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SHAPE PROPERTIES OF THE BOUNDARY OF ATTRACTORS

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Dedicated to Sibe Mardešić in the occasion of his 80^{th} *anniversary and in recognition of his guidance in the realm of geometric topology and shape theory*

Abstract. Let M be a locally compact metric space endowed with a continuous flow $\varphi : M \times \mathbb{R} \to M$. Assume that K is a stable attractor for φ and $P \subseteq \mathcal{A}(K)$ is a compact positively invariant neighbourhood of K contained in its basin of attraction. Then it is known that the inclusion $K \hookrightarrow P$ is a shape equivalence and the question we address here is whether there exists some relation between the shapes of ∂K and ∂P. The general answer is negative, as shown by example, but under certain hypotheses on K the shape domination $\mathrm{Sh}(\partial K) \geq \mathrm{Sh}(\partial P)$ or even the equality $\mathrm{Sh}(\partial K) = \mathrm{Sh}(\partial P)$ hold. However we also put under study interesting situations where those hypotheses are not satisfied, albeit other techniques such as Lefschetz's duality render results relevant to our question.

1. Introduction

1.1. *General setting.* Let $\varphi : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ be a continuous semiflow such that there exists a compact n–manifold $P \subseteq \mathbb{R}^n$ with the property that every orbit through ∂P enters P (for increasing time). This situation was considered by Hastings in his paper [15], and he proved –as some kind of generalized Poincaré–Bendixson theorem– that there exists a compact invariant set $K \subseteq \mathrm{int}(P)$ which is positively asymptotically stable and such that the inclusion $K \hookrightarrow P$ is a shape equivalence.

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The result of Hastings was the first one of a series of papers by different and unrelated authors (for example [2, 12, 14, 25, 26]) who analyze similar situations. Although all of them have different goals in mind, they share an underlying idea which can be considered a variant of Hasting's argument: if $\varphi : M \times \mathbb{R} \to M$ is a continuous flow defined on a locally compact metric space M and K is an attractor in M, then for every positively invariant neighbourhood P of K contained in its basin of attraction $\mathcal{A}(K)$ the inclusion $i : K \hookrightarrow P$ is a shape equivalence. This will be, therefore, our general setting.

In view of the result about the shapes of K and P stated in the previous paragraph, it is only natural to ask whether any relation can be ascertained between the shapes of their boundaries ∂K and ∂P. Let us remark that in [24], some topological properties of the boundary of the region of attraction $\mathcal{A}(K)$ were studied and the present article could be considered a continuation of [24]. The paper by Robinson and Tearne [22] contains a proof of the fact that ∂K agrees with the ω–limit of ∂P, which shows a dynamical connection between them. However we want our point of view to be more shape theoretical in nature, and this is what we develop in this article.

1.2. *Preliminary definitions and results.* We shall devote some short lines to recall the basic tools and notions which will be needed. The reader should be aware that due to the lack of a universally accepted notation, ours can depart from the one used in the cited references.

For any subset $S \subseteq M$, its alpha and omega–limit sets are defined by $\alpha(S) = \bigcap_{t \leq 0} \overline{S \cdot (-\infty, t]}$ and $\omega(S) = \bigcap_{t \geq 0} \overline{S \cdot [t, +\infty)}$ respectively, they are always closed and invariant. It is not difficult to show that when S is a compactum and $\alpha(S)$ (respectively $\omega(S)$) is compact, then it is connected too.

A compact invariant set K is an (asymptotically stable) *attractor* if it possesses a neighbourhood U such that $K = \omega(U)$, and in this case its *basin of attraction* $\mathcal{A}(K) = \{x \in M : \emptyset \neq \omega(x) \subseteq K\}$ is an open neighbourhood of K containing U. A useful characterization of attractors (see [1, Theorem 2.13, p. 73] for a precise statement) is by means of Lyapunov functions. A *Lyapunov function* for K is a continuous $\Phi : \mathcal{A}(K) \to [0, +\infty)$ such that (1) $\Phi|_K \equiv 0$, (2) Φ is strictly decreasing along trajectories in $\mathcal{A}(K) - K$ (explicitly, $\Phi(x \cdot t) < \Phi(x)$ for any $x \in \mathcal{A}(K) - K$ and $t > 0$). A good reference for this elementary theory is [1].

Following Conley we shall deal only with *isolated* compact invariant sets. These are compact invariant sets K which possess a so–called *isolating neighbourhood*, that is, a compact neighbourhood N such that K is the maximal invariant set in N. An *isolating block* N is an isolating neighbourhood such that there are compact sets $N^i, N^o \subseteq \partial N$, called the *entrance* and *exit* sets, satisfying

1. $\partial N = N^i \cup N^o$,

2. for every $x \in N^i$ there exists $\varepsilon > 0$ such that $x \cdot [-\varepsilon, 0) \subseteq M - N$ and for every $x \in N^o$ there exists $\delta > 0$ such that $x \cdot (0, \delta] \subseteq M - N$,
3. for every $x \in \partial N - N^i$ there exists $\varepsilon > 0$ such that $x \cdot [-\varepsilon, 0) \subseteq \operatorname{int} N$ and for every $x \in \partial N - N^o$ there exists $\delta > 0$ such that $x \cdot (0, \delta] \subseteq \operatorname{int} N$.

These blocks form a neighbourhood basis of K in M (see [8, 10]). Moreover, when M is a differentiable n–manifold and the flow is also differentiable, there exist isolating blocks N which are n–manifolds (with boundary ∂N) and such that $N^i, N^o \subseteq \partial N$ are also $(n-1)$–manifolds with common boundary $N^i \cap N^o$ (in ∂N).

The *Conley index* of K is defined as the homotopy type $h(K)$ of the pointed quotient $\big(N/N^o, [N^o]\big)$, where N is any isolating block for N; this can be shown to be independent of the particular N chosen. The monography of Conley [9] is probably the most readable reference although correct proofs should be sought for in [23]. In a similar fashion one can define the *shape index* of K, which is the shape $s(K)$ of the pointed quotient $\big(N/N^o, [N^o]\big)$, or the *cohomological index* of K as $C\check{H}(K) = \check{H}^*\big(N/N^o, [N^o]\big)$.

The book by Spanier [27] is our main reference for algebraic topology. Let us note here that whenever cohomology or homology are used, reduced Čech theory is intended. This theory agrees with the singular one when the underlying space is an ANR (for information about them see [3, 16]), in particular a differentiable manifold. Finally, the essentials of shape theory are contained in [5] or, for more exhaustive information, [11, 18, 19] and the books [17, 20].

2. Sufficient conditions for the domination $\mathrm{Sh}(\partial K) \geq \mathrm{Sh}(\partial P)$

Let us start our programme by showing that, in a general case, no relation can be expected to hold between the shapes of ∂K and ∂P. The following example furnishes a proof of this fact.

Example 2.1. Consider a three dimensional solid torus $P \subseteq \mathbb{R}^3$ and a necklace $K \subseteq \operatorname{int}(P)$ consisting of infinitely many solid balls, tangent to each other and centered at the core circumference of P, which we shall denote L (see Figure 1). It is not difficult to construct a differentiable flow which enters P transversally through its boundary ∂P and has K as a stable attractor. Here the boundary ∂K of K consists of infinitely many hollow spheres (tangent to each other) and $\check{H}^1(\partial K) \cong \mathbb{Z}$, whereas $\check{H}^2(\partial K)$ is not finitely generated. Hence, since $\check{H}^1(\partial P) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\check{H}^2(\partial P) \cong \mathbb{Z}$, neither ∂K dominates ∂P nor the latter dominates the former.

There is a natural way to move points of ∂P into any neighbourhood of ∂K, just by pushing them with the flow (this defines a shape morphism from ∂P to ∂K). But to obtain some domination relation another shape morphism in the opposite sense is needed, and this cannot be afforded by the

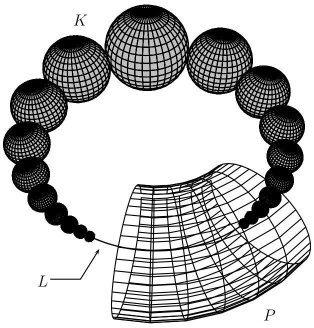

FIGURE 1. An attractor K with complicated ∂K.

flow alone. Therefore let us introduce the following definition, whose interest will be justified by Theorem 2.3.

DEFINITION 2.2. *Let $P \subseteq M$ be a compact set. A homotopical spine for P is a compact set $L \subseteq \mathrm{int}(P)$ such that the inclusion $\partial P \hookrightarrow P - L$ is a homotopy equivalence.*

The notions of *spine* and *pseudospine* of a manifold make frequent appearance in mathematical literature, but there does not seem to be a universally accepted definition of them. However it appears that any of the classical definitions which can be found imply that any spine or pseudospine is a homotopical spine. Hence the latter is the less restrictive of the three, although it will be powerful enough for our purposes.

THEOREM 2.3. *Let K be an attractor in a locally compact metric space M and $P \subseteq \mathcal{A}(K)$ a positively invariant compact neighbourhood of K. Assume that $\mathrm{int}(K)$ contains a homotopical spine L of P. Then $\mathrm{Sh}(\partial K) \geq \mathrm{Sh}(\partial P)$.*

PROOF. Consider a basis of open neighborhoods

$$P - L \supseteq W_1 \supseteq W_2 \supseteq \ldots \supseteq W_n \supseteq \ldots \supseteq \partial K$$

of ∂K in M. Each union $U_n = W_n \cup K$ is an open neighborhood of K in M and, since P is contained in the basin of attraction of K, there exists an increasing sequence of times $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq \ldots$ such that $P \cdot [t_n, +\infty) \subseteq U_n$. Moreover, since $M - K$ is invariant by the flow and $\partial P \subseteq M - K$ we necessarily have that $\partial P \cdot [t_n, +\infty) \subseteq U_n \cap (M - K) \subseteq W_n$. Thus, setting

$$f_n(x) = x \cdot t_n$$

for every $x \in \partial P$ we obtain a family of maps $f_n : \partial P \to W_n$ which define an approximative map

$$\mathbf{f} = \{f_n : \partial P \to \partial K\}$$

in the sense of K. Borsuk [5]. As P is positively invariant we have that

$$j_n \circ f_n \simeq k$$

where $j_n : W_n \hookrightarrow P - L$ and $k : \partial P \hookrightarrow P - L$ are inclusions.

Since L is a homotopical spine for P, the inclusion k is a homotopy equivalence and has a homotopy inverse $\psi : P - L \to \partial P$ which induces by restriction a map $g = \psi|_{\partial K} : \partial K \to \partial P$. Let us call α and β the shape morphisms induced by $\mathbf{f}$ and g respectively. Observe that $\{\psi|_{W_n} \circ f_n : \partial P \to \partial P\}$ is an approximative map for the composition $\beta \circ \alpha$ and

$$\psi|_{W_n} \circ f_n = \psi \circ j_n \circ f_n \simeq \psi \circ k \simeq \mathrm{id}_{\partial P}$$

whence we deduce that $\beta \circ \alpha$ is the identity shape morphism in ∂P. This proves the claim that $\mathrm{Sh}(\partial K) \geq \mathrm{Sh}(\partial P)$. ∎

It is clear that any ball in $\mathbb{R}^n$ has any interior point as a homotopical spine. Therefore the following corollary holds:

Corollary 2.4. *Let K be an attractor in $\mathbb{R}^n$ contained in the interior of a positively invariant n–ball B. If* $\mathrm{int}(K) \neq \emptyset$ *then* $\mathrm{Sh}(\partial K) \geq \mathrm{Sh}(\mathbb{S}^{n-1})$.

Another natural situation where the hypotheses of Theorem 2.3 above hold can be described thus:

Corollary 2.5. *Let K be an attractor such that ∂K can be bicollared (for example, this happens if the phase space is a differentiable manifold and ∂K is an orientable hypersurface). Then* $\mathrm{Sh}(\partial K) \geq \mathrm{Sh}(\partial P)$ *for any positively invariant compact neighbourhood $P \subseteq \mathcal{A}(K)$ of K.*

Proof. We shall assume without loss of generality (by Proposition 5.1) that ∂P is a section of the flow in $\mathcal{A}(K) - K$. Let $\mathrm{int}(P) \supseteq U \supseteq \partial K$ be the bicollar for ∂K, which means that there exists a homeomorphism $h : \partial K \times (-1, 1) \to U$ such that $h(x, 0) = x$ for all $x \in \partial K$ and U is an open neighbourhood of ∂K. Denote $\frac{1}{2}U$ the image under h of the set $\partial K \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, which is half the bicollar U: then it is easy to construct a homeomorphism $k : P \to P$ such that $k\left(K \cup \frac{1}{2}U\right) \subseteq K$ and k is the identity outside U. Observe that moreover, since $K \cup \frac{1}{2}U$ is open in P, the set $k\left(K \cup \frac{1}{2}U\right)$ is open in P too, hence $k\left(K \cup \frac{1}{2}U\right) \subseteq \mathrm{int}_P(K) = \mathrm{int}(K)$.

Now $K \cup \frac{1}{2}U$ is an open neighbourhood of K so there exists $T > 0$ such that $P \cdot [T, +\infty) \subseteq K \cup \frac{1}{2}U$. Let $L = k\big(P \cdot [T, +\infty)\big) \subseteq k\left(K \cup \frac{1}{2}U\right) \subseteq \mathrm{int}(K)$, which is clearly a compact set. We assert that L is a homotopical spine for P. In fact, the assumption that ∂P be a section of the flow implies $P - P \cdot [T, +\infty) = \partial P \cdot [0, T)$ which deformation retracts onto ∂P, hence the inclusion $\partial P \hookrightarrow P - P \cdot [T, +\infty)$ is a homotopy equivalence. Then in turn $\partial P = k(\partial P) \hookrightarrow k\big(P - P \cdot [T, +\infty)\big) = P - L$ is a homotopy equivalence too and L is a homotopical spine for P. The theorem above applies and thus the corollary follows. ∎

Example 2.1 shows an instance where K contains a homotopical spine of P (the core circumference L) but the domination relation does not hold. Observe however that it is not possible to find a homotopical spine for P entirely contained in $\mathrm{int}(K)$, as required by Theorem 2.3.

3. Sufficient conditions for the equality $\mathrm{Sh}(\partial K) = \mathrm{Sh}(\partial P)$

THEOREM 3.1. *Let K be an attractor in a locally compact metric space M and $P \subseteq \mathcal{A}(K)$ a positively invariant compact neighbourhood of K. Assume that ∂K is an attractor for the flow restricted to K whose dual repeller in K is contained in some homotopical spine $L \subseteq \mathrm{int}(K)$ of P. Then* $\mathrm{Sh}(\partial K) = \mathrm{Sh}(\partial P)$.

PROOF. Let us show that the hypotheses guarantee that ∂K is an attractor in $\mathcal{A}(K) - R$, where R denotes the dual repeller of K for the flow restricted to ∂K. A very short way of proving this is via Lyapunov functions: since K is an attractor in $\mathcal{A}(K)$, there exists a continuous function $\Phi_1 : \mathcal{A}(K) \to [0, +\infty)$ which is strictly decreasing along the segments of trajectories contained in $\mathcal{A}(K) - K$ and identically zero on K. Similarly, another Lyapunov function $\Phi_2 : K - R \to [0, +\infty)$ can be found with analogous properties because ∂K is an attractor in K with basin of attraction $K - R$. Now Φ_1 agrees with Φ_2 on ∂K (both are identically zero), hence they can be glued together to obtain a well–defined continuous function $\Phi : \mathcal{A}(K) - R \to [0, +\infty)$ given by

$$\Phi(x) = \begin{cases} \Phi_1(x) & \text{if } x \notin K \\ \Phi_2(x) & \text{if } x \in K \end{cases}$$

which is still strictly decreasing along trajectory segments contained in $\mathcal{A}(K) - R$ and disjoint from ∂K, for any such segment is connected and must lie either in $K - R$ or in $\mathcal{A}(K) - K$. Therefore ∂K is an attractor whose basin of attraction contains $\mathcal{A}(K) - R$, and in fact agrees with it because R and $\partial\mathcal{A}(K)$ are invariant and so not attracted by ∂K.

We shall keep the notations of Theorem 2.3 and let $j : \partial K \hookrightarrow P - L$ be the inclusion. Observe that, by the very definition of k and ψ, the composition $k \circ g = k \circ \psi \circ j \simeq_H j$ via a homotopy $H : \partial K \times [0, 1] \to P - L$. More explicitly stated, H is a continuous map such that $H(x, 0) = (k \circ g)(x) = g(x)$ and $H(x, 1) = j(x) = x$ for all $x \in \partial K$.

Since ∂K is an attractor in $\mathcal{A}(K) - R$, we can assume without loss of generality that the neighbourhoods W_n considered in Theorem 2.3 are positively invariant. This, together with the fact that $\mathrm{im}(H)$ is a compact set in $P - L \subseteq \mathcal{A}(K) - R$, implies that there exists an increasing sequence of times $0 \le s_1 \le s_2 \le \ldots \le s_n \le \ldots$ which satisfies $\mathrm{im}(H) \cdot [s_n, +\infty) \subseteq W_n$. Letting $H_n : \partial K \times [0, 1] \to W_n$ be the continuous map given by $H_n(x, t) = H(x, t) \cdot s_n$, we have $H_n(x, 0) = H(x, 0) \cdot s_n = g(x) \cdot s_n$ and $H_n(x, 1) = H(x, 1) \cdot s_n = x \cdot s_n$

for every $x \in \partial K$. Because of the positive invariance of W_n and the invariance of ∂K it is easily seen that $x \mapsto H_n(x,0) = g(x) \cdot s_n$ and $x \mapsto (f_n \circ g)(x) = g(x) \cdot t_n$, as well as $x \mapsto H_n(x,1) = x \cdot s_n$ and $\mathrm{id}_{\partial K}$, are homotopic in W_n. Therefore $f_n \circ g \simeq \mathrm{id}_{\partial K}$ in W_n for every $n \in \mathbb{N}$ so $\alpha \circ \beta = \mathrm{id}_{\partial K}$ in the shape category. In Theorem 2.3 it was proved that $\beta \circ \alpha = \mathrm{id}_{\partial P}$, hence α is a shape equivalence between ∂P and ∂K. □

REMARK 3.2. It is clear that a result analogous to the previous one could be obtained if ∂K were a repeller for the flow restricted to K whose dual attractor A was contained in a homotopical spine $L \subseteq \mathrm{int}(K)$ of P.

Just as we did above, we shall present the reader with some suggestive situations where Theorem 3.1 holds. Let us begin with a situation where dynamical (Conley index–type) conditions are imposed on ∂K. This is motivated by previous work contained in [24].

COROLLARY 3.3. *Let $P \subseteq \mathcal{A}(K)$ be a positively invariant compact neighbourhood of an attractor K for a differentiable flow in an orientable n–manifold M. Assume that ∂K is isolated with cohomological $(n-1)$–dimensional Conley index $C\check{H}^{n-1}(\partial K) = 0$ and that the maximal compact invariant set $A \subseteq \mathrm{int}(K)$ (which always exists under the present hypotheses, as will be shown below) is a homotopical spine L for P. Then* $\mathrm{Sh}(\partial K) = \mathrm{Sh}(\partial P)$.

PROOF. Let $\Sigma \subseteq \mathcal{A}(K) - K$ be any section of the flow: since Σ is a compact retract of the ANR $\mathcal{A}(K) - K$, it is a compact ANR (see [28]). Therefore it has finitely many compact connected components $\Sigma_1, \dots, \Sigma_k$. The equality $\partial K = \omega(\Sigma)$ which follows from [22, Theorem 2.1] implies that $\partial K = \omega(\Sigma_1) \cup \dots \cup \omega(\Sigma_k)$ has at most k components, because the ω–limit of a compactum, when compact, is connected. This allows us to assume that ∂K is connected without any loss of generality because in other case the following argument should be applied to each one of its components.

Since the flow is differentiable and $\mathcal{A}(K) - A$ is an open neighbourhood of the isolated invariant set ∂K, there exists a connected isolating block $N \subseteq \mathcal{A}(K) - A$ which is an n–manifold and such that N^i and N^o are $(n-1)$–submanifolds of ∂N, too. Observe moreover that, since both N and M are n–dimensional, $N - \partial N$ is an open submanifold of M, hence orientable. Therefore we can apply Lefschetz's duality to obtain $H_1(N, N^i) = H^{n-1}(N, N^o) = 0$. Now from the exact sequence

$$0 \to \check{H}^0(N, N^i) \to \check{H}^0(N) \to \check{H}^0(N^i) \to \check{H}^1(N, N^i) = 0 \to \dots$$

and the assumption that N is connected (hence $\check{H}^0(N) = 0$, recall we are using reduced Čech cohomology) it follows that N^i is connected. Since K is an attractor in $\mathcal{A}(K)$, it is clear that $\big(\mathcal{A}(K) - K\big) \cap N^i \neq \emptyset$ and this implies that every point in $\partial N \cap \mathrm{int}(K)$ belongs to N^o, hence ∂K is a repeller in K. Now its dual attractor $(\partial K)^*$ agrees with A (it is clear that $(\partial K)^* \subseteq A$, for the other inclusion note that $A = \omega(A) \subseteq (\partial K)^*$). Then apply Theorem 3.1 or Remark 3.2 and the conclusion follows. □

Remark 3.4. If the assumption on the orientability of M is dropped the whole argument goes through just taking coefficients in $\mathbb{Z}_2$.

Just as an illustration of how the combination of the different techniques we have been using can yield interesting results in our context let us include this corollary.

Corollary 3.5. *Let $P \subseteq \mathcal{A}(K)$ be a positively invariant compact neighbourhood of an attractor K for a differentiable flow in an orientable n–manifold M. Assume that ∂K is an orientable $(n-1)$–manifold which is isolated and $C\check{H}^{n-1}(\partial K) = 0$. Then ∂K and ∂P are homotopy equivalent.*

Proof. The fact that $C\check{H}^{n-1}(\partial K) = 0$ implies that ∂K is a repeller in K, just as in the preceding corollary, denote $A \subseteq \mathrm{int}(K)$ its dual attractor. Now use the fact that ∂K is an orientable $(n-1)$–manifold to construct a bicollar for it and make minor adjustments to the proof of Corollary 2.5 to deduce that there exists a homotopical spine $L \subseteq \mathrm{int}(K)$ which contains A. Finally recall that ∂K and ∂P are ANR's (the former because it is a manifold, the latter because of Corollary 5.5), so not only their shapes agree but also their homotopy types. □

4. Another situation of interest

So far we have been assuming (tacitly) that $\mathrm{int}(K) \neq \emptyset$, because a homotopical spine for P was required to be contained in it. In case $\mathrm{int}(K) \neq \emptyset$ but such a spine cannot be found, Example 2.1 shows that nothing can be asserted. But what if $\mathrm{int}(K) = \emptyset$? The next proposition shows how to resort to other kind of techniques to render results in this new context.

Proposition 4.1. *Let K be a connected attractor for a differentiable flow. Suppose that $\mathrm{int}(K) = \emptyset$ (thus K coincides with its own boundary) and let r denote the number of connected components of ∂P, where $P \subseteq \mathcal{A}(K)$ is any positively invariant compact neighbourhood of K. If $\check{H}_1(K) = 0$ then $\check{H}^{n-1}(K) = \check{H}_{n-1}(K) = \mathbb{Z}^{r-1}$.*

Proof. We can assume without loss of generality by the remark following Proposition 5.1 that P is an n–manifold with boundary ∂P. Let us prove that $H_0(P) = H^0(P) = \mathbb{Z}$, $H_1(P) = H^1(P) = 0$ and $H_n(P) = H^n(P) = 0$. To this end, recall that the inclusion $i : K \hookrightarrow P$ is a shape equivalence (hence it induces isomorphisms in Čech homology and cohomology, which agree with the singular theory on P) so the hypotheses that K be connected and $\check{H}_1(K) = 0$ imply that P is connected (therefore $H_0(P) = H^0(P) = \mathbb{Z}$) and $H_1(P) = \check{H}_1(P) = \check{H}_1(K) = 0$. By means of the universal coefficient theorem, $H^1(P) = \tau H_0(P) \oplus \beta H_1(P) = \tau\mathbb{Z} \oplus \beta 0 = 0$ (we will use τG and βG to denote the torsion and free parts of an abelian group G, respectively). Finally, since P is a connected n–manifold with boundary $H_n(P) = H^n(P) = 0$.

Now consider the following portions of the long exact sequences in homology and cohomology for the pair $(P, \partial P)$:

$$\ldots \to H_n(P) \to H_n(P,\partial P) \to H_{n-1}(\partial P) \to H_{n-1}(P) \to H_{n-1}(P,\partial P) \to \ldots$$

$$\ldots \to H^{n-1}(P,\partial P) \to H^{n-1}(P) \to H^{n-1}(\partial P) \to H^n(P,\partial P) \to H^n(P) \to \ldots$$

By Lefschetz's duality $H^k(P,\partial P) = H_{n-k}(P)$ and $H_k(P,\partial P) = H^{n-k}(P)$ together with the homology and cohomology groups calculated above we get

$$\ldots \to 0 \to \mathbb{Z} \to H_{n-1}(\partial P) \to H_{n-1}(P) \to 0 \to \ldots$$

$$\ldots \to 0 \to H^{n-1}(P) \to H^{n-1}(\partial P) \to \mathbb{Z} \to 0 \to \ldots$$

The boundary ∂P of P is a compact $(n-1)$–manifold (without boundary) with r connected components, hence $H_{n-1}(\partial P) = H^{n-1}(\partial P) = \mathbb{Z}^r$. It only remains to calculate $H^{n-1}(P)$ and $H_{n-1}(P)$, since these agree with $\check{H}^{n-1}(P) = \check{H}^{n-1}(K)$ and $\check{H}_{n-1}(P) = \check{H}_{n-1}(K)$. From the second exact sequence, which is split because $\mathbb{Z}$ is free, it is readily deduced that $\mathbb{Z}^r = H^{n-1}(\partial P) = H^{n-1}(P) \oplus \mathbb{Z}$ so it must be $H^{n-1}(P) = \mathbb{Z}^{r-1}$. For the statement in homology observe that $0 = H^n(P) = \tau H_{n-1}(P) \oplus \beta H_n(P)$ implies that $H_{n-1}(P)$ is free (because it has no torsion and is finitely generated), hence from the first exact sequence it follows that $H_{n-1}(P) = \mathbb{Z}^{r-1}$. □

5. Addendum: some remarks on ∂P

We have devoted this paper to obtain results relating the shapes of ∂K and ∂P when K is an attractor and $P \subseteq \mathcal{A}(K)$ is a positively invariant compact neighbourhood of K. However ∂K is uniquely determined, but not so ∂P because it depends on the particular neighbourhood of K chosen. Therefore Proposition 5.1 should be of interest, if only for the sake of completeness, although in fact it also renders some useful information for our purposes.

Proposition 5.1. *Let K be an attractor and $P \subseteq \mathcal{A}(K)$ any compact positively invariant neighbourhood of K contained in its basin of attraction. Then* $\mathrm{Sh}(\partial P) = \mathrm{Sh}(\Sigma) = \mathrm{Sh}\big(\mathcal{A}(K) - K\big)$, *where $\Sigma \subseteq \mathcal{A}(K) - K$ is any section of the flow.*

Proof. Step 1. There exists a continuous function $\Phi : \mathcal{A}(K) \to [0, +\infty)$ such that (1) $\Phi(x) = 0 \Leftrightarrow x \in P$ and (2) $\Phi(x \cdot t) < \Phi(x)$ whenever $x \cdot [0,t] \subseteq \mathcal{A}(K) - P$.

Proof of Step 1. We shall just outline the proof of this step since it is analogous to the construction of Lyapunov functions. Let d be a bounded metric for the phase space and consider the function $\psi(x) = \sup\big\{\mathrm{d}(x \cdot s, P) : s \geq 0\big\}$ defined on $\mathcal{A}(K)$. It is clear that $\psi(x) = 0 \Leftrightarrow x \in P$ and $\psi(x \cdot t) \leq \psi(x)$ for every $x \in \mathcal{A}(K)$ and $t > 0$. Now to check that ψ is continuous at any point

$x_0 \in \mathcal{A}(K)$ pick $t_0 > 0$ and a neighbourhood U_0 of x_0 such that $U_0 \cdot t_0 \subseteq \operatorname{int}(P)$. Then $\psi(y) = \max\{\mathrm{d}(y \cdot s, P) : 0 \leq s \leq t_0\}$ for any $y \in U_0$ because of the positive invariance of P. With this local expression it is not difficult to see that ψ is continuous.

Although the mapping ψ does not solve the problem because of the strict inequality in (2), this difficulty is easily overcome by means of the following standard argument: let $\Phi(x) = \int_0^{+\infty} e^{-\tau}\psi(x \cdot \tau)\, d\tau$ and observe that clearly Φ is continuous and has property (1). As for (2), an easy calculation of the derivative of $t \mapsto \Phi(x \cdot t)$ shows that Φ decreases along trajectory segments contained in $\mathcal{A}(K) - P$. □

Step 2(a). Let U be any precompact open neighbourhood of ∂P. Then there exists a section $\Sigma_o \subseteq U$ of the flow in $\mathcal{A}(K) - K$ such that $\Sigma_o \cap P = \emptyset$ and for any $x \in \Sigma_o$, whenever $t \geq 0$ and $x \cdot t \notin P$ then $x \cdot t \in U$ (that is, while the positive semitrajectory of x lies outside P, it is contained in U) or, equivalently, $\Sigma_o \cdot [0, +\infty) \subseteq P \cup U$.

Proof of Step 2(a). Let $V = P \cup U$, which is a precompact open neighbourhood of P, and set $\mu = \min\{\Phi(x) : x \in \partial V\}$. Since $\Phi(x) > 0$ for all $x \in \partial V$ and ∂V is compact, it follows that $\mu > 0$, hence the set $\Sigma_o = \{x \in \mathcal{A}(K) : \Phi(x) = \frac{1}{2}\mu\}$ is compact and disjoint from P. The proof will be finished if we show that Σ_o is a section and $\Sigma_o \cdot [0, +\infty) \subseteq V$.

Consider the mapping $\sigma : \Sigma_o \to \Sigma$ which assigns to every $x \in \Sigma_o$ the unique $\sigma(x) \in \gamma(x) \cap \Sigma$ (where $\gamma(x)$ stands for the trajectory of x, that is, the set $x \cdot \mathbb{R}$). It is continuous because Σ is a section of $\mathcal{A}(K) - K$ and, moreover, Σ_o is also a section of $\mathcal{A}(K) - K$ if and only if σ is a homeomorphism or, equivalently, σ is bijective (since Σ_o is compact). Now if it were not injective there would exist $x \in \Sigma_o$ and $t > 0$ such that $x \cdot t \in \Sigma_o$, but then $x \cdot [0, t] \subseteq \mathcal{A}(K) - P$ because of the positive invariance of P and $\frac{1}{2}\mu = \Phi(x \cdot t) < \Phi(x) = \frac{1}{2}\mu$ which is absurd. Finally to check the surjectivity assume that there exists $y \in \Sigma$ such that $y \cdot t \notin \Sigma_o$ for any $t \in \mathbb{R}$, that is either $\Phi(y \cdot t) < \frac{1}{2}\mu$ or $\Phi(y \cdot t) > \frac{1}{2}\mu$ for every $t \in \mathbb{R}$. However $\Phi(y \cdot t) = 0$ for large enough t, hence it must be the case that $\Phi(y \cdot t) < \frac{1}{2}\mu$ for all $t \in \mathbb{R}$. This implies at once that $\gamma(y) \cap \partial V = \emptyset$ and so $y \cdot \mathbb{R} \subseteq V$, therefore $\emptyset \neq \alpha(y) \subseteq \overline{V}$ and $y \in K$ which contradicts $y \in \Sigma$.

Finally, the assertion that $\Sigma_o \cdot [0, +\infty) \subseteq V$ can be proved thus: if there existed $x \in \Sigma_o$ and $t \geq 0$ such that $x \cdot t \notin V$, then (since V is a neighbourhood of the attractor K) for some $0 \leq t \leq s$ we would have $x \cdot [t, s] \subseteq \mathcal{A}(K) - V$ and $x \cdot s \in \partial V$. Thus $\mu \leq \Phi(x \cdot s) \leq \Phi(x \cdot t) \leq \Phi(x) = \frac{1}{2}\mu$, in contradiction with $\mu > 0$. □

Step 2(b). Let U be any precompact open neighbourhood of ∂P. Then there exists a section $\Sigma_i \subseteq U$ of the flow in $\mathcal{A}(K) - K$ such that $\Sigma_i \subseteq \operatorname{int}(P)$ and for any $x \in \Sigma_i$, whenever $t \leq 0$ and $x \cdot t \in \operatorname{int}(P)$ then $x \cdot t \in U$ (that is, while the negative semitrajectory of x lies inside $\mathrm{int}(P)$, it is contained in U) or, equivalently, $\Sigma_i \cdot (-\infty, 0] \cap \mathrm{int}(P) \subseteq U$.

Proof of Step 2(b). The argument is dual to the one explained in Step 2(a) above. Observe that $P^* = \mathcal{A}(K) - \mathrm{int}(P)$ is a negatively invariant closed set and (taking some care because now P^* need not be compact) there exists a continous function $\Phi^* : \mathcal{A}(K) - K \to [0, +\infty)$ which is null exactly on P^* and strictly increasing along trajectory segments contained in $\mathcal{A}(K) - (K \cup P^*)$. Now a careful reading of Step 2(a) together with some adjustments will furnish a proof for this case. ∎

Step 3. Let U be a precompact open neighbourhood of ∂P. Then there exist sections $\Sigma_o, \Sigma_i \subseteq U$ of the flow in $\mathcal{A}(K) - K$ such that the set comprised between them $[\Sigma_o, \Sigma_i] = \{x \in \mathcal{A}(K) : x \cdot t_o \in \Sigma_o \text{ and } x \cdot t_i \in \Sigma_i \text{ for some } t_o \leq 0 \leq t_i\}$ is a compact neighbourhood of ∂P contained in U and the inclusion $[\Sigma_o, \Sigma_i] \hookrightarrow \mathcal{A}(K) - K$ is a homotopy equivalence.

Proof of Step 3. First of all observe that the real numbers $t_o(x) \leq 0 \leq t_i(x)$ such that $x \cdot t_o(x) \in \Sigma_o$ and $x \cdot t_i(x) \in \Sigma_i$ are uniquely determined by $x \in \mathcal{A}(K) - K$ and in fact depend continuously on it since both Σ_o and Σ_i are sections of $\mathcal{A}(K) - K$. A similar assertion holds true for the unique $t(x)$ such that $x \cdot t(x) \in \Sigma$.

For any $x \in \partial P \subseteq P$ the positive invariance of P implies that $x \cdot [0, +\infty) \subseteq P$ hence since $\Sigma_o \cap P = \emptyset$ and $x \cdot t_o(x) \in \Sigma_o$ the inequality $t_o(x) < 0$ must hold. In addition $\mathrm{int}(P)$ is also positively invariant and $x \cdot t_i(x) \in \Sigma_i \subseteq \mathrm{int}(P)$, therefore $t_i(x) > 0$ necessarily. This means that $\partial P \subseteq \{x \in \mathcal{A}(K) - K : t_o(x) < 0 < t_i(x)\}$, which is clearly an open set contained in $[\Sigma_o, \Sigma_i]$. Hence $[\Sigma_o, \Sigma_i]$ is a neighbourhood of ∂P.

We claim that the set $A = \{x \cdot s : x \in \Sigma_o, 0 \leq s \leq t_i(x)\}$ is just an alternative description of $[\Sigma_o, \Sigma_i]$. In fact for any $x \in \Sigma_o$ and $0 \leq s \leq t_i(x)$ we have $t_o(x \cdot s) = t_o(x) - s = -s \leq 0$ and $t_i(x \cdot s) = t_i(x) - s \geq 0$, hence $x \cdot s \in [\Sigma_i, \Sigma_o]$ and this proves $A \subseteq [\Sigma_o, \Sigma_i]$. Reciprocally, if $y \in [\Sigma_i, \Sigma_o]$, then setting $x = y \cdot t_o(y) \in \Sigma_o$ and $s = -t_o(y)$ the inequality $t_i(x) = t_i(y) - t_o(y) \geq -t_o(y) = s$ holds, which together with $y = x \cdot s$ implies that $y \in A$. In a dual fashion it can be shown that $[\Sigma_o, \Sigma_i] = \{x \cdot s : x \in \Sigma_i, t_o(x) \leq s \leq 0\}$.

Now we are in a position to prove that $[\Sigma_o, \Sigma_i]$ is a compact set contained in U. Let M be the maximum of the continuous function t_i on the compact set Σ_o: then it is clear by the above description that $[\Sigma_o, \Sigma_i] \subseteq \Sigma_o \cdot [0, M]$, which is a compact set. Since $[\Sigma_o, \Sigma_i]$ is closed because of the continuity of the t_i and t_o, it follows that it is also compact. Further, $[\Sigma_o, \Sigma_i] - P \subseteq \Sigma_o \cdot [0, +\infty) - P \subseteq (P \cup U) - P \subseteq U$ and similarly $[\Sigma_o, \Sigma_i] \cap \mathrm{int}(P) \subseteq \Sigma_i \cdot (-\infty, 0] \cap \mathrm{int}(P) \subseteq U$, so $[\Sigma_o, \Sigma_i] \subseteq U$.

Finally, the inclusion $i : [\Sigma_i, \Sigma_o] \hookrightarrow \mathcal{A}(K) - K$ is a homotopy equivalence because there exists a deformation retraction

$$H(x,\tau) := \begin{cases} x \cdot \big(t_o(x)\tau\big) & \text{if } t_o(x) > 0, \\ x & \text{if } t_o(x) \leq 0 \leq t_i(x), \\ x \cdot \big(t_i(x)\tau\big) & \text{if } t_i(x) < 0 \end{cases}$$

of $\mathcal{A}(K) - K$ onto $[\Sigma_i, \Sigma_o]$. □

STEP 4. Because of Step 3 there exists a decreasing sequence $(A_k)_{k=1}^{\infty}$ of sets of the form $[\Sigma_o, \Sigma_i]$ which form a compact neighbourhood basis of ∂P. Now every inclusion $A_{k+1} \hookrightarrow A_k$ is a homotopy equivalence because both inclusions $A_k, A_{k+1} \hookrightarrow \mathcal{A}(K) - K$ are. However since $\mathcal{A}(K) - K$ is homeomorphic to $\Sigma \times \mathbb{R}$, it has the same homotopy type as Σ, therefore $\partial P = \bigcap_{k=1}^{\infty} A_k$ is the inverse limit of a sequence of compact spaces all of which are bonded by homotopy equivalences and such that every single one of them is homotopy equivalent to Σ. Then $\mathrm{Sh}(\partial P) = \mathrm{Sh}(\Sigma) = \mathrm{Sh}(\Sigma \times \mathbb{R}) = \mathrm{Sh}\big(\mathcal{A}(K) - K\big)$. □

REMARK 5.2. One of the byproducts of Proposition 5.1 is that it allows one to suppose, when proving results about $\mathrm{Sh}(\partial P)$, that P is especially nice. For example, if $\Sigma \subseteq \mathcal{A}(K) - K$ is any compact section of the flow, the set $Q = K \cup \Sigma \cdot [0, +\infty)$ is a positively invariant compact neighbourhood of K with $\partial Q = \Sigma$ and this justifies the assumption made in the proof of Corollary 2.5. Or, in another instance, if the flow is differentiable (and the phase space is a differentiable n–manifold) then there exists a positively invariant neighbourhood Q which is an n–manifold with boundary (see, for example, [10]). This was used in Proposition 4.1 and allowed Lefschetz's duality theorem to be applied.

COROLLARY 5.3. *Let K be an attractor in an orientable n–manifold M and let $P \subseteq \mathcal{A}(K)$ be a positively invariant compact neighbourhood of K. Then there exists an exact sequence*

$$\ldots \to \check{H}^{n-(k+1)}(K) \to \check{H}_k(\partial P) \to \check{H}_k(K) \to \ldots$$

in unreduced homology and cohomology.

PROOF. Since $\mathcal{A}(K)$ is open in M, it is also an orientable n–manifold. Hence (see [27, Theorem 6.2.17]) there is an isomorphism $H_k\big(\mathcal{A}(K), \mathcal{A}(K) - K\big) = \check{H}^{n-k}(K)$ and so the long exact sequence in homology for the pair $\big(\mathcal{A}(K), K\big)$ gives rise to the exact sequence

$$\ldots \to \check{H}_k\big(\mathcal{A}(K) - K\big) \to \check{H}_k\big(\mathcal{A}(K)\big) \to \check{H}^{n-k}(K) \to \ldots$$

(singular and Čech homology agree on $\mathcal{A}(K)$ and $\mathcal{A}(K) - K$ because they are manifolds). Now it was proved in Proposition 5.1 that $\mathrm{Sh}(\partial P) = \mathrm{Sh}\big(\mathcal{A}(K) - K\big)$ and it is known that $\mathrm{Sh}(K) = \mathrm{Sh}\big(\mathcal{A}(K)\big)$. Hence $\check{H}_k(\partial P) = \check{H}_k\big(\mathcal{A}(K) - K\big)$ and

$\check{H}_k(K) = \check{H}_k(\mathcal{A}(K))$, which upon substitution in the above exact sequence yields the result. □

EXAMPLE 5.4. Let $K \subseteq \mathbb{R}^3$ be an attractor. Assume that K has trivial shape and nonempty interior: then $\mathrm{Sh}(\partial K) \geq \mathrm{Sh}(\mathbb{S}^2)$. To prove this, let $P \subseteq \mathcal{A}(K)$ be a positively invariant compact neighbourhood of K such that ∂P is a compact section of $\mathcal{A}(K) - K$. By means of the exact sequence given by the corollary above it is easy to prove that ∂P has the same homology groups as the 2–sphere $\mathbb{S}^2$. Moreover, we assert that ∂P is homeomorphic to $\mathbb{S}^2$. In fact, $\partial P \times \mathbb{R}$ is homeomorphic to $\mathcal{A}(K) - K$, which is an open 3–manifold and thus a generalized 3–manifold. Therefore ∂P is an orientable compact generalized 2–manifold (see [21, Theorem 6]), in which case it is an orientable compact 2–manifold, and by the classification theorem we conclude that ∂P is a 2–sphere (this argument is taken from [7]). Observe also that it is a bicollared sphere, hence by Schöenflies theorem [6] it follows that P is a 3–ball. Now apply Corollary 2.4.

COROLLARY 5.5. *If K is an attractor in a phase space M which is an ANR, then for any positively invariant compact neighbourhood $P \subseteq \mathcal{A}(K)$ of K the boundary ∂P has the shape of a finite polyhedron.*

PROOF. We shall keep the same notation as in Step 4 of Proposition 5.1. Since $\mathcal{A}(K) - K$ is open in the phase space and therefore an ANR, every A_k turns out to be a compact ANR (being a retract of $\mathcal{A}(K) - K$). A theorem of West [28] guarantees then that every A_k has the shape of a finite polyhedron, hence ∂P does, too. □

This corollary, together with Theorem 3.1, supplies sufficient conditions for the boundary of an attractor to have a polyhedral shape.

REFERENCES

[1] N. P. Bhatia and G. P. Szegö, Stability Theory of Dynamical Systems, Die Grundlehren der mathematischen Wissenschaften, Band 161, Springer–Verlag, New York-Berlin, 1970.

[2] S. A. Bogatyĭ and V. I. Gutsu, *On the structure of attracting compacta*, Differentsialnye Uravneniya **25** (1989), 907–909.

[3] K. Borsuk, Theory of Retracts, PWN–Polish Scientific Publishers, Warszawa, 1967.

[4] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. **62** (1968), 223–254.

[5] K. Borsuk, Theory of Shape, PWN–Polish Scientific Publishers, Warszawa, 1975.

[6] M. Brown, *A proof of the generalized Schönflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76.

[7] W. C. Chewning and R. S. Owen, *Local sections of flows on manifolds*, Proc. Amer. Math. Soc. **49** (1975), 71–77.

[8] R. C. Churchill, *Isolated invariant sets in compact metric spaces*, J. Differential Equations **12** (1972), 330–352.

[9] C. Conley, Isolated Invariant Sets and the Morse Index, CBMS Regional Conference Series in Mathematics **38**, American Mathematical Society, Providence, 1978.

[10] C. Conley and R. Easton, *Isolated invariant sets and isolating blocks*, Trans. Amer. Math. Soc. **158** (1971), 35–61.
[11] J. Dydak and J. Segal, Shape Theory. An Introduction, Lecture Notes in Mathematics **688**, Springer, Berlin, 1978.
[12] A. Giraldo, M. A. Morón, F. R. Ruiz del Portal, and J. M. R. Sanjurjo, *Shape of global attractors in topological spaces*, Nonlinear Anal. **60** (2005), 837–847.
[13] A. Giraldo and J. M. R. Sanjurjo, *On the global structure of invariant regions of flows with asymptotically stable attractors*, Math. Z. **232** (1999),739–746.
[14] B. Günther and J. Segal, *Every attractor of a flow on a manifold has the shape of a finite polyhedron*, Proc. Amer. Math. Soc. **119** (1993), 321–329.
[15] H. M. Hastings, *A higher-dimensional Poincaré-Bendixson theorem*, Glas. Mat. Ser. III **14(34)** (1979), 263–268.
[16] S. Hu, Theory of Retracts, Wayne State University Press, Detroit, 1965.
[17] S. Mardešić, Strong Shape and Homology, Springer-Verlag, Berlin-Heidelberg-New York 2000.
[18] S. Mardešić and J. Segal, *Equivalence of the Borsuk and the ANR-system approach to shapes*, Fund. Math. **72** (1971), 61–68.
[19] S. Mardešić and J. Segal, *Shapes of compacta and ANR-systems*, Fund. Math. **72** (1971), 41–59.
[20] S. Mardešić and J. Segal, Shape Theory, North-Holland Publishing Co., Amsterdam-New York-Oxford 1982.
[21] F. Raymond, *Separation and union theorems for generalized manifolds with boundary*, Michigan Math. J. **7** (1960), 7–21.
[22] J. C. Robinson and O. M. Tearne, *Boundaries of attractors as omega limit sets*, Stoch. Dyn. **5** (2005), 97–109.
[23] D. Salamon, *Connected simple systems and the Conley index of isolated invariant sets*, Trans. Amer. Math. Soc. **291** (1985), 1–41.
[24] J. J. Sánchez-Gabites and J. M. R. Sanjurjo, *On the topology of the boundary of a basin of attraction*, Proc. Amer. Math. Soc., to appear.
[25] J. M. R. Sanjurjo, *Multihomotopy, Čech spaces of loops and shape groups*, Proc. London Math. Soc. (3) **69** (1994), 330–344.
[26] J. M. R. Sanjurjo, *On the structure of uniform attractors*, J. Math. Anal. Appl. **192** (1995), 519–528.
[27] E. H. Spanier, Algebraic Topology, McGraw–Hill Book Co., New York-Toronto-London, 1966.
[28] J. E. West, *Mapping Hilbert cube manifolds to ANR's: a solution of a conjecture of Borsuk*, Ann. Math. (2) **106** (1977), 1–18.

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