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A SPECTRAL SEQUENCE FOR THE GROUP OF SELF-MAPS WHICH INDUCE IDENTITY AUTOMORPHISMS OF HOMOLOGY GROUPS

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Dedicated to Professor Sibe Mardešić on the occasion of his 80th birthday

Abstract. Let $\mathrm{Aut}_*(X)$ denote the group of homotopy classes of self-maps of X which induce identity automorphisms of homology groups. We construct a spectral sequence converging to $\mathrm{Aut}_*(X)$, induced by the cellular decomposition of X, and use it to obtain some structural and computational results.

1. Introduction

Given a pointed CW-complex X let us consider the following H-groups (with respect to the composition): $\mathrm{aut}(X)$, the space of base-point preserving self-maps of X which are homotopy equivalences, $\mathrm{aut}_\sharp(X)$, the subspace of $\mathrm{aut}(X)$ consisting of maps which induce identity automorphisms of homotopy groups of X, and $\mathrm{aut}_*(X)$, the subspace of $\mathrm{aut}(X)$ consisting of maps which induce identity automorphisms of homology groups of X. The corresponding sets of components form the group $\mathrm{Aut}(X) := \pi_0(\mathrm{aut}(X))$, called the *group of self-homotopy equivalences*, and its subgroups $\mathrm{Aut}_\sharp(X) := \pi_0(\mathrm{aut}_\sharp(X))$ and $\mathrm{Aut}_*(X) := \pi_0(\mathrm{aut}_*(X))$. There is an extensive literature on these spaces and groups as can be seen from the excellent survey paper [1].

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The spectral sequence approach to the computation of $\mathrm{Aut}(X)$ appeared very early in Shih [15], and was further developed in several papers by Didierjean [5, 6] on $\mathrm{Aut}_\sharp(X)$ and higher homotopy groups of $\mathrm{aut}(X)$. Both authors use the Postnikov decomposition of a space in order to obtain a filtration of the space $\mathrm{aut}(X)$, which in turn yields a spectral sequence converging to the homotopy groups of $\mathrm{aut}(X)$, and in particular to $\mathrm{Aut}(X)$. The complexity of these results reflects the complexity of the structure of $\mathrm{Aut}(X)$, while the non-effectiveness of the Postnikov decomposition limits their applicability.

When one attempts to dualize these results to cellular decompositions, two important difficulties arise.

First, a cellular decomposition of X does not lead to a corresponding filtration of $\mathrm{Aut}(X)$. Indeed, there are elementary examples of self-equivalences on a space which cannot be represented by a cellular map whose restrictions to skeletons are also self-equivalences (cf. [14, Remark 1.1]). As shown in [13] the situation is more favorable when dealing with $\mathrm{Aut}_*(X)$, the subgroup of $\mathrm{Aut}(X)$ defined above (which can be also seen as the kernel of the obvious representation $\mathrm{Aut}(X) \to \mathrm{Aut}\, H_*(X)$), since in that case a cellular decomposition of X induces a normal series for $\mathrm{Aut}_*(X)$, which is a neccessary condition for the existence of a spectral sequence.

Still, the approach of Shih and Didierjean cannot be applied directly to obtain a spectral sequence converging to $\mathrm{Aut}_*(X)$. In fact, while the naturality of the Postnikov decomposition of X gives rise to a filtration of $\mathrm{aut}(X)$ and $\mathrm{aut}_\sharp(X)$, a cellular decomposition does not do the same for $\mathrm{aut}_*(X)$. We overcome this difficulty by the introduction of the space $\mathrm{caut}(X)$ of self-equivalences which respect cellular structure. This space is related to $\mathrm{aut}_*(X)$ and has nicer properties with respect to the cellular decomposition so that a spectral sequence converging to the corresponding group of self-equivalences can be obtained.

The paper is organized as follows. The second section contains a brief exposition of (a variant of) the Cartan-Eilenberg method for the construction of spectral sequences. In the third section this method is applied to $\mathrm{caut}(X)$ in order to obtain a spectral sequence converging toward the corresponding group $\mathrm{cAut}(X)$ of homotopy classes. Then this group is related to $\mathrm{Aut}_*(X)$ and the special case of spaces with sparse homology is considered. The fourth section contains several structural results: the description of the kernel of the natural homomorphism $\mathrm{cAut}(X) \to \mathrm{Aut}_*(X)$, the proof that $\mathrm{cAut}(X)$ is nilpotent and that it behaves well with respect to the localization. The final section is dedicated to computational examples.

2. Non-abelian Cartan-Eilenberg systems

The classical reference for the Cartan-Eilenberg approach to the construction of spectral sequences is [2]. In its original form this method is equivalent to exact couples, and is well suited for the computation of (co)homology groups of filtered spaces. On the other side, if one attempts to compute the homotopy groups, a difficulty arises, since the homotopy exact sequences are not exact in dimension 0. To avoid this problem, some modifications are required, and are much easier to describe in the Cartan-Eilenberg setting. This was first done by Shih (as reported in [3]), and we will give here a brief description of the relevant part of his approach, following [10].

A *non-abelian Cartan-Eilenberg system* is given by the following data:
– abelian groups $H^{-1}(p,q)$ for $0 \le p \le q \le \infty$,
– groups $H^0(p,q)$ for $0 \le p \le q \le \infty$,
– pointed sets $H^1(p,p+1)$ with base points b_p for $0 \le p < \infty$,
– homomorphisms $\eta\colon H^n(p,q) \to H^n(p',q')$ for $n \in \{-1,0\}$, $p \ge p'$ and $q \ge q'$,
– homomorphisms $\delta\colon H^{-1}(p,q) \to H^0(q,r)$ for $p \le q \le r$,
– actions of $H^0(p,q)$ on $H^1(q,q+1)$, which, through the action on the base-point, define functions $\delta\colon H^0(p,q) \to H^1(q,q+1)$ as $\delta\colon x \mapsto x \cdot b_q$.

These data satisfy the following conditions:
– $\eta\colon H^n(p,q) \to H^n(p,q)$ is the identity;
– whenever $p \ge p' \ge p''$ and $q \ge q' \ge q''$ the following diagram is commutative

$$\begin{tikzcd} H(p,q) \arrow[rr, "\eta"] \arrow[dr, "\eta"'] & & H(p'',q'') \\ & H(p',q') \arrow[ur, "\eta"'] & \end{tikzcd}$$

– whenever $p \le q \le r$ and $p' \le q' \le r'$ the following diagram is commutative

$$\begin{tikzcd} H^n(p',q') \arrow[r, "\delta"] \arrow[d, "\eta"'] & H^{n+1}(q',r') \arrow[d, "\eta"] \\ H^n(p,q) \arrow[r, "\delta"'] & H^{n+1}(q,r) \end{tikzcd}$$

– whenever $p \le q \le r$ the following sequence is exact

$$H^{-1}(q,r) \xrightarrow{\eta} H^{-1}(p,r) \xrightarrow{\eta} H^{-1}(p,q)$$
$$\xrightarrow{\delta} H^0(q,r) \xrightarrow{\eta} H^0(p,r) \xrightarrow{\eta} H^0(p,q)$$

– whenever $p \le q$ the following sequence is exact

$$H^0(p,q+1) \xrightarrow{\eta} H^0(p,q) \xrightarrow{\delta} H^1(q,q+1)\ .$$

The above definition looks quite cumbersome and in fact it could be given in a more elegant and unified fashion. However, this would require a more abstract categorical setting, which we prefer to avoid.

Theorem 2.1. *Every non-abelian Cartan-Eilenberg system gives rise to a spectral sequence* (E_r, d_r) *of cohomological type where*

$$E_r^{p,-p-1} = \mathrm{Im}(\eta\colon H^{-1}(p,p+r) \to H^{-1}(p,p+1)),$$

$$E_r^{p,-p} = \frac{\mathrm{Im}(\eta\colon H^0(p,p+r) \to H^0(p,p+1))}{\mathrm{Im}(\delta\colon H^{-1}(p-r+1,p) \to H^0(p,p+1))},$$

and $E_r^{p,-p+1}$ *is the quotient of* $E_{r-1}^{p,-p+1}$ *with respect to the action of* $E_{r-1}^{p-r+1,-p+r-1}$, *induced by the action of* $H^0(p-r+1,p)$ *on* $H^1(p,p+1)$.

For $p+q \notin \{-1,0,1\}$ *the* $E_r^{p,q}$ *are trivial.*

The differential $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ *is induced by the relation*

$$H^{p+q}(p,p+1) \xrightarrow{\eta^{-1}} H^{p+q}(p,p+r) \xrightarrow{\delta} H^{p+q+1}(p+r,p+r+1).$$

If there exists some P *such that* $H^n(p,p+1)$ *is trivial when* $p > P$, *then this spectral sequence determines the graded group* $H^0(0,\infty)$ *in the sense that the* E_∞*-term corresponds to the subquotients of the filtration*

$$F^pH(0,\infty) = \mathrm{Im}(H(p,\infty) \to H(0,\infty)) = \mathrm{Ker}(H(0,\infty) \to H(0,p)).$$

Finally, if $H^0(p,p+1)$ *and* $H^1(p,p+1)$ *are abelian groups, and if the function*

$$H^0(p,p+1) \to H^1(p+1,p+2), \quad x \mapsto x \cdot b_{p+1} - b_{p+1}$$

is a homomorphism, then the resulting spectral sequence consists of abelian groups and their homomorphisms.

We omit the proof of this theorem, since it requires only a straightforward verification of the assertions.

3. A spectral sequence for $\mathrm{Aut}_*(X)$

The main part of this section is dedicated to the construction of a spectral sequence associated to a cellular decomposition of a CW-complex X. The strategy to avoid the problems mentioned in the Introduction consists of two steps:

– we first introduce the space of maps which induce identity automorphisms of homology groups and preserve the cellular structure

$$\mathrm{caut}(X) := \{f \in \mathrm{aut}_*(X) \mid \forall p\colon f|_{X^{(p)}} \in \mathrm{aut}_*(X^{(p)})\},$$

and show that the Cartan-Eilenberg method can be used for the computation of the group $\mathrm{cAut}(X) = \pi_0(\mathrm{caut}(X))$;

– second, we study relations between $\mathrm{cAut}(X)$ and $\mathrm{Aut}_*(X)$, and in particular give some sufficient conditions for them to coincide.

Generally speaking the space $\mathrm{caut}(X)$ is not a good approximation of $\mathrm{aut}_*(X)$, nonetheless [13, Theorem 1.4] implies that the natural homomorphism

$$\mathrm{cAut}(X) \to \mathrm{Aut}_*(X)$$

is surjective. Moreover, when all homotopies on X can be substituted by cellular ones, e.g., when X does not have cells in consecutive dimensions, the above homomorphism is injective as well.

Let X be a simply-connected, n-dimensional, finite CW-complex, with a canonical decomposition given by the skeletons $\{X^{(p)} \mid p = 0, 1, \ldots, n\}$ and the attaching maps $\{\varphi_p : V^p \to X^{(p)} \mid p = 2, 3, \ldots, n-1\}$ where V^p is a wedge of p-dimensional spheres. That the decomposition of X is *canonical* means that for every p the wedge V^p splits into a union of subwedges $V^p = V^p_r \vee V^p_g$ such that $(\varphi_p|_{V^p_r})_{*p}$ is injective and $\varphi_p(V^p_g) \subseteq X^{(p-1)}$. This technical assumption does not imply any loss of generality since by [14, Theorem 2.3] every simply-connected CW-complex has the cellular homotopy type of a complex with a canonical decomposition. Let c_{p+1} be the number of spheres in V^p, i.e., the number of $(p+1)$-cells in the decomposition of X, and let r_{p+1} and g_{p+1} denote respectively the number of spheres in V^p_r and V^p_g.

The following results prepare the construction of a Cartan-Eilenberg system.

LEMMA 3.1. *The restriction map* $\mathrm{caut}(X^{(p+1)}) \to \mathrm{caut}(X^{(p)})$ *is a fibration.*

PROOF. Obvious, since $X^{(p)} \hookrightarrow X^{(p+1)}$ is a cofibration. □

COROLLARY 3.2. *If $p \leq q$ then the restriction map* $\mathrm{caut}(X^{(q)}) \to \mathrm{caut}(X^{(p)})$ *is a fibration with fibre* $\mathrm{caut}_{X^{(p)}}(X^{(q)}) = \{f \in \mathrm{caut}(X^{(q)}) \mid f|_{X^{(p)}} = 1_{X^{(p)}}\}$.

COROLLARY 3.3. *If $p \leq q \leq r$ then there is a fibration*

$$\mathrm{caut}_{X^{(q)}}(X^{(r)}) \hookrightarrow \mathrm{caut}_{X^{(p)}}(X^{(r)}) \to \mathrm{caut}_{X^{(p)}}(X^{(q)}).$$

These corollaries allow the construction of a Cartan-Eilenberg system: for $p \leq q$ and $k \in \{-1, 0\}$ let $H^k(p,q) := \pi_{-k}(\mathrm{caut}_{X^{(p)}}(X^{(q)}))$; the homomorphisms η are induced by restrictions while the homomorphisms δ are determined by boundary homomorphisms in the homotopy long exact sequences of fibrations

$$\mathrm{caut}_{X^{(q)}}(X^{(r)}) \hookrightarrow \mathrm{caut}_{X^{(p)}}(X^{(r)}) \to \mathrm{caut}_{X^{(p)}}(X^{(q)});$$

for $p \geq 0$ let $H^1(p, p+1)$ be the cokernel of the homomorphism $\Phi_p : [V^p_r, V^p_g] \to [V^p, X^{(p)}]$, which to an $\alpha : V^p_r \to V^p_g$ assigns the composition

$$V^p \xrightarrow{\pi} V^p_r \xrightarrow{\alpha} V^p_g \xhookrightarrow{\imath} V^p \xrightarrow{\varphi_p} X^{(p)},$$

with φ_p as base-point; finally, let $H^0(p,q) = \mathrm{cAut}_{X^{(p)}}(X^{(q)})$ act on $H^1(q, q+1)$ by post-composition.

LEMMA 3.4. *The sequence of groups and sets*

$$\mathrm{cAut}_{X^{(p)}}(X^{(q+1)}) \xrightarrow{\eta} \mathrm{cAut}_{X^{(p)}}(X^{(q)}) \xrightarrow{\delta} [V^q, X^{(q)}]/\Phi_q([V^q_r, V^q_g])$$

is exact.

Proof. Consider the following commutative diagram

$$\begin{tikzcd} V^q \arrow[r, "\varphi_q"] \arrow[d, dashed, "1+\imath\alpha\pi"'] & X^{(q)} \arrow[r] \arrow[d, "f"] & X^{(q+1)} \arrow[d, dashed, "\bar{f}"] \\ V^q \arrow[r, "\varphi_q"] & X^{(q)} \arrow[r] & X^{(q+1)} \end{tikzcd}$$

If $\delta(f) = [\varphi_q] \in [V^q, X^{(q)}]/\Phi_q([V_r^q, V_g^q])$, i.e., if there is a map $\alpha \in [V_r^q, V_g^q]$ such that $f \circ \varphi_q - \varphi_q = \varphi_q \circ \imath \circ \alpha \circ \pi$, then, by the mapping cone property, f extends to a map $\bar{f}\colon X^{(q+1)} \to X^{(q+1)}$. Its effect on homology groups can be deduced from the following diagram ($\xi := (\varphi_q|_{V_r^q})_*$):

$$\begin{tikzcd} H_{q+1}(X^{(q+1)}) \arrow[r, tail] \arrow[d, "\bar{f}_*"'] & H_q(V_g^q) \oplus H_q(V_r^q) \arrow[r, "{[0,\xi]}"] \arrow[d, "1_*+\imath_*\alpha_*\pi_*"] & H_q(X^{(q)}) \arrow[r, two heads] \arrow[d, equal, "f_*"'] & H_q(X^{(q+1)}) \arrow[d, "\bar{f}_*"'] \\ H_{q+1}(X^{(q+1)}) \arrow[r, tail] & H_q(V_g^q) \oplus H_q(V_r^q) \arrow[r, "{[0,\xi]}"] & H_q(X^{(q)}) \arrow[r, two heads] & H_q(X^{(q+1)}) \end{tikzcd}$$

hence $\bar{f}_{*q} = 1_*$ and $\bar{f}_{*(q+1)} = 1_*$. Consequently $\bar{f} \in \mathrm{caut}_*(X^{(q+1)})$ and $\operatorname{Ker}\delta \subseteq \operatorname{Im}\eta$. Vice versa, if a map $\bar{f} \in \mathrm{caut}(X^{(q+1)})$ satisfies $\bar{f}|_{X^{(q)}} = f$, i.e., $f = \eta(\bar{f})$, then by [8, Theorem 7.3'] there exists a map $\epsilon\colon V^q \to V^q$ such that $f \circ \varphi_q = \varphi_q \circ \epsilon$. From the above diagram we conclude that $\epsilon = 1 + \imath \circ \alpha \circ \pi$ for some $\alpha \in [V_r^q, V_g^q]$, hence $\operatorname{Im}\eta \subseteq \operatorname{Ker}\delta$. ∎

It follows that the above data define a non-abelian Cartan-Eilenberg system so we can study the corresponding spectral sequence.

Theorem 3.5. *Let X be a simply-connected, finite CW complex with a canonical decomposition $\{X^{(p)}\}$ where $X^{(p+1)}$ is the cone of the map $\varphi_p : V^p = V_r^p \vee V_g^p \to X^{(p)}$. Then there exists an abelian spectral sequence of cohomological type whose initial term is given as follows:*

$$\begin{aligned} E_1^{p,-1-p} &= \pi_{p+2}(X^{(p+1)})^{c_{p+1}}, \\ E_1^{p,-p} &= \pi_{p+1}(X^{(p+1)})^{r_{p+1}} \oplus (\operatorname{Ker} h_{p+1})^{g_{p+1}}, \\ E_1^{p,1-p} &= \pi_p(X^{(p)})^{c_{p+1}}/\Phi_p([V_r^p, V_g^p]), \end{aligned}$$

where $h_{p+1} : \pi_{p+1}(X) \to H_{p+1}(X)$ is the Hurewicz homomorphism, and c_p, r_p, g_p, Φ_p are as above.

The spectral sequence converges to $\mathrm{cAut}(X)$, i.e., the groups $E_\infty^{p,-p}$ are subquotients of the filtration

$$F^p(\mathrm{cAut}(X)) = \operatorname{Im}(\mathrm{cAut}_{X^{(p)}}(X) \to \mathrm{cAut}(X)).$$

Proof. The existence and the convergence of the sequence follow from Theorem 2.1 and the previous discussion, so it only remains to compute the initial term.

Observe that $\operatorname{caut}_{X^{(p)}} X^{(p+1)} = \operatorname{aut}_{*X^{(p)}} X^{(p+1)} := \{f \in \operatorname{aut}_*(X^{(p+1)}) \mid f|_{X^{(p)}} = 1_{X^{(p)}}\}$. Let $\nu : X^{(p+1)} \to X^{(p+1)} \vee \Sigma V^p$ be the standard coaction. The function

$$\lambda : \operatorname{map}(\Sigma V^p, X^{(p+1)}) \to \operatorname{map}_{X^{(p)}}(X^{(p+1)}, X^{(p+1)})$$

which to an $f : \Sigma V^p \to X^{(p+1)}$ assigns the composition

$$X^{(p+1)} \xrightarrow{\nu} X^{(p+1)} \vee \Sigma V^p \xrightarrow{1 \vee f} X^{(p+1)} \vee X^{(p+1)} \xrightarrow{\nabla} X^{(p+1)}$$

is a homotopy equivalence, and moreover $\lambda(f)_* = 1_* + f_* \circ q_*$ where $q : X^{(p+1)} \to \Sigma V_g^p$ is the canonical projection. It follows that $\operatorname{caut}_{X^{(p)}}(X^{(p+1)})$ is homotopy equivalent to

$$\operatorname{map}_*(\Sigma V^p, X^{(p+1)}) := \{f \in \operatorname{map}(\Sigma V^p, X^{(p+1)}) \mid f_* \circ q_* = 0\}.$$

The component of the base-point of $\operatorname{map}_*(\Sigma V^p, X^{(p+1)})$ is $\operatorname{map}_0(\Sigma V^p, X^{(p+1)})$, the space of null-homotopic maps, hence

$$\begin{aligned} E_1^{p,-1-p} &= H^{-1}(p, p+1) = \pi_1(\operatorname{caut}_{X^{(p)}}(X^{(p+1)})) = \pi_1(\operatorname{map}_0(\Sigma V^p, X^{(p+1)})) \\ &= \pi_0(\operatorname{map}_0(\Sigma^2 V^p, X^{(p+1)})) = \pi_{p+2}(X^{(p+1)})^{c_{p+1}}. \end{aligned}$$

As a set, $E_1^{p,-p} = \pi_0(\operatorname{map}_*(\Sigma V^p, X^{(p+1)}))$ consists of classes of maps in $\operatorname{map}(\Sigma V^p, X^{(p+1)})$ satisfying $f_* \circ q_* = 0$ on homology groups. Moreover, by [12, Lemma 2.3], the group operation induced by the coH-structure on ΣV^p and that induced through λ by the composition in $\operatorname{cAut}_{X^{(p)}}(X^{(p+1)})$ coincide.

The homomorphism $f_* \circ q_*$ induced in homology by a map $f : \Sigma V^p \to X^{(p+1)}$ can be determined from the following diagram (h is the Hurewicz homomorphism):

$$\begin{tikzcd} \pi_{p+1}(X^{(p+1)}) \arrow[r, "q_\sharp"] \arrow[d, "h"] & \pi_{p+1}(\Sigma V^p) \arrow[r, "f_\sharp"] \arrow[d, "h"] & \pi_{p+1}(X^{(p+1)}) \arrow[d, "h"] \\ H_{p+1}(X^{(p+1)}) \arrow[r, "q_* \oplus 0"'] & H_{p+1}(\Sigma V_g^p) \oplus H_{p+1}(\Sigma V_r^p) \arrow[r, "f_*"'] & H_{p+1}(X^{(p+1)}) \end{tikzcd}$$

Clearly, $f_* \circ q_* = 0$ if and only if $(f|_{V_g^p})_\sharp$ factors through the kernel of the Hurewicz homomorphism. A short computation yields

$$\begin{aligned} &\pi_0(\operatorname{map}_*(\Sigma V^p, X^{(p+1)})) \\ &\quad = \pi_{p+1}(X^{(p+1)})^{r_{p+1}} \oplus \operatorname{Ker}(h : \pi_{p+1}(X^{(p+1)}) \to H_{p+1}(X^{(p+1)}))^{g_{p+1}}. \end{aligned}$$

Moreover, the comparison of the long exact sequences in homotopy and homology of the pair $(X, X^{(p+1)})$ shows that

$$\operatorname{Ker}(h : \pi_{p+1}(X^{(p+1)}) \to H_{p+1}(X^{(p+1)})) = \operatorname{Ker}(h : \pi_{p+1}(X) \to H_{p+1}(X)).$$

The description of $E_1^{p,1-p} = H^1(p, p+1)$ follows directly from the definition.

To prove the abelianity, it is sufficient to show that $\delta : H^0(p, p+1) \to H^1(p+1, p+2)$ is a homomorphism. An easy calculation yields $\delta(f) = f \circ q \circ \varphi_{p+1}$ for $f : \Sigma V^p \to X^{(p+1)}$. By [16, Theorem 8.2] this operation is additive, hence a homomorphism. ∎

We have already observed that the natural homomorphism $\mathrm{cAut}(X) \to \mathrm{Aut}_*(X)$ is always surjective. If the homology of X is *sparse*, i.e. if for every i at least one of the groups $H_i(X)$, $H_{i+1}(X)$ is trivial, then this homomorphism is also injective:

PROPOSITION 3.6. *If a simply-connected, finite-dimensional CW-complex X has sparse homology then* $\mathrm{cAut}(X) \cong \mathrm{Aut}_*(X)$.

PROOF. We may assume that X has a cellular decomposition which does not have cells in consecutive dimensions. It follows that every homotopy between two maps in $\mathrm{aut}_*(X)$ can be substituted by a cellular one, hence $\mathrm{cAut}(X) \to \mathrm{Aut}_*(X)$ is injective. ∎

Since under the assumptions of the above proposition all homology groups of X are free, the corresponding spectral sequence for $\mathrm{Aut}_*(X)$ has a simpler description.

THEOREM 3.7. *Assume that a simply-connected finite CW-complex X has sparse homology. Then there exists an abelian spectral sequence of cohomological type, which converges to* $\mathrm{Aut}_*(X)$. *Its initial term is given as*

$$\begin{aligned} E_1^{p,-1-p} &= \pi_{p+2}(X^{(p+1)})^{c_{p+1}}, \\ E_1^{p,-p} &= \mathrm{Ker}(h \colon \pi_{p+1}(X) \to H_{p+1}(X))^{c_{p+1}}, \\ E_1^{p,1-p} &= \pi_p(X^{(p)})^{c_{p+1}}, \end{aligned}$$

where c_p is the rank of the free group $H_p(X)$.

4. THE STRUCTURE OF cAut(X)

It is possible to improve Theorem 3.5 by describing the kernel of the homomorphism $\mathrm{cAut}(X) \to \mathrm{Aut}_*(X)$. Let X be as in Theorem 3.5, and let $G^p = G^p(X)$ be the subgroup of $\mathrm{Aut}_*(X)$ consisting of classes represented by maps f with the property $f|_{X^{(p)}} = 1_{X^{(p)}}$. By [13, Theorem 1.5], G^p is a normal subgroup of $\mathrm{Aut}_*(X)$ and if X is of dimension n then

$$1 \triangleleft G^{n-1} \triangleleft \ldots \triangleleft G^2 \triangleleft \mathrm{Aut}_* X$$

is a finite normal series for $\mathrm{Aut}_*(X)$. This filtration of $\mathrm{Aut}_*(X)$ is parallel to the filtration $F^p = F^p(\mathrm{cAut}(X))$ of $\mathrm{cAut}(X)$ described in Theorem 3.5.

Indeed, there is a ladder of epimorphisms

$$\begin{tikzcd} F^{n-1} \arrow[r] \arrow[d, two heads] & \cdots & F^2 \arrow[r] \arrow[d, two heads] & \mathrm{cAut}(X) \arrow[d, two heads] \\ G^{n-1} \arrow[r] & \cdots & G^2 \arrow[r] & \mathrm{Aut}_*(X) \end{tikzcd}$$

whose kernels $K^p := \mathrm{Ker}(F^p \to G^p)$ form a filtration of the kernel of the epimorphism $\mathrm{cAut}(X) \to \mathrm{Aut}_*(X)$. We are going to compute the subquotients of the latter filtration, and in order to do so we will need the following lemma, which is essentially taken from [11, p. 291-292], (see also [13, Lemma 1.2]).

Lemma 4.1. *Let*

$$V \xrightarrow{\varphi} A \xhookrightarrow{i} X$$

be the cofibration sequence of the map φ from a wedge of n-dimensional spheres V to a simply-connected CW-complex A of dimension at most n. Then
– there exists an action of the group $[A,V]$ on the monoid $[A,A]$ such that if $\alpha \in [A,V]$ and $f \in [A,A]$, then on homology $(\alpha \cdot f)_* = \varphi_* \circ \alpha_* + f_*$;
– whenever $i \circ f \simeq i \circ g$ for some $f, g \in [A,A]$ then there is an $\alpha \in [A,V]$ such that $f \simeq \alpha \cdot g$.

Proposition 4.2. *Let X be as in Theorem 3.5. Then the kernel of the epimorphism $F_p/F_{p+1} \to G_p/G_{p+1}$ is isomorphic to the subgroup of $\mathrm{cAut}_{X^{(p)}}(X^{(p+1)})$ consisting of elements which can be extended to some element of $\mathrm{cAut}(X)$ and which are of the form $\alpha \cdot 1_{X^{(p+1)}}$ for some $\alpha \in [X^{(p+1)}, V^{p+1}]$ satisfying $(\varphi_{p+1} \circ \alpha)_* = 0$.*

Proof. Observe that the homomorphism $F_p/F_{p+1} \to \mathrm{cAut}_{X^{(p)}}(X^{(p+1)})$ given by $f \mapsto f|_{X^{(p+1)}}$ is injective and that its image consists precisely of elements which can be extended to elements of $\mathrm{cAut}(X)$.

By definition, an element of F^p/F^{p+1} represented by the class of some $f \in F^p$ is in $\mathrm{Ker}(F_p/F_{p+1} \to G_p/G_{p+1})$ if f is homotopic (by a homotopy which is not necessarily cellular) to some $g \in \mathrm{Aut}_*(X)$ satisfying $g|_{X^{(p+1)}} = 1_{X^{(p+1)}}$. It follows that $i \circ f|_{X^{(p+1)}} \simeq i : X^{(p+1)} \hookrightarrow X$. By the cellular approximation theorem $i \circ f|_{X^{(p+1)}} \simeq i : X^{(p+1)} \hookrightarrow X^{(p+2)}$, so by Lemma 4.1 there is an $\alpha : X^{(p+1)} \to V^{p+1}$ which satisfies $(\varphi_{p+1} \circ \alpha)_* = 0$ and $f|_{X^{(p+1)}} = \alpha \cdot 1_{X^{(p+1)}}$. ∎

Since $\mathrm{caut}(X)$ is in many respects dual to $\mathrm{aut}_\sharp(X)$, it is not a surprise that the folowing analogues of the main theorems in [7, 11] hold. Indeed, the proofs are just duals of the proofs of the mentioned results.

Theorem 4.3. *If X is a finite, simply-connected CW-complex, then $\mathrm{cAut}(X)$ is nilpotent.*

Proof. We will give only a sketch of the proof as it is dual to the proofs of [7, Theorems B and C] and requires only a few straightforward induction arguments.

First consider the exact sequence

$$\pi_1(\mathrm{caut}(X^{(p)})) \to \mathrm{cAut}_{X^{(p)}}(X^{(p+1)}) \to \mathrm{cAut}(X^{(p+1)}) \to \mathrm{cAut}(X^{(p)})$$

derived from the fibration

$$\mathrm{caut}_{X^{(p)}}(X^{(p+1)}) \hookrightarrow \mathrm{caut}(X^{(p+1)}) \longrightarrow \mathrm{caut}(X^{(p)}).$$

Clearly $\mathrm{cAut}(X)$ acts nilpotently on $\mathrm{cAut}(X^{(1)}) = \{1\}$, so let us assume that $\mathrm{cAut}(X)$ acts nilpotently on $\mathrm{cAut}(X^{(p)})$. If we can deduce that $\mathrm{cAut}(X)$ acts nilpotently on $\mathrm{cAut}(X^{(p+1)})$, then by induction $\mathrm{cAut}(X)$ acts nilpotently on itself, hence it is nilpotent. In order to do so it is sufficient to show that $\pi_1(\mathrm{caut}(X^{(p)})) \to \mathrm{cAut}_{X^{(p)}}(X^{(p+1)})$ is a $\mathrm{cAut}(X)$-homomorphism of groups on which $\mathrm{cAut}(X)$ acts nilpotently.

Observe that $\mathrm{cAut}(X)$ acts nilpotently on the homology groups of X, and thus, by an easy induction using the Whitehead tower, it acts nilpotently on the homotopy groups of X. The group $\mathrm{cAut}_{X^{(p)}}(X^{(p+1)})$ has been computed in the proof of Theorem 3.5, and it is a $\mathrm{cAut}(X)$-subgroup of a product of homotopy groups of X, so the $\mathrm{cAut}(X)$-action on it is also nilpotent.

To see that $\mathrm{cAut}(X)$ acts nilpotently on $\pi_1(\mathrm{caut}(X^{(p+1)}))$ consider the homotopy exact sequence of the fibration

$$\mathrm{cmap}_{X^{(q)}}(X^{(q+1)}, X^{(p)}) \hookrightarrow \mathrm{cmap}(X^{(q+1)}, X^{(p)}) \longrightarrow \mathrm{cmap}(X^{(q)}, X^{(p)}),$$

where $\mathrm{cmap}(X^{(q)}, X^{(p)})$ denotes the component of the space of cellular maps from $X^{(q)}$ into $X^{(p)}$ which contains the inclusion map. Once again we use the computations of Theorem 3.5 to see that the homotopy groups of $\mathrm{cmap}_{X^{(q)}}(X^{(q+1)}, X^{(p)})$ are just products of homotopy groups of $X^{(p)}$, so by induction $\mathrm{cAut}(X)$ acts nilpotently on all homotopy groups of $\mathrm{caut}(X^{(p)})$ and in particular on π_1. ∎

A nilpotent group can be localized with respect to a set of primes and the following result gives a precise description of the localization of $\mathrm{cAut}(X)$.

Theorem 4.4. *If P is any set of primes, then the P-localization $X \to X_P$ of a simply-connected finite CW-complex X induces a homomorphism $\mathrm{cAut}(X) \to \mathrm{cAut}(X_P)$ which is a P-localization of nilpotent groups.*

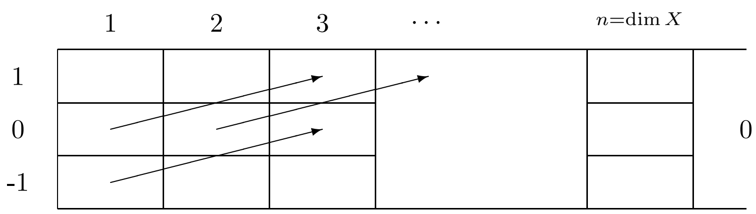
Proof. The proof is by comparison of the corresponding spectral sequences. Theorem 3.5 cannot be directly applied to $\mathrm{cAut}(X_P)$ because X_P is not a finite complex. Nevertheless, if we consider the local cellular decomposition of X_P, i.e., the decomposition obtained by substituting localized cells to the ordinary ones in the decomposition of X, it is not difficult to check that the construction of the spectral sequence goes through. Moreover, the localization $X \to X_P$ induces the localization between the corresponding spectral sequences, which in turn implies that $\mathrm{cAut}(X) \to \mathrm{cAut}(X_P)$ is a P-localization. ∎

Remark 4.5. Note that Theorem 4.3, together with the fact that $\mathrm{cAut}(X) \to \mathrm{Aut}_*(X)$ is an epimorphism yields a direct proof of (a restricted version of) [7, Theorem D], namely that the group $\mathrm{Aut}_*(X)$ is nilpotent when X is finite and simply-connected.

Moreover, Theorem 4.4 together with Proposition 4.2 allows us to prove the main theorem of [11], namely that the natural homomorphism $\mathrm{Aut}_*(X) \to \mathrm{Aut}_*(X_P)$ is a P-localization. In fact, it is sufficient to observe that the kernels described in Proposition 4.2 localize.

5. Applications and examples

In order to simplify the graphical description of the results, we are going to represent the spectral sequence in a way different from the usual. In fact, the sequence is concentrated on three lines, so the corresponding data can be given as follows. In column p and line q we put the group $\widetilde{E}_r^{p,q}$ which corresponds to the group $E_r^{p-1,-p+q+1}$ of the spectral sequence (notice the shift in the index p), while the differential $\tilde{d}_r : \widetilde{E}_r^{p,q} \to \widetilde{E}_r^{p+r,q+1}$ is just the differential $d_r : E_r^{p-1,-p+q+1} \to E_r^{p+r-1,-p-r+q+2}$. For example, a typical $\widetilde{E}_2^{*,*}$-term is given as:



As we have already seen, for a general X our spectral sequence has a quite complicated initial term and gives only an approximation of the group $\mathrm{Aut}_*(X)$. For that reason we will consider only spaces X which have sparse homology. Then we have a spectral sequence with

$$\begin{aligned} \widetilde{E}_1^{p,-1} &= \pi_{p+1}(X^{(p)})^{c_p}, \\ \widetilde{E}_1^{p,0} &= \mathrm{Ker}(h\colon \pi_p(X) \to H_p(X))^{c_p}, \\ \widetilde{E}_1^{p,1} &= \pi_{p-1}(X^{(p-1)})^{c_p}, \end{aligned}$$

where c_p is the rank of the free abelian group $H_p(X)$. The terms $\widetilde{E}_\infty^{p,0}$ are subquotients of a filtration of $\mathrm{Aut}_*(X)$.

Let us consider a few examples:

Example 5.1. The complex projective spaces satisfy the conditions of Theorem 3.7. It follows from the fibration

$$S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$$

that the homotopy groups of $\mathbb{C}P^n$ are given by $\pi_2(\mathbb{C}P^n) \cong \mathbb{Z}$ and $\pi_i(\mathbb{C}P^n) \cong \pi_i(S^{2n+1})$ for $i > 2$. Since the kernel of the Hurewicz homomorphism $h : \pi_2(\mathbb{C}P^n) \to H_2(\mathbb{C}P^n)$ is trivial and $r_p = 0$ when $p > 2n$, we get $\widetilde{E}_1^{p,0} = 0$ for all p, hence $\mathrm{Aut}_*(\mathbb{C}P^n) = \{1\}$.

Example 5.2. The quaternionic projective spaces also satisfy the assumptions of Theorem 3.5, but in this case the spectral sequence is more complicated. Note that by a result of P. Kahn [9], $\mathrm{Aut}(\mathbb{H}P^n) \cong \mathrm{Aut}_*(\mathbb{H}P^n)$. From the fibration

$$S^3 \hookrightarrow S^{4n+3} \to \mathbb{H}P^n$$

we conclude that the homotopy groups of $\mathbb{H}P^n$ are given as $\pi_4(\mathbb{H}P^n) \cong \mathbb{Z}$ and $\pi_i(\mathbb{H}P^n) \cong \pi_{i-1}(S^3)$ when $4 < i < 4n+3$. The groups $\widetilde{E}_1^{p,0}$ are trivial, except when p is of the form $p = 4k$, when we have $\widetilde{E}_1^{4,0} = 0$ because the Hurewicz homomorphism is injective and

$$\widetilde{E}_1^{4k,0} \cong \mathrm{Ker}(h\colon \pi_{4k}\mathbb{H}P^n \to H_{4k}\mathbb{H}P^n) \cong \pi_{4k-1}S^3$$

for $k > 1$ as $\pi_{4k}\mathbb{H}P^n \cong \pi_{4k-1}S^3$ is a torsion group. An estimate of the order of the group $\mathrm{Aut}(\mathbb{H}P^n)$ can be derived:

Proposition 5.3. *The order of the group* $\mathrm{Aut}(\mathbb{H}P^n)$ *divides the product of the orders of the groups* $\pi_{4k-1}(S^3)$ *for* $k = 2, \ldots, n$. *In particular,* $\mathrm{Aut}(\mathbb{H}P^n)$ *is always a finite group.*

Example 5.4. If X has only one non-trivial homology group it is homotopy equivalent to a wedge of spheres, hence $\mathrm{Aut}_*(X) = \{1\}$, which is obvious without the use of the spectral sequence.

The first interesting general case is when X has two non-trivial homology groups, say in dimensions $p < q$. Then the sequence can be represented as follows:

		p		q	
1		0		$(\pi_{q-1}X^{(q-1)})^{c_q}$	
0		0		$(\mathrm{Ker}\, h_q)^{c_q}$	
-1		$(\pi_{p+1}X^{(p)})^{c_p}$	d_{q-p}	$(\pi_{q+1}X)^{c_q}$	

Thus we get that $\mathrm{Aut}_*(X) \cong \mathrm{Coker}(d_{q-p} : (\pi_{p+1}X^{(p)})^{c_p} \to (\mathrm{Ker}\, h_q)^{c_q}$. Note that when $n \geq 3$ then $(\pi_{p+1}X^{(p)})^{c_p} \cong (\mathbb{Z}_2)^{c_p}$, so except for the 2-torsion, $\mathrm{Aut}_*(X)$ is isomorphic to $(\mathrm{Ker}\, h_q)^{c_q}$. Moreover, a straightforward computation shows that d_{q-p} can be described explicitly as follows: if we identify $\pi_{p+1}(X^{(p)})^{c_p}$ with $[\Sigma X^{(p)}, X^{(p)}]$ then $d_{q-p}(H)$ of an $H : \Sigma X^{(p)} \to X^{(p)}$ is the class of the composition

$$\Sigma V^q \xrightarrow{\Sigma\varphi_q} \Sigma X^{(p)} \xrightarrow{H} X^{(p)} \hookrightarrow X^{(q)} = X.$$

This description allows a precise description of $\mathrm{Aut}_*(X)$ by composition methods.

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