most $O(2^{2r})$ ways. Taking into account the total number $O(L \log L)$ of nodes (for the quadtree $T(a,b)$), we obtain an upper running time bound for the pair (a,b)

$$O\left(L \log L \times (m+2)^{8r}((4r)!)^2(2r)^{4r} \times 2^{2r}\right). \tag{5}$$

Derandomization. The dynamic programming procedure described above finds an approximate solution of the well-rounded Min-2-SCCP for each pair a, b, which is related to the quadtree $T(a,b)$. We denote by $APP(a,b)$ the weight of such a solution. As follows from Theorem 2

$$P\left(APP(a,b) \leqslant (1 + \frac{1}{c})OPT\right) \geqslant 1/2,$$

for the probability measure induced by the distribution of a and b. Therefore, there is a pair $(a^*, b^*) \in \mathbb{N}_L$, for which the following inequality

$$OPT \leqslant APP(a^*, b^*) \leqslant (1 + 1/c)OPT$$

is valid. Such a pair can be found by exhaustive search in the time $O(L^2)$. Taking into account that $m = O(c \log n)$, $r = O(c)$, and $L = O(n)$, we have proved the following Theorem 3.

Theorem 3. *The well-rounded Min-2-SCCP in the plane has a PTAS with a time complexity of*

$$O(n^3 (\log n)^{O(c)}). \tag{6}$$

Combining the claims of Lemma 1 Theorem 3 we obtain our main result.

Corollary 1. *The Euclidean Min-2-SCCP in the plane has a PTAS with running time* (6).

It should be noted that time complexity of the proposed PTAS for the Min-2-SCCP equals the complexity of PTAS proposed in [5] for the Euclidean TSP and it differs (as follows from (5)) from it by the constant factor $2^{O(c)}$.

6. Conclusion

For any fixed $k \geqslant 1$, we have proved the intractability of the Min-k-SCCP combinatorial optimization problem and two of its special cases the Metric Min-k-SCCP and the Euclidean Min-k-SCCP.

For the Metric Min-k-SCCP, a 2-approximation polynomial-time algorithm is proposed. For the Euclidean Min-2-SCCP on the plane, a polynomial-time approximation scheme is developed.

While the obtained results implies that the Metric Min-k-SCCP belongs to APX, and the Euclidean Min-2-SCCP belongs to PTAS complexity classes, several issues remain open. In particular, we are interested in constructing approximation algorithms for the Metric Min-k-SCCP with improved approximation ratios. Also, we hope that the presented PTAS can be extended to the Euclidean Min-k-SCCP for any $k > 2$ and any dimension $d > 2$.

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