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HOMOTOPY CHARACTERIZATION OF G-ANR'S

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Dedicated to Professor Sibe Mardešić on the occasion of his 80th birthday

Abstract. Let G be a compact Lie group. We prove that if each point $x \in X$ of a G-space X admits a G_x-invariant neighborhood U which is a G_x-ANE then X is a G-ANE, where G_x stands for the stabilizer of x. This result is further applied to give two equivariant homotopy characterizations of G-ANR's. One of them sounds as follows: a metrizable G-space Y is a G-ANR iff Y is locally G-contractible and every metrizable closed G-pair (X, A) has the G-equivariant homotopy extension property with respect to Y. In the same terms we also characterize G-ANR subsets of a given G-ANR space.

1. Introduction

This paper is devoted to homotopy characterization of equivariant absolute neighborhood retracts or G-ANR's under the assumption that the acting group G is compact Lie. The non-equivariant analogs of the results presented here are well known (see Borsuk [6], Hu [10] and van Mill [12]).

It was proved in Jaworowski [11] that a finite-dimensional metrizable G-space is a G-ANR iff it is locally G-contractible. Local G-contractibility alone is not sufficient to characterize the G-ANR's of arbitrary dimension even if G is the trivial group (see [6, Chapter V, §11] for a counterexample). It turns out (see Theorem 5.3(b)) that local G-contractibility together with the

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G-homotopy extension property (short: G-HEP) characterizes the G-ANR's among metrizable G-spaces of abitrary dimension. We prove in Theorem 5.1 a "controlled" equivariant version of Borsuk's homotopy extension theorem. In Section 4 we define the property $\mathcal{P}(G, \mathcal{V})$ - a stronger property than the G-HEP, which alone characterizes the G-ANR's among all metrizable G-spaces (Theorem 4.4). We should mention here that all these characterizations are based on the following local characterization of G-ANE's obtained in Theorem 3.2: a G-space X is a G-ANE if and only if each point $x \in X$ admits a G_x-invariant neighborhood U which is a G_x-ANE, where G_x stands for the stabilizer of x. In last Theorem 5.4 we prove that a closed invariant subset A of a G-ANR space X is a G-ANR iff the pair (X, A) satisfies the G-HEP with respect to any G-space.

2. Preliminaries

Throughout the paper the letter "G" will always denote a compact Lie group (though some of the results presented here are valid also in the case of an arbitrary compact acting group G).

"A space" will mean a completely regular Hausdorff topological space.

The monographs [7, 13] are our main references for the basic notions of the theory of transformation groups. For the equivariant theory of retracts the reader can see, for instance, [1, 2, 4].

For the convenience of the reader we recall, however, some more special definitions and facts below.

By an action of the group G on a space X we mean a continuous map $(g, x) \mapsto gx$ of the product $G \times X$ into X such that $(gh)x = g(hx)$ and $ex = x$ whenever $x \in X$, g, $h \in G$ and e is the unity of G. A space X together with a fixed action of the group G is called a G-space.

By a normed linear G-space we shall mean a real normed linear space L on which G acts by means of linear isometries, i.e., $g(\lambda x+\mu y) = \lambda(gx)+\mu(gy)$ and $\|gx\| = \|x\|$ for all $g \in G$, $x, y \in L$ and λ, $\mu \in \mathbb{R}$.

A continuous map $f : X \to Y$ of G-spaces is called an equivariant map or, for short, a G-map, if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$. If G acts trivially on Y then we use the term "invariant map" instead of "equivariant map". By a G-embedding we shall mean a topological embedding $X \hookrightarrow Y$ which is a G-map.

Let X be a G-space. For any $x \in X$, we denote by G_x the stabilizer of x defined by $G_x = \{g \in G \,|\, gx = x\}$. A G-fixed point is a point $x \in X$ with $G_x = G$.

For a subset $S \subset X$ and for a subgroup $H \subset G$, the H-hull (or H-saturation) of S is defined as follows: $H(S)= \{hs \,|\, h \in H, s \in S\}$. If S is the one point set $\{x\}$, then the H-hull $H(S)$ is usually denoted by $H(x)$ and called the H-orbit of x. The set X/H of all H-orbits endowed with the quotient topology is called the H-orbit space. A subset $A \subset X$ is called H-invariant, or simply, an H-subset if it coincides with its H-hull, i.e., $A = H(A)$. We shall often use the term "invariant subset" for a "G-invariant subset".

A subset $S \subset X$ is called an H-slice in X, if: (1) S is H-invariant, (2) the G-hull $G(S)$ is open in X, (3) if $g \in G \setminus H$, then $gS \cap S = \emptyset$, (4) S is closed in $G(S)$.

If, in addition, $G(S) = X$, then S is called *a global* H-slice of X.

For each H-slice S, the G-hull $G(S)$ is G-homeomorphic to the twisted product $G \times_H S$ (see [7, Chapter II, Theorem 4.2]); we will use this fact in what follows without a specific reference.

Recall that, for an H-space Y, the twisted product $G \times_H Y$ is defined to be the H-orbit space of the H-space $G \times Y$, where H acts on $G \times Y$ by $h(g, y) = (gh^{-1}, hy)$. Furthermore, there is a natural action of G on $G \times_H Y$ given by $g'[g, y] = [g'g, y]$, where $[g, y]$ denotes the H-orbit of $(g, y) \in G \times Y$ and $g' \in G$. We shall identify Y, as an H-space, with the H-invariant subset $\{[e, y] \mid y \in Y\}$ of $G \times_H Y$.

The following result plays a central rule in the theory of topological transformation groups (see [7, Chapter II, Theorem 5.4]):

THEOREM 2.1 (Slice theorem). *Let X be a G-space, $x \in X$ and U a neighborhood of x. Then there exists a G_x-slice $S_x \subset X$ such that $x \in S_x \subset U$.*

A G-space Y is called an *absolute neighborhood G-extensor* (notation: $Y \in G$-ANE) if, for any closed invariant subset A of a metrizable G-space X and any G-map $f : A \to Y$, there exist an invariant neighborhood U of A in X and a G-map $\psi\colon U \to Y$ that extends f. If, in addition, one can always take $U = X$, then we say that Y is an *absolute G-extensor* (notation: $Y \in G$-AE). The map ψ is called a G-extension of f.

A metrizable G-space Y is called an *absolute neighborhood G-retract* (notation: $Y \in G$-ANR), provided that for any closed G-embedding of Y in a metrizable G-space X, there exists a G-retraction $r\colon U \to Y$, where U is an invariant neighborhood of Y in X. If, in addition, one can always take $U = X$, then we say that Y is an *absolute G-retract* (notation: $Y \in G$-AR).

It is known [2] that a metrizable G-space is a G-ANR (resp., a G-AR) iff it is a G-ANE (resp., a G-AE); we shall often use this fact throughout the paper without an additional reference.

As usual, the letter I will stand for the closed interval $[0, 1]$.

Let X and Y be G-spaces. A homotopy $F_t\colon X \to Y$, $t \in I$, is called a *G-homotopy*, if $F_t(gx) = gF_t(x)$ for every $x \in X$, $g \in G$ and $t \in I$. Two G-maps $f, \varphi\colon X \to Y$ are *G-homotopic*, if there exists a G-homotopy $F_t\colon X \to Y$ such that $F_0 = f$ and $F_1 = \varphi$.

Let γ be an open covering of the G-space Y. Then a G-homotopy F_t : $X \to Y$, $t \in I$, is said to be *limited by* γ, or simply, a γ-G-homotopy provided for any $x \in X$, there exists $\Gamma \in \gamma$ such that $F_t(x) \in \Gamma$ for all $t \in I$. In such a case F_0 and F_1 are called γ-G-homotopic G-maps.

A G-subset A of a G-space X is called *G-contractible* in X if the identity inclusion $A \hookrightarrow X$ is G-homotopic to a constant map $A \to \{x_0\}$, where $x_0 \in X$ is a G-fixed point. Respectively, X is called locally G-contractible at the point $x \in X$ if for every G_x-invariant neighborhood U of x there exists a G_x-invariant neighborhood V of x such that V is G_x-contractible in U. A G-space X is called *locally G-contractible* if it is locally G_x-contractible at each point $x \in X$.

In the sequel we will need the following known results:

Proposition 2.2. *Let K be a closed subgroup of G, and S a K-space. Then S is a neighborhood K-retract of the twisted product $G \times_K S$.*

Proof. See [5, Proposition 4.1]. □

Proposition 2.3. *If a G-space Y is the union of a family of invariant open G-*ANE *subsets $Y_\mu \subset Y$, $\mu \in \mathcal{M}$, then Y is a G-*ANE *as well.*

Proof. See [4, Corollary 5.7]. □

Proposition 2.4. *Let K be a closed subgroup of G, and S a K-space. Then every K-map $f : S \to Y$ in a G-space Y induces a G-map $f' : G \times_K S \to Y$ according to the formula: $f'([g, s]) = gf(s)$ for any $[g, s] \in G \times_K S$.*

Proof. See [8, Chapter I, Proposition 4.3]. □

Proposition 2.5. *Let K be a closed subgroup of G, and S a global K-slice of the G-space X. If S is a K-*ANE *then X is a G-*ANE*.*

Proof. See [13, Corollary 1.7.16]. □

3. Local G-ANE's

Definition 3.1. *A G-space X is called a local G-*ANE *if each point $x \in X$ admits a G_x-invariant neighborhood U which is a G_x-*ANE*.*

The following local characterization of G-ANE's plays a fundamental role in the paper:

Theorem 3.2. *A G-space X is a G-*ANE *if and only if X is a local G-*ANE*.*

Proof. If X is a G-ANE then X is also an H-ANE for any closed subgroup $H \subset G$ (see [14, Corollary 4.5]). In particular, X is a G_x-ANE for any $x \in X$.

Now assume that X is a local G-ANE. For any $x \in X$, let U be a G_x-invariant neighborhood of x which is a G_x-ANE. By Theorem 2.1, one can choose a G_x-slice S_x such that $x \in S_x \subset U$. Since the G-hull $G(S_x)$ is G-homeomorphic to the twisted product $G \times_{G_x} S_x$, by Proposition 2.2, S_x is a G_x-retract of some G_x-invariant neighborhood W of S_x in $G(S_x)$. Since U is G_x-invariant, without loss of generality one can assume that $W \subset U$. Then, being a G_x-invariant open subset of the G_x-ANE space U, the set W is itself a G_x-ANE. This yields that S_x is a G_x-ANE too. Next, by virtue of Proposition 2.5, the G-hull $G(S_x)$ is a G-ANE.

Now, since X is the union of its open invariant G-ANE subsets $G(S_x)$, $x \in X$, then it follows from Proposition 2.3 that X is a G-ANE. □

4. Homotopy characterization of G-ANR's

Recall that a covering $\mathcal{U}$ of a G-space Y is called a *G-covering* if $gU \in \mathcal{U}$ for every $U \in \mathcal{U}$ and $g \in G$. Two continuous maps $f, \varphi : X \to Y$ are called $\mathcal{U}$-near, if for every $x \in X$ there exists $U \in \mathcal{U}$ such that $\{f(x), \varphi(x)\} \subset U$.

Definition 4.1. *Let Y be a G-space and let $\mathcal{U}$ and $\mathcal{V}$ be open G-coverings of Y such that $\mathcal{V}$ is a refinement of $\mathcal{U}$. We say that Y satisfies the property $\mathcal{P}(G,\mathcal{U},\mathcal{V})$ if for any two $\mathcal{V}$-near G-maps $f, \varphi : X \to Y$ defined on a metrizable G-space X and any $\mathcal{V}$-G-homotopy $j_t : A \to Y$, $t \in I$, defined on a closed G-subset A of X with $j_0 = f|_A$ and $j_1 = \varphi|_A$, there exists a $\mathcal{U}$-G-homotopy $J_t : X \to Y$, $t \in I$, with $J_0 = f$, $J_1 = \varphi$ and $J_t|_A = j_t$ for every $t \in I$.*

If $\mathcal{U} = \{Y\}$ is the one element covering, then we shall write $\mathcal{P}(G,\mathcal{V})$ instead of $\mathcal{P}(G,\mathcal{U},\mathcal{V})$

Theorem 4.2. *If Y is a G-*ANR *and $\mathcal{U}$ a given open G-covering of Y, then there exists an open G-covering $\mathcal{V}$ of Y which is a refinement of $\mathcal{U}$ such that Y satisfies the property $\mathcal{P}(G,\mathcal{U},\mathcal{V})$ from Definition* 4.1.

Proof. By [2, Corollary 5], we can assume that Y is an invariant closed subset of a normed linear G-space L. Since Y is a G-ANR, there exists an invariant neighborhood M of Y in L and an equivariant retraction $r : M \to Y$. Consider the open covering $r^{-1}(\mathcal{U}) = \{r^{-1}(U) \mid U \in \mathcal{U}\}$ of M. Let $\mathcal{W}$ consist of all open balls of L each of which is contained in an element of $r^{-1}(\mathcal{U})$. Clearly, $\mathcal{W}$ is an open G-covering of M which refines $r^{-1}(\mathcal{U})$. Put $\mathcal{V} = \{W \cap Y \mid W \in \mathcal{W}\}$. We claim that $\mathcal{V}$ is the required G-covering of Y.

Indeed, let X be a metrizable G-space and A a closed G-subset of X. Assume further that f, $\varphi : X \to Y$ are any two $\mathcal{V}$-near G-maps defined on X and $j_t : A \to Y$, $t \in I$, is a given $\mathcal{V}$-G-homotopy defined on A with $j_0 = f|_A$ and $j_1 = \varphi|_A$.

We construct a $\mathcal{W}$-G-homotopy $\psi_t : X \to M$, $t \in I$, by putting

$$\psi_t(x) = (1-t)f(x) + t\varphi(x)$$

for every $x \in X$ and every $t \in I$.

Consider the closed G-subset

$$T = (X \times \{0\}) \cup (A \times I) \cup (X \times \{1\})$$

of the topological product $P = X \times I$ endowed with the G-action: $g(x,t) = (gx,t)$. Define a G-map $\Phi : T \times I \to Y$ by the rule:

$$\Phi(x,t) = \begin{cases} f(x), & \text{if } x \in X \text{ and } t = 0 \\ j_t(x), & \text{if } x \in A \text{ and } t \in I \\ \varphi(x), & \text{if } x \in X \text{ and } t = 1 \end{cases}.$$

Since Y is a G-ANR, it follows that Φ has a G-extension $\Psi : N \to Y$ over a G-neighborhood N of T in P.

By means of compactness of the unit interval I, one can easily prove the existence of an open neighborhood C' of A in X, such that $C' \times I$ is contained in N and that the homotopy $\xi'_t : C' \to Y$, $t \in I$, defined by

$$\xi'_t(x) = \Psi(x,t), \quad x \in C',\ t \in I$$

is a $\mathcal{V}$-homotopy. Since G is compact, one can choose an invariant neighborhood C of A in X such that $C \subset C'$. Then the restriction $\xi_t = \xi'_t|_C$, $t \in I$, is a $\mathcal{V}$-G-homotopy.

Further, choose an open invariant set B in X such that

$$A \subset B \subset \overline{B} \subset C.$$

Then, by the equivariant Urysohn lemma, there exists an invariant map $s : X \to I$ such that

$$s(x) = \begin{cases} 0, & \text{if } x \in X \setminus B \\ 1, & \text{if } x \in A. \end{cases}$$

Define a G-homotopy $\theta_t : X \to M$, $t \in I$, by the rule:

$$\theta_t(x) = \begin{cases} \big(1 - s(x)\big)\psi_t(x) + s(x)\xi_t(x), & \text{if } x \in C \\ \psi_t(x), & \text{if } x \in X \setminus B. \end{cases}$$

Each θ_t is a G-map since ψ_t and ξ_t are so and G acts linearly on L.

Let us prove that θ_t is a $\mathcal{W}$-homotopy. For this purpose, let x be an arbitrary point of X. We will prove the existence of a $W_\mu \in \mathcal{W}$ such that

$$\theta_t(x) \in W_\mu \quad \text{for every} \quad t \in I.$$

Consider two cases.

Case I. $s(x) = 0$. In this case, we have $\theta_t(x) = \psi_t(x)$ for every $t \in I$. Since ψ_t is a $\mathcal{W}$-homotopy, there is a $W_\mu \in \mathcal{W}$ such that $\theta_t(x) = \psi_t(x) \in W_\mu$ for every $t \in I$.

Case II. $s(x) > 0$. In this case, we have $x \in B \subset C$. Since ξ_t is a $\mathcal{W}$-homotopy, there exists a $W_\mu \in \mathcal{W}$ such that $\xi_t(x) \in W_\mu$ for every $t \in I$. In particular, W_μ contains both points

$$\xi_0(x) = f(x) \quad \text{and} \quad \xi_1(x) = \varphi(x).$$

Since W_μ is a convex set, it follows that $\psi_t(x) \in W_\mu$ for every $t \in I$. Now, since the convex set W_μ contains both points $\psi_t(x)$ and $\xi_t(x)$, it must also contain $\theta_t(x)$ for every $t \in I$. Thus, we have proved that

$$\theta_t : X \to M, \quad t \in I$$

is a $\mathcal{W}$-homotopy.

Finally, define a G-homotopy $J_t : X \to Y$, $t \in I$, by taking

$$J_t(x) = r\big(\theta_t(x)\big), \quad x \in X \text{ and } t \in I.$$

Since θ_t is a $\mathcal{W}$-homotopy and $\mathcal{W}$ is a refinement of $r^{-1}(\mathcal{U})$, it follows that J_t is a $\mathcal{U}$-homotopy. On the other hand, since r is a retraction, it is easy to verify that $J_0 = f$, $J_1 = \varphi$, and $J_t|_A = j_t$ for every $t \in I$. ∎

PROPOSITION 4.3. *Let Y be a G-space and $\mathcal{V}$ an open G-covering of Y. If Y satisfies the property $\mathcal{P}(G, \mathcal{V})$ then it also satisfies the property $\mathcal{P}(K, \mathcal{V})$ for every closed subgroup $K \subset G$.*

PROOF. Let X be a metrizable K-space and A a closed K-invariant subset of X. Assume that $f, \varphi : X \to Y$ are two $\mathcal{V}$-near K-maps and $j_t : A \to Y$, $t \in I$, is a $\mathcal{V}$-K-homotopy with $j_0 = f|_A$ and $j_1 = \varphi|_A$. Then the twisted product $X' = G \times_K X$ is a metrizable G-space and $A' = G \times_K A$ is a G-invariant closed subset of X'.

Now, by Proposition 2.4, the K-maps f, φ and the K-homotopy j_t induce G-maps $f', \varphi' : X' \to Y$ and a G-homotopy $j'_t : A' \to Y, t \in I$, respectively.

Let us check first that f' and φ' are $\mathcal{V}$-near. Indeed, $f'([g, x]) = gf(x)$ and $\varphi'([g, x]) = g\varphi(x)$ for any $[g, x] \in G \times_K X$. Since f and φ are $\mathcal{V}$-near, then there exists an element $V \in \mathcal{V}$ which contains both points $f(x)$ and $\varphi(x)$. Consequently, $gf(x)$, $g\varphi(x) \in gV$, and since $gV \in \mathcal{V}$ (remember that $\mathcal{V}$ is a G-covering), we conclude that the G-maps f' and φ' are $\mathcal{V}$-near.

Next, let us check that $j'_t : A' \to Y$, $t \in I$, is a $\mathcal{V}$-homotopy. Indeed, $j'_t([g, a]) = gj_t(a)$ for any $[g, a] \in G \times_K A$ and $t \in I$. Since $j_t : A \to Y, t \in I$, is a $\mathcal{V}$-homotopy, there exists an element $W \in \mathcal{V}$ such that $j_t(a) \in W$ for all $t \in I$. Consequently, $j'_t([g, a]) = gj_t(a) \in gW$ for all $t \in I$, and since $gW \in \mathcal{V}$, we infer that $j'_t : A' \to Y, t \in I$, is a $\mathcal{V}$-homotopy.

Now, since the G-space Y satisfies the property $\mathcal{P}(G, \mathcal{V})$, there must exist a G-homotopy $J'_t : X' \to Y$, $t \in I$, with $J'_0 = f'$, $J'_1 = \varphi'$ and $J'_t|_{A'} = j'_t$ for every $t \in I$. Evidently, the restriction $J_t = J'_t|_X : X \to Y$, $t \in I$, is a K-equivariant homotopy with $J_0 = f$, $J_1 = \varphi$ and $J_t|_A = j_t$ for every $t \in I$, as required. This completes the proof. ∎

It turns out that in the class of all metrizable G-spaces the property $\mathcal{P}(G, \mathcal{V})$ characterizes the G-ANR's. In fact, we have the following

THEOREM 4.4. *A necessary and sufficient condition for a metrizable G-space Y to be a G-*ANR *is the existence of an open G-covering $\mathcal{V}$ of Y such that Y satisfies the property $\mathcal{P}(G, \mathcal{V})$.*

PROOF. The necessity condition follows from Theorem 4.2 by taking $\mathcal{U} = \{Y\}$ – the covering consisting of a single open set Y.

To prove the sufficiency of the condition $\mathcal{P}(G, \mathcal{V})$, by virtue of Theorem 3.2, it suffices to show that Y is local G-ANE.

For, let $y \in Y$ and let $V \in \mathcal{V}$ be an element that contains y. By compactness of the group G_y, we can and do choose a G_y-invariant neighborhood S of y such that $S \subset V$. Define two G_y-maps $\phi, \psi : S \to Y$ and a G_y-homotopy $\theta_t : \{y\} \to Y$, $t \in I$, by putting

$$\begin{cases} \phi(s) = y, & \text{if } s \in S \\ \psi(s) = s, & \text{if } s \in S \\ \theta_t(y) = y, & \text{if } t \in I. \end{cases}$$

Obviously, ϕ and ψ are $\mathcal{V}$-near G_y-maps, and θ_t, $t \in I$, is a $\mathcal{V}$-G_y-homotopy. According to Proposition 4.3, Y considered as a G_y-space satisfies the condition $\mathcal{P}(G_y, \mathcal{V})$.

Now, since S is a metrizable G_y-space and $\{y\}$ is a closed G_y-subset of S, it follows from $\mathcal{P}(G_y, \mathcal{V})$ that there exists a G_y-homotopy $j_t : S \to Y$, $t \in I$, with $j_0 = \phi$, $j_1 = \psi$, and $j_t(y) = y$ for every $t \in I$.

Since the unit interval I is compact and since $j_t(y) = y \in V$ for every $t \in I$, there exists an open G_y-invariant neighborhood U of y such that $U \subset S$ and $j_t(U) \subset V$ for every $t \in I$. We will prove that U is a G_y-ANE.

To this end, let $f : A \to U$ be any G_y-map defined on a closed G_y-subspace A of a metrizable G_y-space X. Define two G_y-maps $\xi, \eta : X \to Y$ and a G_y- homotopy $J_t : A \to Y$, $t \in I$, by taking

$$\xi(x) = y = \eta(x), \quad x \in X$$

and

$$J_t(x) = \begin{cases} j_{2t}\big(f(x)\big), & \text{if } x \in A,\ 0 \le t \le \frac{1}{2} \\ j_{2-2t}\big(f(x)\big), & \text{if } x \in A,\ \frac{1}{2} \le t \le 1. \end{cases}$$

Obviously, ξ and η are $\mathcal{V}$-near G_y-maps and J_t is a $\mathcal{V}$-G_y-homotopy. Hence, by the condition $\mathcal{P}(G_y, \mathcal{V})$, there exists a G_y-homotopy $R_t : X \to Y$, $t \in I$, with $R_0 = \xi$, $R_1 = \eta$, and $R_t|_A = J_t$ for every $t \in I$.

Consider the G_y-map $r = R_{\frac{1}{2}} : X \to Y$. By the construction of r, one can clearly see that $r|_A = f$. Let $W = r^{-1}(U)$. Then, W is an open G_y-neighborhood of A in X and the restriction $r|_W : W \to U$ is a G_y-extension of f over W. This proves that U is a G_y-ANE, and hence, Y is a local G-ANE, as required. ∎

5. Equivariant homotopy extension property

By a *G-pair* we shall mean a couple (X, A) where X is a metrizable G-space and A a closed G-subset of X.

A G-pair (X, A) is said to have the *equivariant homotopy extension property* (abbreviated: G-HEP) with respect to a G-space Y iff every partial G-homotopy

$$h_t : A \to Y, \ t \in I$$

of an arbitrary G-map $f : X \to Y$ has a G-extension

$$f_t : X \to Y, \ t \in I \quad \text{such that} \quad f_0 = f.$$

The G-pair (X, A) is said to have the *absolute equivariant homotopy extension property* (abbreviated: G-AHEP) iff it has the G-HEP with respect to every G-space Y. In this case one says also that the inclusion $A \hookrightarrow X$ is a *G-cofibration* (see [8, p. 96]).

An immediate consequence of the G-HEP of (X, A) with respect to Y is that the equivariant extension problem of a G-map $f : A \to Y$ over X depends only on the G-homotopy class of f. In other words, if $f, \phi : A \to Y$ are G-homotopic G-maps and if f is G-extendable over X, then so is ϕ.

Equivariant version of the well known Borsuk homotopy extension theorem states that if Y is a G-ANR, then every G-pair (X, A) has the G-HEP with respect to Y (see [1, Theorem 5]). Our next theorem establishes a "controlled" version of this result:

THEOREM 5.1. *Let Y be a G-*ANR *and $\mathcal{U}$ an open G-covering of Y. Assume that A is a closed G-subset of a metrizable G-space X and $j_t : A \to Y$, $t \in I$, a partial $\mathcal{U}$-G-homotopy. If j_0 can be extended to a G-map $f : X \to Y$, then there exists a $\mathcal{U}$-G-homotopy $J_t : X \to Y$ such that $J_0 = f$ and $J_t|_A = j_t$ for all $t \in I$.*

PROOF. By the above quoted equivariant Borsuk homotopy extension theorem (see [1, Theorem 5]), there exists a G-homotopy $F_t : X \to Y$, $t \in I$ such that $F_0 = f$ and $F_t|_A = j_t$. For each $a \in A$, there exists $U_a \in \mathcal{U}$ containing $F_t(a) = j_t(a)$ for all $t \in I$. By means of compactness of the unit interval I, there exists a neighborhood W_a of a in X such that

$$F_t(W_a) \subset U_a, \quad \text{for all} \quad t \in I. \tag{5.1}$$

Put $W = \bigcup_{a \in A} W_a$. Then W is a neighborhood of A in X. Due to the compactness of the acting group G, there exists a G-invariant neighborhood V of A such that $V \subset W$.

Next we choose an invariant Urysohn function $\lambda : X \to I$ such that $\lambda|_A = 1$ and $\lambda|_{X \setminus V} = 0$. Define $J_t : X \to Y$, $t \in I$, as follows:

$$J_t(x) = F_{\lambda(x) \cdot t}(x), \quad x \in X.$$

Then, clearly, $J_t(x)$ depends continuously upon the pair $(x, t) \in X \times I$, J_t is equivariant and $J_t|_A = j_t$ for all $t \in I$. In addition,

$$J_0(x) = F_0(x) = f(x)$$

for every $x \in X$, so $J_0 = f$. It remains to prove that the G-homotopy J_t, $t \in I$, is limited by $\mathcal{U}$. Indeed, take an arbitrary $x \in X$. If $x \in V$ then there exists $a \in A$ such that $x \in W_a$. Consequently, by (5.1), for each $t \in I$ one has:

$$J_t(x) = F_{\lambda(x)\cdot t}(x) \in U_a.$$

If $x \notin V$, then $\lambda(x) = 0$, from which it follows that

$$J_t(x) = F_0(x) = f(x), \quad t \in I.$$

Since $\mathcal{U}$ is a covering of Y, there exists an element $U \in \mathcal{U}$ that contains $f(x)$, and therefore, $J_t(x)$ is contained in U for all $t \in I$, as required. ☐

Proposition 5.2. *Let Y be a G-space such that every G-pair has the G-*HEP *with respect to Y. Then for every closed subgroup $K \subset G$, every K-pair (X, A) has the K-*HEP *with respect to Y considered as a K-space.*

Proof. The proof is quite similar to the one of Proposition 4.3. ☐

Local G-contractibility or G-HEP alone cannot characterize G-ANR's even in the case of the trivial acting group G. Corresponding counterexamples can be found in Borsuk [6, Chapter V, §11] and Hanner [9].

However, we have the following convenient characterization of G-ANR's:

Theorem 5.3. *For a given metrizable G-space Y, the following three statements are equivalent:*

*(a) Y is a G-*ANR.
*(b) Y is locally G-contractible, and every G-pair (X, A) has the G-*HEP *with respect to Y.*
(c) Every point $y \in Y$ has a G_y-invariant neighborhood V such that any G_y-map $f : A \to V$ defined on a closed G_y-subset A of a metrizable G_y-space X has a G_y-extension $\phi : X \to Y$.

Proof. (a) $\Rightarrow$ (b). The G-HEP follows from Theorem 5.1 if we take $\mathcal{U} = \{Y\}$ – the one element covering. Let us prove that Y is locally G-contractible. According to [2, Corollary 5], one can assume that Y is a closed G-subset of a normed linear G-space Z. Since Y is a G-ANR, there must exist an open G-subset $U \subset Z$ and a G-retraction $r : U \to Y$. Now, take a point $y \in Y$ and a G_y-neighborhood W of y in Y. Since $r^{-1}(W)$ is an open subset of Z, we can choose an open ball $B(y, \varepsilon)$ centered at y and having the radius $\varepsilon > 0$ such that $B(y, \varepsilon) \subset r^{-1}(W)$. Put $V = B(y, \varepsilon) \cap Y$. Since G acts on Z by means of linear isometries we infer that the ball $B(y, \varepsilon)$, and hence, also V is a G_y-invariant set. Next we define a G_y-homotopy $f_t : V \to W$, $t \in I$, by the formula:

$$f_t(v) = r\big(ty + (1 - t)v\big), \quad v \in V.$$

Clearly f_t, $t \in I$, is a G_y-contraction of V in W to the G_y-fixed point y.

(b) $\Rightarrow$ (c). Let y be an arbitrary point in Y. Since Y is locally G-contractible, there exists a G_y-invariant neighborhood V of y which is G_y-contractible in Y to a G_y-fixed point $z \in Y$ (in general, z may be different from y). To prove that V satisfies (c), let $f : A \to V$ be a G_y-map defined on a closed G_y-subset A of a metrizable G_y-space X. Since V is G_y-contractible to the G_y-fixed point z, it follows that f, considered as a G_y-map into Y, is G_y-homotopic to the constant G_y-map $c : A \to Y$ which carries all A into the point $z \in Y$. Now, observe that by Proposition 5.2, (X, A) satisfies the G_y-HEP with respect to Y. Therefore, since c can be G_y-extended over X, it then follows that f has a G_y-extension $\phi : X \to Y$.

(c) $\Rightarrow$ (a). By Theorem 3.2, it suffices to show that Y is a local G-ANE. Let $y \in Y$ be an arbitrary point and let V be a G-invariant neighborhood of y which satisfies (c). We will prove that V is an G_y-ANE. For this purpose, let $f : A \to V$ be any G_y-map defined on a closed G_y-subset A of a metrizable G_y-space X. By (c), f has a G_y-extension $\phi : X \to Y$. The inverse image $U = \phi^{-1}(V)$ is a G_y-invariant open set in X containing A, and the restriction $\phi|_U : U \to V$ is a G_y-extension of $f : A \to V$ over U. □

Our last result characterizes invariant closed G-ANR subsets in a G-ANR space; more precisely, we have the following

Theorem 5.4. *Let X be a G-*ANR*. Then an invariant closed subset A of X is a G-*ANR *iff the G-pair (X, A) has the G-*AHEP.

For the proof we shall need the following two lemmas.

Lemma 5.5. *A G-pair (X, A) has the G-*AHEP *iff the invariant closed subset*

$$T = (X \times \{0\}) \cup (A \times I)$$

of the G-space $P = X \times I$ is a G-retract of P.

Proof. *The "only if" part.* Let $f : X \to T$ denote the G-map defined by

$$f(x) = (x, 0),\ x \in X.$$

Define a partial G-homotopy $h_t : A \to T,\ t \in I$, of f by putting

$$h_t(a) = (a, t) \quad \text{for} \quad a \in A,\ t \in I.$$

Since (X, A) has the G-AHEP and $h_0 = f|_A$, we infer that h_t has an equivariant extension $f_t : X \to T,\ t \in I$, such that $f_0 = f$. Let $r : P \to T$ denote the G-map defined by

$$r(x, t) = f_t(x), \quad x \in X,\ t \in I.$$

Then r is a G-retraction of P onto T, and hence, T is a G-retract of P.

The "if" part. Assume that T is a G-retract of P with a G-retraction $r : P \to T$. To prove the G-AHEP of (X, A), let $f : X \to Y$ be any G-map to a G-space Y and $h_t : A \to Y$, $t \in I$, a partial G-homotopy of f. Define a G-map $H : T \to Y$ by taking

$$H(x,t) = \begin{cases} f(x), & \text{if } x \in X \text{ and } t = 0 \\ h_t(x), & \text{if } x \in A \text{ and } t \in I. \end{cases}$$

Then h_t has a G-extension $f_t : X \to Y$, $t \in I$, defined by

$$f_t(x) = H\big(r(x,t)\big) \quad \text{for every} \quad x \in X,\ t \in I.$$

Clearly, $f_0 = f$, and hence, (X, A) has the G-AHEP. ∎

Lemma 5.6. *If X is a G-*ANR *and A is an invariant closed G-*ANR *subset of X, then the invariant closed subset*

$$T = (X \times \{0\}) \cup (A \times I)$$

of the G-space $P = X \times I$ is a G-retract of P.

Proof. Since $X \times \{0\}$ and $A \times I$ are invariant closed G-ANR subsets of T and their intersection $A \times \{0\}$ is also a G-ANR, it follows from [1, Theorem 4(2)] that T is a G-ANR. Hence, the identity G-map $i : T \to T$ has an equivariant extension $j : U \to T$ over an invariant neighborhood U of T in P.

Let us show that then there exists a G-retraction $r : P \to T$. Indeed, due to compactness of the interval I, one can find a neighborhood V of A in X such that $V \times I \subset U$. Because of compactness of the acting group G one can assume that V is invariant. Next, since A and $X \setminus V$ are disjoint invariant closed subsets of X, using normality of the orbit space X/G (which is in fact even metrizable), one can find an invariant function $\lambda : X \to I$ such that

$$\lambda(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in X \setminus V. \end{cases}$$

Define a G-map $r : P \to T$ by putting

$$r(x,t) = j\big(x, \lambda(x)t\big)$$

for every $x \in X$ and every $t \in I$. Then r is a G-retraction of P onto T. ∎

Proof of Theorem 5.4. The *"only if"* part is a simple combination of Lemmas 5.5 and 5.6.

The "if" part. Assume that the G-pair (X, A) has the G-AHEP. Then, by Lemma 5.5, T is an equivariant retract of P. Since $P = X \times I$ is a G-ANR then it follows that T is also a G-ANR.

Next, A may be identified, as a G-space, with the invariant closed subspace $A \times \{1\}$ of T. Evidently, the set

$$V = \{(a,t) \in T \mid a \in A,\ t > 0\}$$

is an invariant neighborhood of A in T. Let $s : V \to A$ denote the G-map defined by

$$s(a,t) = (a,1), \quad a \in A,\ 0 < t \leq 1.$$

Since s is clearly a G-retraction of V onto A, we infer that A is a neighborhood G-retract of T. Since T is a G-ANR, then it follows that A is a G-ANR. This completes the proof. ∎

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