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LOCAL CHARACTERIZATION OF ABSOLUTE CO-EXTENSORS

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The authors wish to dedicate this article to Professor Sibe Mardešić on the anniversary of his 80*th birthday*

Abstract. Suppose that K is a space and X is a paracompact space. We show that X is an absolute co-extensor for K (i.e., K is an absolute extensor for X) if and only if it is a local absolute co-extensor for K. We also provide a similar characterization using a weaker local extension property. The property that X is an absolute co-extensor for K is inherited by closed subsets but not necessarily by open subsets of X. We also present several extension results for open subsets of stratifiable spaces if K is a CW-complex.

1. Introduction

Suppose that A is a subspace of a topological space X and $f : A \to K$ is a map (i.e., continuous function). Then the problem of determining whether f has a continuous extension to X, i.e. whether there is a map $F : X \to K$ such that $F|A = f$, is called the "extension problem". The most investigated case occurs when A is a closed subset of X, where X belongs to a certain class of spaces and K is a CW-complex or a polyhedron. The subject is then referred to as *extension theory* (refer, e.g., to [2, 4]). Just to illustrate, among the most widely used theorems which are the solutions of appropriate extension problems, are the Tietze Extension Theorem and the Homotopy Extension Theorem. Let us mention that typically the class of spaces studied

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is at least within the class of paracompacta, but that in [1], extension theory with respect to Tychonoff spaces was visited.

In extension theory we use the following notation and terminology. One says that K is an *absolute extensor* for X, $K \in AE(X)$, or that X is an *absolute co-extensor* for K, $X\tau K$, if for each closed subset A of X and map $f : A \to K$, there exists a map $F : X \to K$ such that F is an extension of f. If A is a closed subset of X and $X\tau K$ holds, then $A\tau K$ holds too. So the property $X\tau K$ is inherited by closed subsets but not necessarily by arbitrary subsets. We note, however, that for the class of stratifiable spaces (Section 3), a subspace theorem for K a CW-complex was proved in [5, Theorem 3.6], i.e., if X is stratifiable, K a CW-complex, and $X\tau K$, then for *any* subspace A of X, $A\tau K$.

In Section 2 of this paper we show that if a paracompact space X has the extension property locally, meaning that every point of X has an open neighborhood U with $U\tau K$, then $X\tau K$. We also show that the converse is true (see Theorem 2.3).

Restricting ourselves in Section 3 to the class of stratifiable spaces, a subclass of the class of paracompact spaces, we present several extension results associated with collections of open subsets of such a space.

2. Absolute co-extensors and paracompacta

If X is a locally finite union of closed subsets that are absolute co-extensors for K then using transfinite induction one shows that the following proposition holds true (see [2, Proposition 1.18]).

Proposition 2.1. *Let X, K be spaces, and $\mathcal{D}$ a locally finite closed cover of X such that $D\tau K$ for each $D \in \mathcal{D}$. Then $X\tau K$.*

Following [8], we shall say that a topological space X has a certain property $\mathcal{P}$ locally if every point of X has an open neighborhood U which has property $\mathcal{P}$. In our case $\mathcal{P}$ is the property that $U\tau K$. We prove that for paracompact spaces [3] (paracompact spaces are assumed to be Hausdorff) property $\mathcal{P}$ holds globally if and only if it holds locally.

Theorem 2.2. *Let K be a space and X be a paracompact space. Then $X\tau K$ if and only if X has an open cover $\mathcal{U}$ such that for each $U \in \mathcal{U}$, $U\tau K$.*

Proof. If $X\tau K$, then let $\mathcal{U} = \{X\}$. Conversely, let $\mathcal{D} = \{D_U \mid U \in \mathcal{U}\}$ be a closed, locally finite shrinking of an open cover $\mathcal{U}$ of X where $U\tau K$ for each $U \in \mathcal{U}$. Thus, D_U is closed in X, $D_U \subset U$, and $D_U\tau K$ for each $U \in \mathcal{U}$; and $\mathcal{D}$ is a locally finite closed cover of X. From Proposition 2.1 it follows that $X\tau K$. ∎

We now define a weaker local extension property for a given open subset U of a space X. We write $U\tau^* K$ if for each closed subset C of X with $C \subset U$

and map $f : C \to K$, there exists a map $f^* : U \to K$ that extends f. Next we show that for any paracompact space X, the global property $X\tau K$ is characterized by the local property $U\tau^* K$ for all elements U of some open cover of X.

THEOREM 2.3. *Let K be a space and X be a paracompact space. Then $X\tau K$ if and only if X has an open cover $\mathcal{U}$ such that for each $U \in \mathcal{U}$, $U\tau^* K$.*

PROOF. If $X\tau K$, then let $\mathcal{U} = \{X\}$. Conversely, let $\mathcal{U}$ be an open cover such that $U\tau^* K$ for each $U \in \mathcal{U}$. As in the proof of Theorem 2.2 let $\mathcal{D} = \{D_U \mid U \in \mathcal{U}\}$ be a closed, locally finite shrinking of the open cover $\mathcal{U}$ of X where $U\tau^* K$ for each $U \in \mathcal{U}$. Thus, D_U is closed in X, $D_U \subset U$ for each $U \in \mathcal{U}$, and $\mathcal{D}$ is a locally finite closed cover of X. Clearly $U\tau^* K$ implies $D_U \tau K$, so $X\tau K$ follows from Proposition 2.1. □

COROLLARY 2.4. *Let X be a paracompact space and K a space. Then the following statements are equivalent:*

(1) *There exists an open cover $\mathcal{U}$ of X such that for each $U \in \mathcal{U}$, $U\tau K$.*
(2) *There exists an open cover $\mathcal{U}$ of X such that for each $U \in \mathcal{U}$, $U\tau^* K$.*

3. ABSOLUTE CO-EXTENSORS AND STRATIFIABLE SPACES

Restricting ourselves to the class of stratifiable spaces X, we recall (see [7]) some inherent properties.

(S1) X is hereditarily stratifiable,
(S2) X is paracompact (hereditarily paracompact because of (S1)),
(S3) if K is a CW-complex, then K is an absolute neighborhood extensor for X, and
(S4) X (because of (S3)) satisfies the homotopy extension property with respect to CW-complexes.

Let us state a corollary that follows from Theorem 2.2.

COROLLARY 3.1. *Let X be a stratifiable space and $\mathcal{U}$ a collection of open subsets of X such that each member of $\mathcal{U}$ is an absolute co-extensor for K. Then $\bigcup \mathcal{U}$ is an absolute co-extensor for K.*

Since the class of metrizable spaces is a subclass of the class of stratifiable spaces (again refer to [7]), Corollary 3.1 holds true in particular for metrizable spaces.

It is known that closed subsets of X in the definition of absolute co-extensor for K cannot be replaced by open sets. But, the property that X is an absolute co-extensor for K can be characterized in term of pairs of open subsets of X and maps on one of them. The following characterization for a stratifiable space X to be an absolute co-extensor for a CW-complex K

is obtained in [6, Theorem 5.5], where the property "to be an absolute co-extensor" was studied relative to the inverse limit of an inverse sequence of stratifiable spaces. Here we quote this theorem.

THEOREM 3.2. *Let X be a stratifiable space and K a CW-complex. Then X is an absolute co-extensor for K if and only if for each pair U, V of open subsets of X and map $f : U \to K$, there exist open subsets U^*, V^* of X, $U^* \subset U$, $V^* \subset V$, and a map $f^* : U^* \cup V^* \to K$ such that,*

(1) *$U^* \cup V^* = U \cup V$, and*
(2) *$f^*|U^*$ is homotopic to $f|U^*$.*

Restricting ourselves to nonempty open sets one has the following.

PROPOSITION 3.3. *If a stratifiable space X is an absolute co-extensor for a CW-complex K, then for each nonempty open subset U of X and map $f : U \to K$, there exist a nonempty open subset U^* of X, $U^* \subset U$, and a map $f^* : X \to K$ such that $f^*|U^* \simeq f|U^*$.*

PROOF. Let $p \in U$ and put $V = X \setminus \{p\}$. Then V is open because stratifiable spaces are T_1-spaces by definition. Observe that $U \cup V = X$ and apply Theorem 3.2. Thus there are open subsets U^* and V^* of X, $U^* \cup V^* = U \cup V = X$, $U^* \subset U$, $V^* \subset V$ and a map $f^* : X \to K$ such that (2) of that theorem is true. Clearly $p \in U^*$, so $U^* \neq \emptyset$. □

In Proposition 3.3 one can have a true extension on a smaller nonempty open subset of X, so we state,

PROPOSITION 3.4. *If a stratifiable space X is an absolute co-extensor for a CW-complex K, then for each nonempty open subset U of X and map $f : U \to K$ there exist a nonempty open subset U^* of X, $U^* \subset U$, and a map $f^* : X \to K$ such that $f^*|U^* = f|U^*$.*

PROOF. Consider the open subsets V, U^*, V^* obtained in the proof of Proposition 3.3. Then $\{U^*, V^*\}$ is an open cover of X. Property (S2) implies that X is normal, so an application of the shrinking theorem allows us to assume without loss of generality that $\operatorname{cl} U^* \subset U$. Since $X \tau K$, then $f|\operatorname{cl} U^* : \operatorname{cl} U^* \to K$ extends to a map $f^* : X \to K$, and again $U^* \neq \emptyset$. □

The next simply formulated sufficient condition for a metrizable space X to be an absolute co-extensor for a CW-complex K, given in terms of open sets, is due to Professor Akira Koyama (oral communication). Because of properties (S3) and (S4) of stratifiable spaces, the same argument holds if X is a stratifiable space.

PROPOSITION 3.5. *Let K be a CW-complex and X a stratifiable space. If for each open set U of X and map $f : U \to K$, there exists a map $F : X \to K$ such that $F|U \simeq f$, then $X \tau K$.*

PROOF. Let B be a closed subset of X and $\varphi : B \to K$ a map. Extend φ to a map $f : U \to K$ of an open neighborhood U of B using the previously mentioned property (S3) of X. Accordingly there is a map $F : X \to K$ such that $F|U \simeq f$. Then $\varphi \simeq F|B$, so apply property (S4) of X to obtain a map $\Phi : X \to K$ such that $\Phi|B = \varphi$, proving that $X\tau K$. □

Before stating our concluding proposition, we remind the reader of [5, Corollary 3.4].

PROPOSITION 3.6. *Let X be a stratifiable space, K a CW-complex, and $h : A \to K$ a map from an arbitrary subspace A of X. Then for some open neighborhood U of A, there exists a map $f : U \to K$ such that $f|A \simeq h$.*

PROPOSITION 3.7. *Let X be a stratifiable space and K a CW-complex. Then the following are equivalent:*

(1) *for every subset A of X and map $f : A \to K$, there exists a map $F : X \to K$ such that $F|A \simeq f$,*
(2) *for every open subset U of X and map $f : U \to K$, there exists a map $F : X \to K$ such that $F|U \simeq f$.*

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