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THE COARSE SHAPE

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Dedicated to Academician Sibe Mardešić for his 80th birthday

Abstract. Given a category $\mathcal{C}$, a certain category pro^*-$\mathcal{C}$ on inverse systems in $\mathcal{C}$ is constructed, such that the usual pro-category pro-$\mathcal{C}$ may be considered as a subcategory of pro^*-$\mathcal{C}$. By simulating the (abstract) shape category construction, $Sh_{(\mathcal{C},\mathcal{D})}$, an (abstract) *coarse shape category* $Sh^*_{(\mathcal{C},\mathcal{D})}$ is obtained. An appropriate functor of the shape category to the coarse shape category exists. In the case of topological spaces, $\mathcal{C} = HTop$ and $\mathcal{D} = HPol$ or $\mathcal{D} = HANR$, the corresponding realizing category for Sh^* is pro^*-$HPol$ or pro^*-$HANR$ respectively. Concerning an operative characterization of a coarse shape isomorphism, a full analogue of the well known Morita lemma is proved, while in the case of inverse sequences, a useful sufficient condition is established. It is proved by examples that for $\mathcal{C} = Grp$ (groups) and $\mathcal{C} = HTop$, the classification of inverse systems in pro^*-$\mathcal{C}$ is strictly coarser than in pro-$\mathcal{C}$. Therefore, the underlying *coarse shape theory* for topological spaces makes sense.

1. Introduction

The standard homotopy theory has successfully solved many classifying problems for some classes of locally nice spaces (polyhedra, CW-complexes ANR's, ...). Unfortunately, when one is to study a class of locally bad spaces it cannot help significantly. To overcome this defect, K. Borsuk [1, 2] was founded in 1968 the shape theory of (metrizable) compacta. The corresponding classification of compacta is generally coarser than the homotopy type classification, while on the subclass of locally nice spaces (compact polyhedra, finite CW-complexes, compact ANR's, ...) it coincides with the homotopy

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type classification. The most significant step forward in this course was made by S. Mardešić and J. Segal [14]. They had successfully used the inverse system approach and the language of a pro-category to describe the shape theory. They also had extended the shape theory to the class of compact Hausdorff spaces. Finally, Mardešić [10] and K. Morita [18] had extended the shape theory to all topological spaces.

Since 1976 a few new classifications of compacta have been considered. For instance, Borsuk [3] introduced the relations of quasi-affinity and quasi-equivalence, while Mardešić [11] introduced the S-equivalence relation between compacta. All of them are shape type invariant relations. These classifications are strictly coarser than the shape type classification [3, 6, 9]. Moreover, the quasi-equivalence and S-equivalence on compact ANR's and compact polyhedra coincide with the homotopy type classification. However, the mentioned relations, being defined only on the class of objects, were not supported by the appropriate with them associated theories. In other words, it was not clear whether these relations are categorical. Furthermore, if such an equivalence relation admits a category characterization, there should exist a functor relating the shape category and the new category.

The reason why these new classifications was, for example, the problem of the shape types of fibers of a shape fibration. In 1977 D. Coram and P. F. Duvall [4] introduced and studied the approximate fibrations between compact ANR's. These are a shape analogue of the standard (Hurewicz) fibrations. In 1978 S. Mardešić and T. B. Rushing [13] generalized approximate fibrations to shape fibrations between metric compacta. The following important question was asking for the answer (analogously to the same homotopy type of the fibers of a fibration): Whether all the fibers of a shape fibration (over a continuum) have the same shape? In 1979 J. Keesling and S. Mardešić [9] gave a negative answer. However, Mardešić [11] had proved before that all those fibers are mutually S-equivalent. He also proved that some shape invariant classes of compacta (FANR's, movable compacta, compacta having shape dimension $\leq n, \dots$) are actually S-invariant.

In recent years the interest for the mentioned relations has arisen. So N. Uglešić [20] studied the Borsuk's quasi-equivalence and quasi-affinity and introduced some new ones, Mardešić and Uglešić [16] described the S^*-equivalence (a uniformization of the S-equivalence) in a category framework, Uglešić and B. Červar [21, 22] derived the S_n-equivalences, $n \in \mathbb{N}$, from the S-equivalence and constructed a categorical subshape spectrum for compacta, while A. Kadlof, N. Koceić Bilan and Uglešić [8] proved (solved the problem stated in [3]) that the Borsuk quasi-equivalence is not transitive.

In this paper, the results of Mardešić and Uglešić [16] for metric compacta are fully generalized to all topological spaces. More precisely, in the first step, the Mardešić-Uglešić category $\underline{\mathcal{S}}^*$ is described as a kind of the pro-category on inverse sequences of compacta (Theorems 3.1 and 3.2), such that the usual morphism sets are significantly enriched. Then, in the second step, it is noticed that this description enables us to apply the construction to any category $\mathcal{C}$ and obtain a category, denoted by tow^*-$\mathcal{C}$, on the inverse sequences in $\mathcal{C}$. Even more, as a third step, the construction admits a generalization from inverse sequences to arbitrary inverse systems in $\mathcal{C}$ to obtain a category, denoted by pro^*-$\mathcal{C}$, so that one may consider pro-$\mathcal{C}$ to be its subcategory having the same object class. Further, given a category pair $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D}$ is dense in $\mathcal{C}$ (in the shape-theoretical sense), the construction of the abstract "shape" category $Sh^*_{(\mathcal{C},\mathcal{D})}$ can be fully simulated by means of the "pro-category" pro^*-$\mathcal{D}$, i.e.,

$$Sh^*_{(\mathcal{C},\mathcal{D})}(X,Y) \approx pro^*\text{-}\mathcal{D}(\boldsymbol{X},\boldsymbol{Y}),$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are any $\mathcal{D}$-expansions of the $\mathcal{C}$-objects X and Y respectively. There also exists an abstract "shape" functor

$$S^*_{(\mathcal{C},\mathcal{D})} : \mathcal{C} \to Sh^*_{(\mathcal{C},\mathcal{D})},$$

which keeps the objects fixed, while, for every $\mathcal{C}$-morphism f, $S^*_{(\mathcal{C},\mathcal{D})}(f)$ is represented by a unique pro^*-$\mathcal{D}$ equivalence class $\langle \boldsymbol{f}^* \rangle$. The functor $S^*_{(\mathcal{C},\mathcal{D})}$ factorizes through the abstract shape category $Sh_{(\mathcal{C},\mathcal{D})}$, i.e., $S^*_{(\mathcal{C},\mathcal{D})} = J_{(\mathcal{C},\mathcal{D})}S_{(\mathcal{C},\mathcal{D})}$, where $S_{(\mathcal{C},\mathcal{D})} : \mathcal{C} \to Sh_{(\mathcal{C},\mathcal{D})}$ is the abstract shape functor, and

$$J_{(\mathcal{C},\mathcal{D})} : Sh_{(\mathcal{C},\mathcal{D})} \to Sh^*_{(\mathcal{C},\mathcal{D})}.$$

is the "inclusion" (faithful) functor. The classification of objects of $\mathcal{D}$ in $Sh^*_{(\mathcal{C},\mathcal{D})}$ is the same as in $Sh_{(\mathcal{C},\mathcal{D})}$ as well as in $\mathcal{D}$, while the classification of objects of $\mathcal{C}$ in $Sh^*_{(\mathcal{C},\mathcal{D})}$ is coarser than in $Sh_{(\mathcal{C},\mathcal{D})}$. Therefore, one might speak of an underlying *abstract coarse shape theory* (for $\mathcal{C}$).

In the case $\mathcal{C} = HTop$ (the homotopy category of topological spaces) and $\mathcal{D} = HPol$ (the homotopy category of polyhedra) (or $\mathcal{D} = HANR$ - the homotopy category of ANR's for metric spaces), one obtains the *coarse shape category* of topological spaces

$$Sh^* \equiv Sh^*_{(HTop,HPol)} \cong Sh^*_{(HTop,HANR)},$$

realized via the category pro^*-$HPol$ (or pro^*-$HANR$), and the *coarse shape functor*

$$S^* : HTop \to Sh^*,$$

which factorizes through the shape category Sh of topological spaces, i.e., $S^* = JS$, where $S : HTop \to Sh$ is the shape functor and

$$J : Sh \to Sh^*$$

is the "inclusion" (faithful) functor. The underlying theory is called the *coarse shape theory* for topological spaces. In the subspecial case $\mathcal{C} = HcM \subseteq HTop$ (the homotopy category of compact metric spaces) and $\mathcal{D} = HcPol \subseteq HPol$ (the homotopy category of compact polyhedra) (or $\mathcal{D} = HcANR \subseteq HANR$

- the homotopy category of compact ANR's), one obtains the coarse shape category of compacta

$$Sh^*(cM) \equiv Sh^*_{(HcM,HcPol)} \cong Sh^*_{(HcM,HcANR)},$$

realized "sequentially" via the category tow^*-$HcPol$ (or tow^*-$HcANR$), and the (restriction of the) coarse shape functor

$$S^* : HcM \to Sh^*(cM),$$

such that $S^* = JS$, where $S : HcM \to Sh(cM)$ is the shape functor on compacta, while

$$J : Sh(cM) \to Sh^*(cM)$$

is the corresponding "inclusion" functor. We have proved (Corollary 5.3) that, in general, the coarse shape classification of compacta is indeed (strictly) coarser than the shape classification.

Since the problem of an easy recognition of an isomorphism in pro^*-$\mathcal{C}$ is the most important, we have established an analogue (Theorem 6.1) of the well known Morita lemma in *pro*-Top [17], such that its reduction to the pro-category is exactly the Morita lemma. Hereby an analogue in pro^*-$\mathcal{C}$ of the "reindexing theorem" in *pro*-$\mathcal{C}$ has been needed, since a morphism of pro^*-$\mathcal{C}$, in general, does not admit a level representative. Concerning inverse sequences (a level representative is not indispensable), we have got a very operative sufficient condition (Theorem 6.4) for an isomorphism in tow^*-$\mathcal{C}$.

At the end we have constructed a pair of pro-groups $\boldsymbol{G}, \boldsymbol{H}$ such that $\boldsymbol{G} \cong \boldsymbol{H}$ in pro^*-Grp, while $\boldsymbol{G}$ and $\boldsymbol{H}$ are not isomorphic in *pro*-Grp (Example 7.1), as well as a pair of pro-spaces $\boldsymbol{X}, \boldsymbol{Y}$ such that $\boldsymbol{X} \cong \boldsymbol{Y}$ in pro^*-$HPol \subseteq pro^*$-$HTop$, while $\boldsymbol{X}$ and $\boldsymbol{Y}$ are not isomorphic in *pro*-$HTop$ (Example 7.2). They confirm that the coarse shape theory is indeed a new nontrivial tool for studying and classifying locally bad spaces.

2. Preliminaries

For the sake of completeness, let us briefly recall the well known notions and main facts concerning a pro-category and a shape category (see [15]) as well as the recently constructed category $\mathcal{S}^*$ (see [16]). The category language follows [7].

Let $\mathcal{C}$ be a category. An *inverse system* in $\mathcal{C}$, denoted by $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, consists of a directed preordered set $(\Lambda, \leq)$, called the *index set*, of $\mathcal{C}$-objects X_λ for each $\lambda \in \Lambda$, called the *terms* of $\boldsymbol{X}$, and of $\mathcal{C}$-morphisms $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ ($p_{\lambda\lambda} = 1_{X_\lambda}$), for each related pair $\lambda \leq \lambda'$ in Λ, called the *bonding morphisms* of $\boldsymbol{X}$, such that

$$p_{\lambda\lambda'} p_{\lambda'\lambda''} = p_{\lambda\lambda''},$$

whenever $\lambda \leq \lambda' \leq \lambda''$. A *morphism of inverse systems* $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ consists of a function $f : M \to \Lambda$, called the *index function*, and of $\mathcal{C}$-morphisms $f_\mu : X_{f(\mu)} \to Y_\mu$ for each $\mu \in M$, such that, for every related pair $\mu \le \mu'$, there exists $\lambda \in \Lambda$, $\lambda \ge f(\mu), f(\mu')$, for which

$$f_\mu p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{f(\mu')\lambda}.$$

The *composition* of morphisms of inverse systems is defined as follows: Given any $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ and any $(g, g_\nu) : \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$, then $(g, g_\nu)(f, f_\mu) = (h, h_\nu) : \boldsymbol{X} \to \boldsymbol{Z}$, where $h = fg : N \to \Lambda$ and $h_\nu = g_\nu f_{g(\nu)} : X_{h(\nu)} \to Z_\nu$. Finally, the *identity* morphism on $\boldsymbol{X}$ is $(1_\Lambda, 1_{X_\lambda}) : \boldsymbol{X} \to \boldsymbol{X}$. In this way is obtained a category, denoted by *inv*-$\mathcal{C}$, whose objects are all inverse systems in $\mathcal{C}$ and whose morphisms are all morphisms of inverse systems described above.

Notice that, for every index set Λ, there exists a full subcategory $\mathcal{C}^\Lambda$ of *inv*-$\mathcal{C}$ determined by all inverse systems indexed by Λ. If $\Lambda = \mathbb{N}$, then $\mathcal{C}^{\mathbb{N}} \subseteq$ *inv*-$\mathcal{C}$ is the full subcategory of all inverse sequences in $\mathcal{C}$.

A morphism $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *equivalent to* a morphism $(f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$, denoted by $(f, f_\mu) \sim (f', f'_\mu)$, provided each $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), f'(\mu)$, such that

$$f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

This defines an equivalence relation on each set *inv*-$\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$. The equivalence class $[(f, f_\mu)]$ of (f, f_μ) is denoted by $\boldsymbol{f}$. Furthermore, the equivalence relation respects the composition in *inv*-$\mathcal{C}$, i.e., if $(f, f_\mu) \sim (f', f'_\mu)$ and $(g, g_\nu) \sim (g', g'_\nu)$, then $(g, g_\nu)(f, f_\mu) \sim (g', g'_\nu)(f', f'_\mu)$, whenever these compositions are defined. Therefore, there exists the corresponding quotient category (*inv*-$\mathcal{C}$)$/\sim$, denoted by *pro*-$\mathcal{C}$ and called the *pro-category* for the category $\mathcal{C}$. Its objects are all inverse systems $\boldsymbol{X}$ in $\mathcal{C}$ and its morphisms are all equivalence classes $\boldsymbol{f}$ of morphisms of *inv*-$\mathcal{C}$. The full subcategory of *pro*-$\mathcal{C}$ determined by all inverse sequences in $\mathcal{C}$ (corresponding to $\mathcal{C}^{\mathbb{N}}/\sim$) is usually called the *tow-category* of $\mathcal{C}$ and is denoted by *tow*-$\mathcal{C}$.

Recall that, if the index set M of an inverse system $\boldsymbol{Y}$ is *cofinite* (every $\mu \in M$ has at most finitely many predecessors), then every $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ admits a representative (f, f_μ), such that the index function $f : M \to \Lambda$ is increasing and, for every related pair $\mu \le \mu'$,

$$f_\mu p_{f(\mu)f(\mu')} = q_{\mu\mu'} f_{\mu'}.$$

Such a representative is called a *simple morphism* of inverse systems. A simple morphism $(1_\Lambda, f_\lambda)$, belonging to a subcategory $\mathcal{C}^\Lambda$, is called a *level morphism.* Finally, recall that every inverse system $\boldsymbol{X}$ admits an isomorphic (in *pro*-$\mathcal{C}$) $\boldsymbol{X}'$ having a cofinite index set.

Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. Given $X \in Ob\mathcal{C}$, a $\mathcal{D}$-*expansion* of X is a morphism $\boldsymbol{p} = [(c, p_\lambda)] : X \to \boldsymbol{X}$ of *pro*-$\mathcal{C}$ (X is a rudimentary system and c is the constant function), where $\boldsymbol{X}$ belongs to *pro*-$\mathcal{D}$, such that, for every $\boldsymbol{Y}$ in *pro*-$\mathcal{D}$ and every $\boldsymbol{p}' : X \to \boldsymbol{Y}$ in *pro*-$\mathcal{C}$, there exists a unique morphism

$\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ (in *pro*-$\mathcal{D}$) satisfying $\boldsymbol{f}\boldsymbol{p} = \boldsymbol{p}'$. $\mathcal{D}$ is said to be *dense* in $\mathcal{C}$ provided every $\mathcal{C}$-object X admits a $\mathcal{D}$-expansion $\boldsymbol{p} : X \to \boldsymbol{X}$.

Every two $\mathcal{D}$-expansions of the same object are naturally isomorphic (as the objects of *pro*-$\mathcal{D}$, by a unique isomorphism), and every system which is isomorphic to a $\mathcal{D}$-expansion of X is also a $\mathcal{D}$-expansion of X. A $\mathcal{D}$-expansion $\boldsymbol{p} : X \to \boldsymbol{X}$ is characterized by the following two properties:

(E1) for every $P \in Ob(\mathcal{D})$ and every $g : X \to P$ in $\mathcal{C}$, there exist $\lambda \in \Lambda$ and an $f : X_\lambda \to P$ in $\mathcal{D}$, such that $fp_\lambda = g$;
(E2) if $f, f' : X_\lambda \to P$ in $\mathcal{D}$ satisfy $fp_\lambda = f'p_\lambda$, then there exists $\lambda' \geq \lambda$ such that $fp_{\lambda\lambda'} = f'p_{\lambda\lambda'}$.

Let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{p}' : X \to \boldsymbol{X}'$ be $\mathcal{D}$-expansions of the same object X of $\mathcal{C}$, and let $\boldsymbol{q} : Y \to \boldsymbol{Y}$ and $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ be $\mathcal{D}$-expansions of the same object Y of $\mathcal{C}$. Then there exist two natural isomorphisms $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{X}'$ and $\boldsymbol{j} : \boldsymbol{Y} \to \boldsymbol{Y}'$. A morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *pro*-$\mathcal{D}$ *equivalent to* a morphism $\boldsymbol{f}' : \boldsymbol{X}' \to \boldsymbol{Y}'$, denoted by $\boldsymbol{f} \sim \boldsymbol{f}'$, provided the following diagram in *pro*-$\mathcal{D}$ commutes:

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\boldsymbol{i}} & \boldsymbol{X}' \\ \boldsymbol{f} \downarrow & & \downarrow \boldsymbol{f}' \\ \boldsymbol{Y} & \xrightarrow{\boldsymbol{j}} & \boldsymbol{Y}' \end{array}$$

It defines an equivalence relation on the appropriate subclass of $Mor($*pro*-$\mathcal{D})$. The equivalence class of $\boldsymbol{f}$ is denoted by $\langle \boldsymbol{f} \rangle$. If $\boldsymbol{f} \sim \boldsymbol{f}'$ and $\boldsymbol{g} \sim \boldsymbol{g}'$, then $\boldsymbol{g}\boldsymbol{f} \sim \boldsymbol{g}'\boldsymbol{f}'$ whenever it is defined. Further, given $\boldsymbol{p}$, $\boldsymbol{p}'$, $\boldsymbol{q}$, $\boldsymbol{q}'$ and $\boldsymbol{f}$, there exists a unique $\boldsymbol{f}'$ such that $\boldsymbol{f} \sim \boldsymbol{f}'$.

For given pair $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D}$ is dense in $\mathcal{C}$, one defines the (*abstract*) *shape category* $Sh_{(\mathcal{C},\mathcal{D})}$ *for* $(\mathcal{C}, \mathcal{D})$ as follows. The objects of $Sh_{(\mathcal{C},\mathcal{D})}$ are all the objects of $\mathcal{C}$. A morphism $F \in Sh_{(\mathcal{C},\mathcal{D})}(X, Y)$ is the *pro*-$\mathcal{D}$ equivalence class $\langle \boldsymbol{f} \rangle$ of a morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, with respect to any choice of a pair of $\mathcal{D}$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$. In other words, a *shape morphism* $F : X \to Y$ is given by a diagram

$$\begin{array}{ccc} \boldsymbol{X} & \xleftarrow{\boldsymbol{p}} & X \\ \boldsymbol{f} \downarrow & & \downarrow F \\ \boldsymbol{Y} & \xleftarrow{\boldsymbol{q}} & Y \end{array}$$

The *composition* of $F : X \to Y$, $F = \langle \boldsymbol{f} \rangle$ and $G : Y \to Z$, $G = \langle \boldsymbol{g} \rangle$, is well defined by the representatives, i.e., $GF : X \to Z$, $GF = \langle \boldsymbol{g}\boldsymbol{f} \rangle$. The *identity shape morphism* on an object X, $1_X : X \to X$, is the *pro*-$\mathcal{D}$ equivalence class $\langle \boldsymbol{1_X} \rangle$ of the identity morphism $\boldsymbol{1_X}$ in *pro*-$\mathcal{D}$. Since

$$Sh_{(\mathcal{C},\mathcal{D})}(X, Y) \approx pro - \mathcal{D}(\boldsymbol{X}, \boldsymbol{Y})$$

is a set, the shape category $Sh_{(\mathcal{C},\mathcal{D})}$ is well defined. One often says that *pro*-$\mathcal{D}$ is the *realizing* category for the shape category $Sh_{(\mathcal{C},\mathcal{D})}$.

For every $f : X \to Y$ in $\mathcal{C}$ and every pair of $\mathcal{D}$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$, there exists a unique $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in *pro*-$\mathcal{D}$, such that the following diagram in *pro*-$\mathcal{C}$ commutes:

$$\begin{array}{ccc} \boldsymbol{X} & \stackrel{\boldsymbol{p}}{\longleftarrow} & X \\ \boldsymbol{f} \downarrow & & \downarrow f \\ \boldsymbol{Y} & \stackrel{\boldsymbol{q}}{\longleftarrow} & Y \end{array} .$$

The same f and another pair of $\mathcal{D}$-expansions $\boldsymbol{p}' : X \to \boldsymbol{X}'$, $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ yield a unique $\boldsymbol{f}' : \boldsymbol{X}' \to \boldsymbol{Y}'$ in *pro*-$\mathcal{D}$. Then, however, $\boldsymbol{f} \sim \boldsymbol{f}'$ in *pro*-$\mathcal{D}$ must hold. Thus, every morphism $f \in \mathcal{C}(X,Y)$ yields a unique *pro*-$\mathcal{D}$ equivalence class $\langle \boldsymbol{f} \rangle$, i.e., a unique shape morphism $F \in Sh_{(\mathcal{C},\mathcal{D})}(X,Y)$. If one defines $S(X) = X$, $X \in Ob\mathcal{C}$, and $S(f) = F = \langle \boldsymbol{f} \rangle$, $f \in Mor\mathcal{C}$, then

$$S : \mathcal{C} \to Sh_{(\mathcal{C},\mathcal{D})}$$

becomes a functor, called the (*abstract*) *shape functor*. The restriction of S to $\mathcal{D}$ into the full subcategory of $Sh_{(\mathcal{C},\mathcal{D})}$, determined by $Ob\mathcal{D}$, is a category isomorphism. Therefore, P and Q are isomorphic objects of $\mathcal{D}$ if and only if they are isomorphic in $Sh_{(\mathcal{C},\mathcal{D})}$, i.e., they are of the same shape. Finally, if $X \in Ob\mathcal{C}$ and $P \in Ob\mathcal{D}$, then every shape morphism $F : X \to P$ admits a unique morphism $f : X \to P$ in $\mathcal{C}$ such that $S(f) = F$. Thus, the restriction function (of S)

$$S|\cdot : \mathcal{C}(X,P) \to Sh_{(\mathcal{C},\mathcal{D})}(X,P)$$

is a bijection. The most interesting example of the above construction is $\mathcal{C} = HTop$ - the homotopy category of topological spaces and $\mathcal{D} = HPol$ - the homotopy category of polyhedra (or $\mathcal{D} = HANR$ - the homotopy category of ANR's for metric spaces, which yields the same theory, since $Ob(Pol)$ and $Ob(ANR)$ are homotopy equivalent classes). Namely, the (full) subcategory $HPol \subseteq HTop$ is dense in $HTop$, since every space X admits a $HPol$-expansion $\boldsymbol{p} = ([p_\lambda]) : X \to \boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$, which is obtained by applying the homotopy functor to a polyhedral resolution $(p_\lambda) : X \to \underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of X, [12]. In this case, one speaks about the (ordinary or standard) *shape category* $Sh_{(HTop,HPol)} \equiv Sh$ of topological spaces and (ordinary or standard) *shape functor* $S : Htop \to Sh$. Clearly, the realizing category for Sh is the pro-category *pro*-$HPol$ (or *pro*-$HANR$). The underlying theory is called the (ordinary or standard) *shape theory* for topological spaces.

Let $HcM \subseteq HTop$ denote the homotopy subcategory of compact metric spaces, and let $HcPol \subseteq HPol$ denote the homotopy subcategory of compact polyhedra. Since $HcPol \subseteq HcM$ is a "*sequentially*" dense subcategory (every compactum X admits a $HcPol$-expansion $\boldsymbol{p} = ([p_i]) : X \to \boldsymbol{X} = (X_i, [p_{ii'}], \mathbb{N})$, which is obtained by applying the homotopy functor to the limit $(p_i) : X \to \underline{X} = (X_i, p_{ii'})$ of an inverse *sequence* of compact polyhedra, $X = \lim \underline{X}$, [5]), there exists the shape category of compacta,

$Sh_{(HcM,HcPol)} \equiv Sh(cM)$, which is a full subcategory of Sh. Notice that the realizing category for $Sh(cM)$ is the tow-category *tow-HcPol*. Clearly, since the classes $Ob(cPol)$ and $Ob(cANR)$ (all compact ANR's for metric spaces) are homotopy equivalent, the tow-category *tow-HcANR* may also serve as the realizing category for the shape category $Sh(cM)$.

Let us now recall the Mardešić-Uglešić category $\mathcal{S}^*$, [16]. Its object class (indeed a set) consists of all compact metric spaces, while the morphisms are more sophisticated than the corresponding shape morphisms. First, an S^**-mapping* $(f,[f_j^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ of an inverse sequence of metric compacta $\boldsymbol{X} = (X_i,[p_{ii'}]) \in Ob\left(HcM^{\mathbb{N}}\right)$ to another one $\boldsymbol{Y} = (Y_j,[q_{jj'}])$ consists of an increasing and unbounded function $f : \mathbb{N} \to \mathbb{N}$ and of a set of homotopy classes $[f_j^n]$ of mappings $f_j^n : X_{f(j)} \to Y_j$, $n \in \mathbb{N}$, $j \in \mathbb{N}$, such that there exists an increasing unbounded function $\gamma : \mathbb{N} \to \mathbb{N}$, called the commutativity radius for $(f,[f_j^n])$, which has the property that, for every $n \in \mathbb{N}$, the following (finite) diagram is commutative:

$$\begin{array}{ccccccc} X_{f(1)} & \leftarrow & X_{f(2)} & \leftarrow & \cdots & \leftarrow & X_{f(\gamma(n))} \\ {\scriptstyle [f_1^n]}\downarrow & & \downarrow{\scriptstyle [f_2^n]} & & \cdots & & \downarrow{\scriptstyle [f_{\gamma(n)}^n]} \\ Y_1 & \leftarrow & Y_2 & \leftarrow & \cdots & \leftarrow & Y_{\gamma(n)} \end{array} . \tag{1}$$

(The case $\gamma(n) = 1$ is trivial, i.e., the diagram consisting of a single homotopy class $[f_1^n]$ is also considered to be commutative.) If $(f,[f_j^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g,[g_k^n]) : \boldsymbol{Y} \to \boldsymbol{Z}$ are the S^*-mappings, then their composition $(h,[h_k^n]) : \boldsymbol{X} \to \boldsymbol{Z}$ is defined by $h = fg$ and $h_k^n = g_k^n f_{g(k)}^n$. For every $\boldsymbol{X}$, the S^*-mapping $(1_{\mathbb{N}},[1_{X_j}^n]) : \boldsymbol{X} \to \boldsymbol{X}$, where $1_{X_j}^n : X_j \to X_j$ is the identity mapping for every pair $j,n \in \mathbb{N}$, is the S^*-identity mapping on $\boldsymbol{X}$.

An S^*-mapping $(f,[f_j^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *homotopic to* an S^*-mapping $(f',[f_j'^n]) : \boldsymbol{X} \to \boldsymbol{Y}$, $(f,[f_j^n]) \simeq (f',[f_j''^{n'}])$, provided there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$, $\sigma \geq f,\ f'$, called the *shift function* for $(f,[f_j^n])$, $(f',[f_j'^n])$, and there exists an increasing and unbounded function $\chi : \mathbb{N} \to \mathbb{N} \cup \{0\}$, called the *homotopy radius* for $(f,[f_j^n])$, $(f',[f_j'^n])$, such that, for every $n \in \mathbb{N}$ and every j, $1 \leq j \leq \chi(n)$,

$$[f_j^n][p_{f(j)\sigma(j)}] = [f_j'^n][p_{f'(j)\sigma(j)}].$$

The homotopy relation $\simeq$ is a natural equivalence relation on the class of all S^*-mappings. The homotopy class $[(f,[f_j^n])]$ of an S^*-mapping $(f,[f_j^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ is briefly denoted by $\boldsymbol{f}^*$. The composition of such homotopy classes is well defined by putting $\boldsymbol{g}^*\boldsymbol{f}^* = \boldsymbol{h}^* \equiv [(h,[h_k^n])]$, where $(h,[h_k^n]) = (g,[g_k^n])(f,[f_j^n])$. Let $\underline{\mathcal{S}}^*$ be the collection consisting of the class $Ob\left((HcM)^{\mathbb{N}}\right) = Ob(tow - HcM)$ of objects and of the class $Mor\underline{\mathcal{S}}^*$ of all the sets $\underline{\mathcal{S}}^*(\boldsymbol{X},\boldsymbol{Y})$ of all $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$. Then $\underline{\mathcal{S}}^*$, endowed with the above composition and all the identities $\boldsymbol{1}_{\boldsymbol{X}}^*$, makes a category. There, also, exists a functor $\underline{J} : tow\text{-}HcM \to \underline{\mathcal{S}}^*$, which keeps the objects fixed and, for every

$\boldsymbol{f} = [(f, f_j)] : \boldsymbol{X} \to \boldsymbol{Y}$ in *tow-HcM*, $\underline{J}(\boldsymbol{f}) = \boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ in $\underline{\mathcal{S}}^*$, where $\boldsymbol{f}^*$ is the homotopy class of the S^*-mapping $(f, [f_j^n])$, $f_j^n = f_j$ for every $n \in \mathbb{N}$. (In [16], the functor $\underline{J}$ is denoted by $\underline{S}^*$!)

Now, let the objects of $\mathcal{S}^*$ be all compact metric spaces. The morphisms of $\mathcal{S}^*$ are defined quite analogously to the shape morphisms in $Sh(cM)$. Let $\boldsymbol{p} = ([p_i]) : X \to \boldsymbol{X}$, $\boldsymbol{p}' = ([p_i']) : X \to \boldsymbol{X}'$ and $\boldsymbol{q} = ([q_j]) : Y \to \boldsymbol{Y}$, $\boldsymbol{q}' = ([q_j']) : Y \to \boldsymbol{Y}'$ be two sequential $HcANR$-expansions of X and Y respectively. Let $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{X}'$ and $\boldsymbol{j} : \boldsymbol{Y} \to \boldsymbol{Y}'$ be the (unique) natural isomorphisms in *tow-HcANR*. $\boldsymbol{f}^* \in \underline{\mathcal{S}}^*(\boldsymbol{X}, \boldsymbol{Y})$ is said to be equivalent to $\boldsymbol{f}^{*\prime} \in \underline{\mathcal{S}}^*(\boldsymbol{X}', \boldsymbol{Y}')$, denoted by $\boldsymbol{f}^* \sim \boldsymbol{f}^{*\prime}$, provided $\underline{\mathcal{S}}^*(\boldsymbol{j})\boldsymbol{f}^* = \boldsymbol{f}^{*\prime}\underline{\mathcal{S}}^*(\boldsymbol{i})$. This relation is a natural equivalence relation on $Mor\underline{\mathcal{S}}^*$. The equivalence class of $\boldsymbol{f}^*$ is denoted by $\langle \boldsymbol{f}^* \rangle$. Finally, a morphism $F^* \in \mathcal{S}^*(X, Y)$ is the $\underline{\mathcal{S}}^*$ equivalence class $\langle \boldsymbol{f}^* \rangle$ of a morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$, with respect to any choice of a pair of sequential $HcANR$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$. In other words, a morphism $F^* : X \to Y$ in $\mathcal{S}^*$ is given by a diagram

$$\begin{array}{ccc} \boldsymbol{X} & \xleftarrow{\boldsymbol{p}} & X \\ {\scriptstyle \boldsymbol{f}^*} \downarrow & & \downarrow {\scriptstyle F^*} \\ \boldsymbol{Y} & \xleftarrow{\boldsymbol{q}} & Y \end{array} .$$

The composition of $F^* : X \to Y$, $F^* = \langle \boldsymbol{f}^* \rangle$ and $G^* : Y \to Z$, $G^* = \langle \boldsymbol{g}^* \rangle$, is defined by the representatives, i.e., $G^*F^* : X \to Z$, $G^*F^* = \langle \boldsymbol{g}^*\boldsymbol{f}^* \rangle$. The identity on an object X in $\mathcal{S}^*$, $1_X^* : X \to X$, is the $\underline{\mathcal{S}}^*$ equivalence class $\langle \mathbf{1}_{\boldsymbol{X}}^* \rangle$ of the identity morphism $\mathbf{1}_{\boldsymbol{X}}^*$ in $\underline{\mathcal{S}}^*$. Since

$$\mathcal{S}^*(X, Y) \approx \underline{\mathcal{S}}^*(\boldsymbol{X}, \boldsymbol{Y})$$

is a set, the category $\mathcal{S}^*$ is well defined. One may say that $\underline{\mathcal{S}}^*$ is the *realizing* category for the category $\mathcal{S}^*$. There also exists a functor $J : Sh(cM) \to \mathcal{S}^*$ which keeps the objects fixed and, for every $F = \langle \boldsymbol{f} \rangle \in Sh(X, Y)$, $\boldsymbol{f} = [(f, f_j)] : \boldsymbol{X} \to \boldsymbol{Y}$, $J(F) = F^* = \langle \boldsymbol{f}^* \rangle$, $\boldsymbol{f}^* = \underline{J}(\boldsymbol{f})$. (In [16], the functor J is denoted by S^*!)

The category $\mathcal{S}^*$ classifies compacta (by its isomorphisms) strictly coarser than the shape category, [16]. This classification coincides with the classification of compacta by the Mardešić-Uglešić S^*-equivalence, which is a uniformization of the Mardešić S-equivalence, [11]. On compact polyhedra and compact ANR's, it coincides with the shape type classification, i.e., with the homotopy type classification.

3. The pro*-categories

3.1. *The category tow*-$\mathcal{C}$.* First of all, we shall characterize the basic conditions for the category $\underline{\mathcal{S}}^*$ by means of conditions which are quite similar to those for the usual tow-category. This description will indicate how to generalize, for any category $\mathcal{C}$, the whole "S^*-structure" to obtain a category

tow^*-$\mathcal{C}$ on the inverse sequences in $\mathcal{C}$ as well as a category pro^*-$\mathcal{C}$ on the inverse systems in $\mathcal{C}$.

THEOREM 3.1. *Let* $\boldsymbol{X} = (X_i, [p_{ii'}])$ *and* $\boldsymbol{Y} = (Y_j, [q_{jj'}])$ *be inverse sequences of metric compacta. Let* $f : \mathbb{N} \to \mathbb{N}$ *be an increasing unbounded function and let, for every* $n \in \mathbb{N}$ *and for every* $j \in \mathbb{N}$, $f_j^n : X_{f(j)} \to Y_j$ *be a mapping. Then the following three conditions are equivalent:*

(i) $(f, [f_j^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ *is an* S^**-mapping.*

(ii) For every related pair $j \leq j'$ *in* $\mathbb{N}$, *there exists* $n \in \mathbb{N}$ *such that, for every* $n' \geqslant n$,

$$[f_j^{n'}][p_{f(j)f(j')}] = [q_{jj'}][f_{j'}^{n'}]. \tag{2}$$

(iii) For every $j \in \mathbb{N}$, *there exists* $n \in \mathbb{N}$ *such that, for every* $n' \geq n$,

$$[f_j^{n'}][p_{f(j)f(j+1)}] = [q_{jj+1}][f_{j+1}^{n'}]. \tag{2'}$$

PROOF. Let $(f, [f_j^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ be an S^*-mapping. Let $\gamma : \mathbb{N} \to \mathbb{N}$ be a commutative radius for $(f, [f_j^n])$. Let any pair $j \leq j'$ in $\mathbb{N}$ be given. Since γ is unbounded, there exists $n \in \mathbb{N}$ such that $\gamma(n) \geqslant j'$. Let $n' \geq n$. Since γ increases, diagram (1) implies that

$$[f_j^{n'}][p_{f(j)f(j')}] = [q_{jj'}][f_{j'}^{n'}].$$

Therefore, (i) implies (ii). The implication (ii) $\Rightarrow$ (iii) is trivial ($j' = j + 1$). Let us now prove that (iii) implies (i). Let an increasing unbounded function $f : \mathbb{N} \to \mathbb{N}$ and a set of mappings $f_j^n : X_{f(j)} \to Y_j$, $n \in \mathbb{N}$, $j \in \mathbb{N}$, be given, such that, for every $j \in \mathbb{N}$, there exists $n \equiv n_j \in \mathbb{N}$ so that, for every $n' \geq n_j$, condition (2′) holds. Let us define a function $\gamma : \mathbb{N} \to \mathbb{N}$ to be a commutative radius for $\left(f, \left[f_j^n\right]\right)$. Consider a strictly increasing sequence (m_k) in $\mathbb{N} \cup \{0\}$, $k \in \mathbb{N} \cup \{0\}$, defined by induction as follows:

$$m_0 = 0,\ m_1 = \max\{n_1, m_0 + 1\}, \ldots,\ m_{k+1} = \max\{n_{k+1}, m_k + 1\}, \ldots.$$

Then put $\gamma(n) = k + 1$, $n \in \mathbb{N}$, where $k = k(n)$ is the unique element of $\mathbb{N} \cup \{0\}$ satisfying $m_k \leq n < m_{k+1}$. Thus, for every n, $m_{\gamma(n)-1} \leq n < m_{\gamma(n)}$. Clearly, the function γ is increasing and unbounded. Let $n \in \mathbb{N}$. If $\gamma(n) = 1$, then there is nothing to prove. Let $\gamma(n) > 1$. Then, by construction, $n \geqslant n_0 = \max\{n_1, \ldots, n_{\gamma(n)-1}\}$, and thus, for every $j = 1, \ldots, \gamma(n) - 1$,

$$[f_j^n][p_{f(j)f(j+1)}] = [q_{jj+1}][f_{j+1}^n].$$

Therefore, the appropriate diagram (1) commutes, i.e., $(f, \left[f_j^n\right])$ is an S^*-mapping of $\boldsymbol{X}$ to $\boldsymbol{Y}$. □

THEOREM 3.2. *An* S^**-mapping* $(f, [f_j^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ *is homotopic to an* S^**-mapping* $(f', [f_j'^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ *if and only if, for every* $j \in \mathbb{N}$, *there exists* $i \in \mathbb{N}$, $i \geq f(j), f'(j)$, *and there exists* $n \in \mathbb{N}$ *such that, for every* $n' \geqslant n$

$$[f_j^{n'}][p_{f(j)i}] = [f_j'^{n'}][p_{f'(j)i}]. \tag{3}$$

PROOF. Let $(f,[f_j^n]) \simeq (f',[f_j'^n])$ be realized via σ and χ. Given any $j \in \mathbb{N}$, put $i = \sigma(j) \in \mathbb{N}$ and choose $n \in \mathbb{N}$ such that $\chi(n) \geqslant j$ (χ is unbounded). Since χ increases, $\chi(n') \geq j$ whenever $n' \geq n$. Hence, relation (3) holds. Conversely, assume that, for every $j \in \mathbb{N}$, there exist $i \equiv i_j, n \equiv n_j \in \mathbb{N}$, $i_j \geq f(j), f'(j)$, such that, for every $n' \geq n_j$, relation (3) holds. Let us define a shift function $\sigma : \mathbb{N} \to \mathbb{N}$ by induction as follows:

$$\sigma(1) = i_1,\ \sigma(2) = \max\{i_2, \sigma(1)\}, \dots,\ \sigma(j+1) = \max\{i_{j+1}, \sigma(j)\}, \dots$$

Then, obviously, σ is increasing and $\sigma \geq f, f'$. Further, consider a strictly increasing sequence (m_k) in $\mathbb{N} \cup \{0\}$, $k \in \mathbb{N} \cup \{0\}$, defined by induction in the following way:

$$m_0 = 0,\ m_1 = \max\{n_1, m_0+1\}, \dots,\ m_{k+1} = \max\{n_{k+1}, m_k+1\}, \dots.$$

Let us define a homotopy radius $\chi : \mathbb{N} \to \mathbb{N} \cup \{0\}$ by putting $\chi(n) = k$, where $k = k(n)$ is the unique element of $\mathbb{N} \cup \{0\}$ satisfying $m_k \leq n < m_{k+1}$. Thus, for every n, $m_{\chi(n)} \leq n < m_{\chi(n)+1}$. Clearly, χ is increasing and unbounded. Let $n \in \mathbb{N}$. If $\chi(n) = 0$, then there is nothing to prove. Let $\chi(n) > 0$. Then, by construction, $n \geq n_0 = \max\{n_1, \dots, n_{\chi(n)}\}$. Thus, for every $j = 1, \dots, \chi(n)$,

$$f_j^n p_{f(j)\sigma(j)} \simeq f_j'^n p_{f'(j)\sigma(j)},$$

which shows that the functions σ and χ realize the homotopy relation $(f,[f_j^n]) \simeq (f',[f_j'^n])$. □

We shall now use condition (iii) of Theorem 3.1 to define an analogue of S^*-mapping of inverse sequences in any category $\mathcal{C}$. The conditions that the index function has to be increasing and that the corresponding rectangles commute will be relaxed in the usual way.

DEFINITION 3.3. *Let $\mathcal{C}$ be a category, and let $\boldsymbol{X} = (X_i, p_{ii'})$ and $\boldsymbol{Y} = (Y_j, q_{jj'})$ be inverse sequences in $\mathcal{C}$. An S^*-**morphism of inverse sequences** in $\mathcal{C}$, $(f, f_j^n) : \boldsymbol{X} \to \boldsymbol{Y}$, consists of a function $f : \mathbb{N} \to \mathbb{N}$, called the **index function**, and of a set of morphisms $f_j^n : X_{f(j)} \to Y_j$, $n \in \mathbb{N}$, $j \in \mathbb{N}$, in $\mathcal{C}$, such that, for every $j \in \mathbb{N}$, there exists $i \in \mathbb{N}$, $i \geqslant f(j), f(j+1)$, and there exists $n \in \mathbb{N}$ so that, for every $n' \geqslant n$,*

$$f_j^{n'} p_{f(j)i} = q_{jj+1} f_{j+1}^{n'} p_{f(j+1)i}.$$

*If the index function f is increasing and, for every $j \in \mathbb{N}$, $i = f(j+1)$, then (f, f_j^n) is said to be a **simple** S^*-morphism. If, in addition, $f = 1_{\mathbb{N}}$, then $(1_{\mathbb{N}}, f_j^n)$ is said to be a **level** S^*-morphism. Further, an S^*-morphism $(f, f_j^n) : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be **commutative**, provided, for every $j \in \mathbb{N}$, one may put $n = 1$.*

REMARK 3.4. (a) One can easily verify that the analogous definition of an "S^*-morphism" of inverse sequences by means of condition (ii) of Theorem 3.1 (i.e., ... for every related pair $j \leq j'$ in $\mathbb{N}$, there exists $i \in \mathbb{N}$, $i \geqslant f(j), f(j')$

and there exists $n \in \mathbb{N}$ such that, for every $n' \geq n$, ...) yields the same notion.

(b) The additional condition for a simple S^*-morphism is a (non-essential) property of an S^*-mapping by its definition.

(c) Notice that a commutative S^*-morphism of inverse sequences (f, f_j^n) : $\boldsymbol{X} \to \boldsymbol{Y}$ yields a sequence of morphisms $(f^n = f, f_j^n) : \boldsymbol{X} \to \boldsymbol{Y}$, $n \in \mathbb{N}$, in $\mathcal{C}^{\mathbb{N}}$. On the other side, every sequence of simple morphisms $(f^n, f_j^n) : \boldsymbol{X} \to \boldsymbol{Y}$, $n \in \mathbb{N}$, in $\mathcal{C}^{\mathbb{N}}$, such that $f^n = f$ for all n, determines the unique commutative S^*-morphism of the inverse sequences $(f, f_j^n) : \boldsymbol{X} \to \boldsymbol{Y}$. This fact indicates the huge difference between the standard morphisms of inverse sequences and the new S^*-morphisms.

LEMMA 3.5. *Let* $(f, f_j^n) : \boldsymbol{X} \to \boldsymbol{Y}$ *and* $(g, g_k^n) : \boldsymbol{Y} \to \boldsymbol{Z} = (Z_k, r_{kk'})$ *be* S^**-morphisms of inverse sequences in* $\mathcal{C}$*. Then* (h, h_k^n)*, where* $h = fg$ *and* $h_k^n = g_k^n f_{g(k)}^n$*, is an* S^**-morphism of* $\boldsymbol{X}$ *to* $\boldsymbol{Z}$*.*

PROOF. Let $k \in \mathbb{N}$. Since (g, g_k^n) is an S^*-morphism, there exists $j \in \mathbb{N}$, $j \geq g(k), g(k+1)$, and there exists $n_0 \in \mathbb{N}$ such that, for every $n' \geq n_0$,

$$g_k^{n'} q_{g(k)j} = r_{kk+1} g_{k+1}^{n'} q_{g(k+1)j}.$$

Since (f, f_j^n) is an S^*-morphism, for the indices $g(k)$, $g(k+1)$, j, there exist $i_1 \geq fg(k), f(g(k)+1)$, $i_2 \geq fg(k+1), f(g(k+1)+1)$, $i_3 \geq f(j), f(j+1)$, and there exist $n_1, n_2, n_3 \in \mathbb{N}$ such that, for every $n' \geq n_1$, every $n' \geq n_2$ and every $n' \geq n_3$, the appropriate relations for (f, f_j^n) hold respectively. Notice that $j+1 \geq g(k)+1, g(k+1)+1$. Put $i = \max\{i_1, i_2, i_3\} \geq fg(k), fg(k+1)$ and $n = \max\{n_0, n_1, n_2, n_3\}$. Then, for every $n' \geq n$, one straightforwardly establishes

$$g_k^{n'} f_{g(k)}^{n'} p_{fg(k)i} = r_{kk+1} g_{k+1}^{n'} f_{g(k+1)}^{n'} p_{fg(k+1)i},$$

which proves that $(h = fg, h_k^n = g_k^n f_{g(k)}^n) : \boldsymbol{X} \to \boldsymbol{Z}$ is an S^*-morphism. □

Lemma 3.5 enables us to define the *composition* of S^*-morphisms (f, f_j^n) : $\boldsymbol{X} \to \boldsymbol{Y}$ and $(g, g_k^n) : \boldsymbol{Y} \to \boldsymbol{Z}$ to be $(fg, g_k^n f_{g(k)}^n) : \boldsymbol{X} \to \boldsymbol{Z}$. Since the composition of functions and composition of morphisms in $\mathcal{C}$ are associative, the composition of S^*-morphisms is associative.

LEMMA 3.6. *The composition of commutative* S^**-morphisms of inverse sequences in* $\mathcal{C}$ *is a commutative* S^**-morphism.*

PROOF. It suffices to observe that in the proof of Lemma 3.5, in this case, one may put $n_0 = n_1 = n_2 = n_3 = 1$. The conclusion follows. □

Given an inverse sequence $\boldsymbol{X} = (X_i, p_{ii'})$ in $\mathcal{C}$, let $(1_{\mathbb{N}}, 1_{X_i}^n)$ be defined by the identity function $1_{\mathbb{N}}$ on $\mathbb{N}$ and by the identity morphisms $1_{X_i} : X_i \to X_i$ in $\mathcal{C}$, $i \in \mathbb{N}$, i.e., $1_{X_i}^n = 1_{X_i}$ for every $n \in \mathbb{N}$. Then $(1_{\mathbb{N}}, 1_{X_i}^n) : \boldsymbol{X} \to \boldsymbol{X}$ is an S^*-morphism (commutative and leveled). One readily sees that, for every

$(f, f_j^n) : \boldsymbol{X} \to \boldsymbol{Y}$ and every $(g, g_i^n) : \boldsymbol{Z} \to \boldsymbol{X}$, $(f, f_j^n)(1_{\mathbb{N}}, 1_{X_i}^n) = (f, f_j^n)$ and $(1_{\mathbb{N}}, 1_{X_i}^n)(g, g_i^n) = (g, g_i^n)$ hold. Thus, $(1_{\mathbb{N}}, 1_{X_i}^n)$ may be called the *identity* S^*-morphism on $\boldsymbol{X}$.

As a summary, for every category $\mathcal{C}$, there exists a category, denoted by $(\mathcal{C}^{\mathbb{N}})^*$, consisting of the object class $Ob(\mathcal{C}^{\mathbb{N}})^* = Ob\mathcal{C}^{\mathbb{N}}$ and of the morphism class $Mor(\mathcal{C}^{\mathbb{N}})^*$ of all the sets $(\mathcal{C}^{\mathbb{N}})^*(\boldsymbol{X}, \boldsymbol{Y})$ of all S^*-morphisms (f, f_j^n) of $\boldsymbol{X}$ to $\boldsymbol{Y}$, endowed with the composition and identities described above. By Lemma 3.6, there exists a subcategory $(\mathcal{C}^{\mathbb{N}})^*_\omega$ of $(\mathcal{C}^{\mathbb{N}})^*$ with the same object class and with the morphism class $Mor(\mathcal{C}^{\mathbb{N}})^*_\omega$ consisting of all commutative S^*-morphisms of inverse sequences in $\mathcal{C}$.

REMARK 3.7. Let (f, f_j) be a morphism in $\mathcal{C}^{\mathbb{N}}$. For every $n \in \mathbb{N}$, put $f^n = f$ and $f_j^n = f_j$ for all $j \in \mathbb{N}$. Consider all such sequences $(f^n, f_j^n)_{n \in \mathbb{N}}$ to be new morphisms, and define the new composition coordinatewise. Then, clearly, the new category with the same object class $Ob(\mathcal{C}^{\mathbb{N}})$ is isomorphic to $\mathcal{C}^{\mathbb{N}}$. On the other hand, it is obvious that the new category is a subcategory of $(\mathcal{C}^{\mathbb{N}})^*_\omega$. Consequently, the category $\mathcal{C}^{\mathbb{N}}$ may be considered as a subcategory of $(\mathcal{C}^{\mathbb{N}})^*$ in a way that the morphism sets are significantly enriched.

We shall now use Theorem 3.2 to define an equivalence relation on a set $(\mathcal{C}^{\mathbb{N}})^*(\boldsymbol{X}, \boldsymbol{Y})$.

DEFINITION 3.8. *An S^*-morphism $(f, f_j^n) : \boldsymbol{X} \to \boldsymbol{Y}$ of inverse sequences in $\mathcal{C}$ is said to be* ***equivalent to*** *an S^*-morphism $(f', f_j'^n) : \boldsymbol{X} \to \boldsymbol{Y}$, denoted by $(f, f_j^n) \sim (f', f_j'^n)$, provided every $j \in \mathbb{N}$ admits $i \in \mathbb{N}$, $i \geqslant f(j), f'(j)$, and $n \in \mathbb{N}$, such that, for every $n' \geqslant n$,*

$$f_j^{n'} p_{f(j)i} = f_j'^{n'} p_{f'(j)i}.$$

LEMMA 3.9. *The relation $\sim$ is an equivalence relation on each set $(\mathcal{C}^{\mathbb{N}})^*(\boldsymbol{X}, \boldsymbol{Y})$. The equivalence class $[(f, f_j^n)]$ of an S^*-morphism $(f, f_j^n) : \boldsymbol{X} \to \boldsymbol{Y}$ is briefly denoted by $\boldsymbol{f}^*$.*

PROOF. The relation $\sim$ is obviously reflexive and symmetric. To prove transitivity, one should take, for given $j \in \mathbb{N}$, the maximums of pairs of indices $\{i_1, i_2\}$ and $\{n_1, n_2\}$, which exist by $(f, f_j^n) \sim (f', f_j'^n)$ (i_1 and n_1) and $(f', f_j'^n) \sim (f'', f_j''^n)$ (i_2 and n_2). ◻

LEMMA 3.10. *Let $(f, f_j^n), (f', f_j'^n) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g, g_k^n), (g', g_k'^n) : \boldsymbol{Y} \to \boldsymbol{Z}$ be S^*-morphisms in $(\mathcal{C}^{\mathbb{N}})^*$. If $(f, f_j^n) \sim (f', f_j'^n)$ and $(g, g_k^n) \sim (g', g_k'^n)$, then $(g, g_k^n)(f, f_j^n) \sim (g', g_k'^n)(f', f_j'^n)$.*

PROOF. According to Lemma 3.9 (transitivity), it suffices to prove that $(g, g_k^n)(f, f_j^n) \sim (g, g_k^n)(f', f_j'^n)$ and $(g, g_k^n)(f, f_j^n) \sim (g', g_k'^n)(f, f_j^n)$. Given $k \in \mathbb{N}$, choose $i \in \mathbb{N}$, $i \geq fg(k), f'g(k)$, and $n \in \mathbb{N}$, by $(f, f_j^n) \sim (f', f_j'^n)$ for $j = g(k)$. Then, for every $n' \geq n$,

$$g_k^{n'} f_{g(k)}^{n'} p_{fg(k)i} = g_k^{n'} f_{g(k)}'^{n'} p_{f'g(k)i}.$$

Thus, $(g,g_k^n)(f,f_j^n) \sim (g,g_k^n)(f',f_j'^n)$. Further, if $(g,g_k^n) \sim (g',g_k'^n)$, then, for given $k \in \mathbb{N}$, there exist $j \geq g(k), g'(k)$ and n_1 such that

$$g_k^{n'} q_{g(k)j} = g_k'^{n'} q_{g'(k)j},$$

whenever $n' \geq n_1$. Since (f,f_j^n) is an S^*-morphism, there exist

$$i \geq \max\{fg(k), f(g(k)+1), fg'(k), f(g'(k)+1), f(j), f(j+1)\}$$

and n_2 large enough, such that, for every $n' \geq n_2$, the n'-coordinate of (f,f_j^n) commutes at the indices $g(k)$, $g'(k)$ and j with "the tail" at i. Thus,

$$f_{j_1}^{n'} p_{f(j_1)i} = q_{j_1 j_2} f_{j_2}^{n'} p_{f(j_2)i},$$

where $j_1 = \min\{g(k), g'(k)\}$ and $j_2 = \max\{g(k), g'(k)\}$. Consequently, for every $n' \geq n = \max\{n_1, n_2\}$,

$$g_k^{n'} f_{g(k)}^{n'} p_{fg(k)i} = g_k'^{n'} f_{g'(k)}^{n'} p_{fg'(k)i}.$$

Therefore, $(g,g_k^n)(f,f_j^n) \sim (g',g_k'^n)(f,f_j^n)$. □

By Lemmata 3.9 and 3.10, one may compose the equivalence classes of S^*-morphisms of inverse squences by putting $\boldsymbol{g}^*\boldsymbol{f}^* = \boldsymbol{h}^* \equiv [(h,h_k^n)]$, where $(h,h_k^n) = (g,g_k^n)(f,f_j^n) = (fg, g_k^n f_{g(k)}^n)$. The corresponding quotient category $\left(\mathcal{C}^{\mathbb{N}}\right)^* /\!\sim$ is denoted by *tow**-$\mathcal{C}$. There exists a subcategory *tow*$^*_\omega$-$\mathcal{C} \subseteq$ *tow**-$\mathcal{C}$ determined by all equivalence classes having commutative representatives. Clearly, *tow*$^*_\omega$-$\mathcal{C}$ is isomorphic to the quotient category $\left(\mathcal{C}^{\mathbb{N}}\right)^*_\omega /\!\sim$. According to Remark 3.7, one may consider *tow*-$\mathcal{C} = (\mathcal{C}^{\mathbb{N}})/\sim$ as a subcategory of *tow*$^*_\omega$-$\mathcal{C}$ and, consequently, as a subcategory of *tow**-$\mathcal{C}$. Namely, by Theorem 3.2, the equivalence relations $\sim$ in $\mathcal{C}^{\mathbb{N}}$ and in $(\mathcal{C}^{\mathbb{N}})^*_\omega$ are of the same kind. (See also Proposition 3.13 below.)

Proposition 3.11. *Every morphism* $\boldsymbol{f}^* = [(f,f_j^n)] : \boldsymbol{X} \to \boldsymbol{Y}$ *in tow**-$\mathcal{C}$ *admits a simple representative* $(f',f_j'^n)$. *Moreover, one can achieve the index function* f' *to be strictly increasing.*

Proof. Let (f,f_j^n) be any representative of $\boldsymbol{f}^*$. Then, for every $j \in \mathbb{N}$, there exists $i \equiv i_j \in \mathbb{N}$, $i \geqslant f(j), f(j+1)$, and there exists $n \equiv n_j \in \mathbb{N}$ so that, for every $n' \geqslant n$,

$$f_j^{n'} p_{f(j)i} = q_{jj+1} f_{j+1}^{n'} p_{f(j+1)i}.$$

Let us define a function $f' : \mathbb{N} \to \mathbb{N}$ by induction as follows:

$$\begin{aligned} &f'(1) = f(1), \\ &f'(2) = \max\{i_1, f'(1)+1\}, \ldots, \\ &f'(j+1) = \max\{i_j, f'(j)+1\}, \ldots \end{aligned}$$

Clearly, f' is strictly increasing and $f' \geq f$. Further, for every $n \in \mathbb{N}$ and every $j \in \mathbb{N}$, let $f'^n_j = f^n_j p_{f(j)f'(j)} : X_{f'(j)} \to Y_j$. Now, given any $j \in \mathbb{N}$, put $i = f'(j+1)$ and $n = n_j$. Then, for every $n' \geq n$, the above relation implies

$$f'^{n'}_j p_{f'(j)f'(j+1)} = q_{jj+1} f'^{n'}_{j+1}.$$

Thus, $(f', f'^n_j) : \boldsymbol{X} \to \boldsymbol{Y}$ is a simple S^*-morphism. Furthermore, given any $j \in \mathbb{N}$, put $i = f'(j)$ and $n = 1$. Then, for every $n' \geqslant 1$,

$$f^{n'}_j p_{f(j)i} = f^{n'}_j p_{f(j)f'(j)} = f'^{n'}_j = f'^{n'}_j p_{f'(j)f'(j)} = f'^{n'}_j p_{f'(j)i},$$

which proves that $(f, f^n_j) \sim (f', f'^n_j)$. □

Let us observe that in the case $\mathcal{C} = HcM$, Proposition 3.11 yields the following corollary:

Corollary 3.12. *The category tow^*-HcM is isomorphic to the category $\underline{\mathcal{S}}^*$. An isomorphism tow^*-$HcM \to \underline{\mathcal{S}}^*$ is given by the identity on the object class and by $[(f, [f^n_j])] \mapsto [(f', [f'^n_j])]$, where $(f', [f'^n_j])$ is a simple representative of $[(f, [f^n_j])]$.*

Let us define a functor $\underline{J} \equiv \underline{J}_{\mathcal{C}} : tow\text{-}\mathcal{C} \to tow^*\text{-}\mathcal{C}$, which keeps the objects fixed, by putting $\underline{J}(\boldsymbol{f}) = \boldsymbol{f}^*$, where $\boldsymbol{f} = [(f, f_j)]$, $\boldsymbol{f}^* = [(f, f^n_j)]$ and (f, f^n_j) is *induced* by (f, f_j) in the following way: The index function f is the same, while, for every $n \in \mathbb{N}$, $f^n_j = f_j$ for all $j \in \mathbb{N}$. It is readily seen that $\underline{J}(\boldsymbol{f})$ is well defined and that $\underline{J}$ is indeed a functor. Notice that every induced $(f, f^n_j = f_j)$ is a commutative S^*-morphism of inverse sequences. Hence, $\underline{J}$ is actually a functor of tow-$\mathcal{C}$ to tow^*_ω-$\mathcal{C} \subseteq tow^*$-$\mathcal{C}$.

Proposition 3.13. *Functor $\underline{J}$ is faithful. Therefore, one may consider tow-$\mathcal{C}$ to be a subcategory of tow^*_ω-$\mathcal{C}$ and, consequently, a subcategory of tow^*-$\mathcal{C}$.*

Proof. Let $\boldsymbol{f}, \boldsymbol{f}' \in tow\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$, $\boldsymbol{f} = [(f, f_j)], \boldsymbol{f}' = [(f', f'_j)]$, be given such that $\underline{J}(\boldsymbol{f}) = \underline{J}(\boldsymbol{f}')$. Then, $(f, f^n_j) \sim (f', f'^n_j)$, where (f, f^n_j), (f', f'^n_j) are induced by (f, f_j), (f', f'_j) respectively. Thus, for every $j \in \mathbb{N}$, there exist $i, n \in \mathbb{N}$ such that, for every $n' \geq n$,

$$f^{n'}_j p_{f(j)i} = f'^{n'}_j p_{f'(j)i}.$$

This means

$$f_j p_{f(j)i} = f'_j p_{f'(j)i},$$

i.e., $(f, f_j) \sim (f', f'_j)$. Therefore, $\boldsymbol{f} = \boldsymbol{f}'$. □

Remark 3.14. Given any category $\mathcal{C}$, the category tow^*-$\mathcal{C}$ is constructed as a quotient category $(\mathcal{C}^{\mathbb{N}})^*/\sim$ on the inverse sequences in $\mathcal{C}$. It is a full analogue of the known category $\underline{\mathcal{S}}^*$ on compact metric inverse *sequences* ($\mathcal{C} = HcM$). According to Remark 3.4(a), one can construct in the same manner, for any *directed preordered set* $(\Lambda, \leq)$, the appropriate category $(\mathcal{C}^{\Lambda})^*$ as well as the corresponding quotient category $(\mathcal{C}^\Lambda)^*/\sim$. Moreover, there also exists a faithful functor of $(\mathcal{C}^\Lambda)/\sim$ to $(\mathcal{C}^\Lambda)^*_\omega/\sim\ \subseteq (\mathcal{C}^\Lambda)^*/\sim$. We shall not do this explicitly because we may even abandon the fixed index set and work in the most general setting of arbitrary inverse systems, i.e., in any inv-category.

3.2. *The category pro***-*$\mathcal{C}$. The next definition is based on condition (ii) of Theorem 3.1 (compare Definition 3.3 and Remark 3.4(a)).

DEFINITION 3.15. *Let* $\mathcal{C}$ *be a category and let* $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *and* $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ *be inverse systems in* $\mathcal{C}$. *An* S^****-morphism of inverse systems***, $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$, *consists of a function* $f : M \to \Lambda$, *called the* ***index function***, *and of a set of morphisms* $f_\mu^n : X_{f(\mu)} \to Y_\mu$, $n \in \mathbb{N}$, $\mu \in M$, *in* $\mathcal{C}$, *such that, for every related pair* $\mu \leq \mu'$ *in* M, *there exists* $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), f(\mu')$, *and there exists* $n \in \mathbb{N}$ *so that, for every* $n' \geqslant n$,

$$f_\mu^{n'} p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^{n'} p_{f(\mu')\lambda}.$$

If the index function f *is increasing and, for every pair* $\mu \leq \mu'$, *one may put* $\lambda = f(\mu')$, *then* (f, f_μ^n) *is said to be a* ***simple*** S^**-morphism. If, in addition,* $M = \Lambda$ *and* $f = 1_\Lambda$, *then* $(1_\Lambda, f_\lambda^n)$ *is said to be a* ***level*** S^**-morphism. Further, an* S^**-morphism of inverse systems* $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ *is said to be* ***commutative***, *provided, for every pair* $\mu \leq \mu'$, *one may put* $n = 1$.

REMARK 3.16. Similarly to Remark 3.4(c), a commutative S^*-morphism of inverse systems $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ yields a sequence of morphisms $(f^n = f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$, $n \in \mathbb{N}$, in *inv*-$\mathcal{C}$. On the other side, every sequence of simple morphisms $(f^n, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$, $n \in \mathbb{N}$, in *inv*-$\mathcal{C}$, such that $f^n = f$ for all n, determines the unique commutative S^*-morphism of the inverse systems $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$. This indicates the significant difference between the standard morphisms of inverse systems and the new S^*-morphisms.

LEMMA 3.17. *Let* $\left(f, f_\mu^n\right) : \boldsymbol{X} \to \boldsymbol{Y}$ *and* $(g, g_\nu^n) : \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$ *be* S^**-morphisms of inverse systems. Then* (h, h_ν^n), *where* $h = fg$ *and* $h_\nu^n = g_\nu^n f_{g(\nu)}^n$, $n \in \mathbb{N}$, $\nu \in N$, *is an* S^**-morphism of* $\boldsymbol{X}$ *to* $\boldsymbol{Z}$.

PROOF. Let $\nu, \nu' \in N$, $\nu \leq \nu'$, be given. Since (g, g_ν^n) is an S^*-morphism, there exists $\mu \in M$, $\mu \geq g(\nu), g(\nu')$, and there exists $n_0 \in \mathbb{N}$ such that, for every $n' \geq n_0$,

$$g_\nu^{n'} q_{g(\nu)\mu} = r_{\nu\nu'} g_{\nu'}^{n'} q_{g(\nu')\mu}.$$

Since $\left(f, f_\mu^n\right)$ is an S^*-morphism, for the pair $g(\nu) \leq \mu$, there exist $\lambda_1 \geq fg(\nu), f(\mu)$ and $n_1 \in \mathbb{N}$ such that, for every $n' \geq n_1$,

$$f_{g(\nu)}^{n'} p_{fg(\nu)\lambda_1} = q_{g(\nu)\mu} f_\mu^{n'} p_{f(\mu)\lambda_1}.$$

Further, for the pair $g(\nu') \leq \mu$, there exist $\lambda_2 \geq fg(\nu'), f(\mu)$ and $n_2 \in \mathbb{N}$ such that, for every $n' \geq n_2$,

$$f_{g(\nu'')}^{n'} p_{fg(\nu')\lambda_2} = q_{g(\nu')\mu} f_\mu^{n'} p_{f(\mu)\lambda_2}.$$

Since Λ is directed, there exists $\lambda \in \Lambda$, $\lambda \geq \lambda_1, \lambda_2$. Put $n = \max\{n_0, n_1, n_2\}$. Then, for every $n' \geq n$, one straightforwardly establishes

$$g_\nu^{n'} f_{g(\nu)}^{n'} p_{fg(\nu)\lambda} = r_{\nu\nu'} g_{\nu'}^{n'} f_{g(\nu')}^{n'} p_{fg(\nu')\lambda},$$

which proves that $(h = fg, h_\nu^n = g_\nu^n f_{g(\nu)}^n) : \boldsymbol{X} \to \boldsymbol{Z}$ is an S^*-morphism. ☐

Lemma 3.17 enables us to define the *composition* of S^*-morphisms of inverse systems: If $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g, g_\nu^n) : \boldsymbol{Y} \to \boldsymbol{Z}$, then $(g, g_\nu^n)(f, f_\mu^n) = (h, h_\nu^n) : \boldsymbol{X} \to \boldsymbol{Z}$, where $h = fg$ i $h_\nu^n = g_\nu^n f_{g(\nu)}^n$. Clearly, this composition is associative.

Lemma 3.18. *The composition of commutative S^*-morphisms of inverse systems in $\mathcal{C}$ is a commutative S^*-morphism.*

Proof. It suffices to observe that in the proof of Lemma 3.17, in this case, one may put $n_0 = n_1 = n_2 = 1$. The conclusion follows. ☐

Given an inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathcal{C}$, let $\left(1_\Lambda, 1_{X_\lambda}^n\right)$, consists of the identity function 1_Λ and of the identity morphisms $1_{X_\lambda}^n = 1_{X_\lambda}$ in $\mathcal{C}$, for every $n \in \mathbb{N}$ and every $\lambda \in \Lambda$. Then $(1_\Lambda, 1_{X_\lambda}^n) : \boldsymbol{X} \to \boldsymbol{X}$ is an S^*-morphism (commutative and leveled). One readily sees that, for every $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ and every $(g, g_\lambda^n) : \boldsymbol{Z} \to \boldsymbol{X}$, $(f, f_\mu^n)(1_\Lambda, 1_{X_\lambda}^n) = (f, f_\mu^n)$ and $(1_\Lambda, 1_{X_\lambda}^n)(g, g_\lambda^n) = (g, g_\lambda^n)$ hold. Thus, $(1_\Lambda, 1_{X_\lambda}^n)$ may be called the *identity* S^*-morphism on $\boldsymbol{X}$.

By summarizing, for every category $\mathcal{C}$, there exists a category, denoted by $(inv\text{-}\mathcal{C})^*$, consisting of the object class $Ob(inv\text{-}\mathcal{C})^* = Ob(inv\text{-}\mathcal{C})$ and of the morphism class $Mor(inv\text{-}\mathcal{C})^*$ of all the sets $(inv\text{-}\mathcal{C})^*(\boldsymbol{X}, \boldsymbol{Y})$ of all S^*-morphisms (f, f_μ^n) of $\boldsymbol{X}$ to $\boldsymbol{Y}$, endowed with the composition and identities described above. By Lemma 3.18, there exists a subcategory $(inv\text{-}\mathcal{C})_\omega^*$ of $(inv\text{-}\mathcal{C})^*$ with the same object class and with the morphism class $Mor(inv\text{-}\mathcal{C})_\omega^*$ consisting of all commutative S^*-morphisms of inverse systems in $\mathcal{C}$.

Similarly to Definition 3.8, we shall use Theorem 3.2 to define an equivalence relation on each set $(inv\text{-}\mathcal{C})^*(\boldsymbol{X}, \boldsymbol{Y})$.

Definition 3.19. *An S^*-morphism $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ of inverse systems in $\mathcal{C}$ is said to be* ***equivalent to*** *an S^*-morphism $(f', f_\mu'^n) : \boldsymbol{X} \to \boldsymbol{Y}$, denoted by $(f, f_\mu^n) \sim (f', f_\mu'^n)$, provided every $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), f'(\mu)$, and $n \in \mathbb{N}$, such that, for every $n' \geqslant n$,*

$$f_\mu^{n'} p_{f(\mu)\lambda} = f_\mu'^{n'} p_{f'(\mu)\lambda}.$$

Lemma 3.20. *The relation $\sim$ is an equivalence relation on each set $(inv\text{-}\mathcal{C})^*(\boldsymbol{X}, \boldsymbol{Y})$. The equivalence class $[(f, f_\mu^n)]$ of an S^*-morphism $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ is briefly denoted by $\boldsymbol{f}^*$.*

Proof. The relation $\sim$ is obviously reflexive and symmetric. To prove transitivity, one should take, for given $\mu \in M$, the maximums of pairs of indices $\{\lambda_1, \lambda_2\}$ and $\{n_1, n_2\}$, which exist by $(f, f_\mu^n) \sim (f', f_\mu'^n)$ (λ_1 and n_1) and $(f', f_\mu'^n) \sim (f'', f_\mu''^n)$ (λ_2 and n_2). □

LEMMA 3.21. *Let $(f, f_\mu^n), (f', f_\mu'^n) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g, g_\nu^n), (g', g_\nu'^n) : \boldsymbol{Y} \to \boldsymbol{Z}$ be S^*-morphisms of inverse systems in $\mathcal{C}$. If $(f, f_\mu^n) \sim (f', f_\mu'^n)$ and $(g, g_\nu^n) \sim (g', g_\nu'^n)$, then $(g, g_\nu^n)(f, f_\mu^n) \sim (g', g_\nu'^n)(f', f_\mu'^n)$.*

PROOF. According to Lemma 3.20 (transitivity), it suffices to prove that $(g, g_\nu^n)(f, f_\mu^n) \sim (g, g_\nu^n)(f', f_\mu'^n)$ and $(g, g_\nu^n)(f, f_\mu^n) \sim (g', g_\nu'^n)(f, f_\mu^n)$. Given $\nu \in N$, choose $\lambda \in \Lambda$, $\lambda \geq fg(\nu), f'g(\nu)$, and $n \in \mathbb{N}$, by $(f, f_\mu^n) \sim (f', f_\mu'^n)$ for $\mu = g(\nu)$. Then, for every $n' \geq n$,

$$g_\nu^{n'} f_{g(\nu)}^{n'} p_{fg(\nu)\lambda} = g_\nu^{n'} f_{g(\nu)}'^{n'} p_{f'g(\nu)\lambda}.$$

Thus, $(g, g_\nu^n)(f, f_\mu^n) \sim (g, g_\nu^n)(f', f_\mu'^n)$. Further, if $(g, g_\nu^n) \sim (g', g_\nu'^n)$, then, for given $\nu \in N$, there exist $\mu \geq g(\nu), g'(\nu)$ and n_1 such that

$$g_\nu^{n'} q_{g(\nu)\mu} = g_\nu'^{n'} q_{g'(\nu)\mu},$$

whenever $n' \geq n_1$. Since (f, f_μ^n) is an S^*-morphism, there exist $\lambda \geq \max\{fg(\nu),\ fg'(\nu), f(\mu)\}$ and n_2 large enough, such that, for every $n' \geq n_2$, the n'-coordinate of (f, f_μ^n) commutes at the pairs $g(\nu) \leq \mu$ and $g'(\nu) \leq \mu$ with "the tail" at λ. Thus,

$$f_{\mu_1}^{n'} p_{f(\mu_1)\lambda} = q_{\mu_1\mu_2} f_{\mu_2}^{n'} p_{f(\mu_2)\lambda},$$

where $\mu_1 = \min\{g(\nu), g'(\nu)\}$ and $\mu_2 = \max\{g(\nu), g'(\nu)\}$. Consequently, for every $n' \geq n = \max\{n_1, n_2\}$,

$$g_\nu^{n'} f_{g(\nu)}^{n'} p_{fg(\nu)\lambda} = g_\nu'^{n'} f_{g'(\nu)}^{n'} p_{fg'(\nu)\lambda}.$$

Therefore, $(g, g_\nu^n)(f, f_\mu^n) \sim (g', g_\nu'^n)(f, f_\mu^n)$. □

By Lemmata 3.20 and 3.21, one may compose the equivalence classes of S^*-morphisms of inverse systems by putting $\boldsymbol{g}^* \boldsymbol{f}^* = \boldsymbol{h}^* \equiv [(h, h_\nu^n)]$, where $(h, h_\nu^n) = (g, g_\nu^n)(f, f_\mu^n) = (fg, g_\nu^n f_{g(\nu)}^n)$. The corresponding quotient category $(inv\text{-}\mathcal{C})^*/_\sim$ is denoted by $pro^*\text{-}\mathcal{C}$. There exists a subcategory $pro_\omega^*\text{-}\mathcal{C} \subseteq pro^*\text{-}\mathcal{C}$ determined by all equivalence classes having commutative representatives. Clearly, $pro_\omega^*\text{-}\mathcal{C}$ is isomorphic to the quotient category $(inv\text{-}\mathcal{C})_\omega^*/_\sim$. Similarly to a tow-category, one may consider $pro\text{-}\mathcal{C} = (inv\text{-}\mathcal{C})/\sim$ as a subcategory of $pro_\omega^*\text{-}\mathcal{C}$ and, consequently, as a subcategory of $pro^*\text{-}\mathcal{C}$ (see also Proposition 3.24 below). First, recall the well known lemma (see [15, Lemma I.1.1]):

LEMMA 3.22. *Let $(\Lambda, \leq)$ be a directed set and let $(M, \leq)$ be a cofinite directed set. Then every function $f : M \to \Lambda$ admits an increasing function $f' : M \to \Lambda$ such that $f \leq f'$.*

LEMMA 3.23. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in $\mathcal{C}$ with M cofinite. Then every morphism $\boldsymbol{f}^* = [(f, f_\mu^n)] : \boldsymbol{X} \to \boldsymbol{Y}$ in $pro^*\text{-}\mathcal{C}$ admits a simple representative $(f', f_\mu'^n) : \boldsymbol{X} \to \boldsymbol{Y}$.*

Proof. Let $\mu \in M$. If μ has no predecessors, choose any $\lambda \in \Lambda$, $\lambda \geq f(\mu)$, and put $\varphi(\mu) = \lambda$. If μ is not an initial element of M, let $\mu_1, \dots, \mu_m \in M$, $m \in \mathbb{N}$, be all the predecessors of μ (M is cofinite). Since (f, f_μ^n) is an S^*-morphism, for every $i = 1, \dots, m$ and every pair $\mu_i \leq \mu$, there exists $\lambda_i \in \Lambda$, $\lambda_i \geq f(\mu_i), f(\mu)$, and there exists $n_i \in \mathbb{N}$, such that, for every $n' \geq n_i$, the appropriate condition holds. Choose any $\lambda \in \Lambda$, $\lambda \geq \lambda_i$ for all $i = 1, \dots, m$ (Λ is directed), and put $\varphi(\mu) = \lambda$. This defines a function $\varphi : M \to \Lambda$. Notice that $f \leq \varphi$. By Lemma 3.22, there exists an increasing function $f' : M \to \Lambda$ such that $\varphi \leq f'$. Hence, $f \leq f'$. Now, for every $\mu \in M$, put $f_\mu'^n = f_\mu^n p_{f(\mu)f'(\mu)}$. One readily verifies that $(f', f_\mu'^n) : \boldsymbol{X} \to \boldsymbol{Y}$ is a simple S^*-morphism and that $(f', f_\mu'^n) \sim (f, f_\mu^n)$. ∎

Let us define a functor $\underline{J} \equiv \underline{J}_{\mathcal{C}} : pro\text{-}\mathcal{C} \to pro^*\text{-}\mathcal{C}$ (an extension of the already defined functor $\underline{J} : tow\text{-}\mathcal{C} \to tow^*\text{-}\mathcal{C}$). Put $\underline{J}(\boldsymbol{X}) = \boldsymbol{X}$, for every inverse system $\boldsymbol{X}$ in $\mathcal{C}$. If $\boldsymbol{f} \in pro\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$ and if (f, f_μ) is any representative of $\boldsymbol{f}$, put

$$\underline{J}(\boldsymbol{f}) = \boldsymbol{f}^* = [(f, f_\mu^n)] \in pro^*\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y}),$$

where (f, f_μ^n) is *induced by* (f, f_μ), i.e., for every $n \in \mathbb{N}$, $f_\mu^n = f_\mu$ *for* all $\mu \in M$. One straightforwardly verifies that $\underline{J}(\boldsymbol{f})$ is well defined and that $\underline{J}$ is indeed a functor. Notice that every induced S^*-morphism is commutative. Therefore, $\underline{J}$ is a functor of $pro\text{-}\mathcal{C}$ to the subcategory $pro^*_\omega\text{-}\mathcal{C} \subseteq pro^*\text{-}\mathcal{C}$.

Proposition 3.24. *The functor* $\underline{J} : pro\text{-}\mathcal{C} \to pro^*_\omega\text{-}\mathcal{C} \subseteq pro^*\text{-}\mathcal{C}$ *is faithful.*

Proof. Let $\boldsymbol{f}^* = \underline{J}(\boldsymbol{f}) = \underline{J}(\boldsymbol{f}') = \boldsymbol{f}'^*$. Let (f, f_μ) and (f', f'_μ) be any representatives of $\boldsymbol{f}$ and $\boldsymbol{f}'$ respectively. By definition of the functor $\underline{J}$, $\boldsymbol{f}^* = [(f, f_\mu^n = f_\mu)]$ and $\boldsymbol{f}'^* = [(f', f_\mu'^n = f'_\mu)]$. Since $(f, f_\mu^n) \sim (f', f_\mu'^n)$, for every $\mu \in M$, there exist $\lambda \geq f(\mu), f'(\mu)$ and n such that, for every $n' \geq n$,

$$f_\mu^{n'} p_{f(\mu)\lambda} = f_\mu'^{n'} p_{f'(\mu)\lambda}.$$

This means

$$f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

Therefore, $(f, f_\mu) \sim (f', f'_\mu)$, i.e., $\boldsymbol{f} = \boldsymbol{f}'$. ∎

Remark 3.25. The functor $\underline{J}$ is not full. For instance, let us consider the restriction $pro\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{T}) \to pro^*_\omega\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{T})$, where $\boldsymbol{T} = (T_0 \equiv T)$ is a rudimentary inverse system. Let $\boldsymbol{f} \in pro\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{T})$. Then every representative (f, f_0) of $\boldsymbol{f}$ is uniquely determined by $\lambda_0 \in \Lambda$ ($f(0) = \lambda_0$) and by a morphism $f_0 \equiv f_{\lambda_0} \in \mathcal{C}(X_\lambda, T)$. However, it is not the case for $\boldsymbol{f}^* \in pro^*_\omega\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{T})$. Indeed, if (f, f_0^n) is a representative of $\boldsymbol{f}^*$, then $f(0) = \lambda_0 \in \Lambda$, while $(f_0^n \equiv f_{\lambda_0}^n)_{n \in \mathbb{N}}$ is a sequence of morphisms $f_{\lambda_0}^n \in \mathcal{C}(X_{\lambda_0}, T)$. Notice that $(f, f_0^n) \sim (f', f_0'^n)$ if and only if

$$(\exists \lambda \geqslant \lambda_0, \lambda'_0)\,(\exists n)\,(\forall n' \geqslant n)\;\; f_0^{n'} p_{\lambda_0 \lambda} = f_0'^{n'} p_{\lambda'_0 \lambda}.$$

By the well known "Mardešić trick", every inverse system $\boldsymbol{X}$ in $\mathcal{C}$ is isomorphic (in *pro*-$\mathcal{C}$) to a cofinite inverse system $\boldsymbol{X}'$. If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{X}'$ is an isomorphism in *pro*-$\mathcal{C}$, then $\underline{J}(\boldsymbol{f}) : \boldsymbol{X} \to \boldsymbol{X}'$ is an isomorphism in *pro**-$\mathcal{C}$. Therefore, the next corollary holds.

COROLLARY 3.26. *Every inverse system $\boldsymbol{X}$ in $\mathcal{C}$ is isomorphic in pro*-$\mathcal{C}$ to a cofinite inverse system $\boldsymbol{X}'$.*

A morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ in *pro**-$\mathcal{C}$ does not admit, in general, a level representative. However, the following "reindexing theorem" will help to overcome some technical difficulties concerning this fact.

THEOREM 3.27. *Let $\boldsymbol{f}^* \in pro^*\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$. Then there exist inverse systems $\boldsymbol{X}'$ and $\boldsymbol{Y}'$ in $\mathcal{C}$ having the same cofinite index set $(N, \leq)$, there exists a morphism $\boldsymbol{f}'^* : \boldsymbol{X}' \to \boldsymbol{Y}'$ having a level representative $(1_N, f_\nu'^n)$ and there exist isomorphisms $\boldsymbol{i}^* : \boldsymbol{X} \to \boldsymbol{X}'$ and $\boldsymbol{j}^* : \boldsymbol{Y} \to \boldsymbol{Y}'$ in pro*-$\mathcal{C}$, such that the following diagram in pro*-$\mathcal{C}$ commutes:*

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\boldsymbol{f}^*} & \boldsymbol{Y} \\ \boldsymbol{i}^* \downarrow & & \downarrow \boldsymbol{j}^* \\ \boldsymbol{X}' & \xrightarrow{\boldsymbol{f}'^*} & \boldsymbol{Y}' \end{array}$$

PROOF. Let $\boldsymbol{f}^* \in pro^*\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$. By Corollary 3.26, there exist cofinite inverse systems $\widetilde{\boldsymbol{X}} = (\widetilde{X}_\alpha, \widetilde{p}_{\alpha\alpha'}, A)$ and $\widetilde{\boldsymbol{Y}} = (\widetilde{Y}_\beta, \widetilde{q}_{\beta\beta'}, B)$, and there exist isomorphisms $\boldsymbol{u}^* : \boldsymbol{X} \to \widetilde{\boldsymbol{X}}$ and $\boldsymbol{v}^* : \boldsymbol{Y} \to \widetilde{\boldsymbol{Y}}$ in *pro**-$\mathcal{C}$. Let $\widetilde{\boldsymbol{f}}^* = \boldsymbol{v}^* \boldsymbol{f}^* (\boldsymbol{u}^*)^{-1} : \widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{Y}}$. By Lemma 3.23, there exists a simple representative (w, w_β^n) of $\widetilde{\boldsymbol{f}}^*$. Let

$$N = \{\nu \equiv (\alpha, \beta) \mid \alpha \in A, \beta \in B, w(\beta) \leq \alpha\} \subseteq A \times B,$$

and define $(N, \leq)$ coordinatewise, i.e., $\nu = (\alpha, \beta) \leq (\alpha', \beta') = \nu'$ if and only if $\alpha \leq \alpha'$ in A and $\beta \leq \beta'$ in B. Clearly, N is preordered. Let any $\nu = (\alpha, \beta), \nu' = (\alpha', \beta') \in N$ be given. Since B is directed, there exists $\beta_0 \geq \beta, \beta'$. Since A is directed, there exists $\alpha_0 \geq \alpha, \alpha', w(\beta_0)$. Then $(\alpha_0, \beta_0) \equiv \nu_0 \in N$ and $\nu_0 \geq \nu, \nu'$. Thus, N is directed. Further, since A and B are cofinite and since $N \subseteq A \times B$ is (pre)ordered coordinatewise, the set N is cofinite too. Let us now construct desired inverse systems $\boldsymbol{X}' = (X'_\nu, p'_{\nu\nu'}, N)$ and $\boldsymbol{Y}' = (Y'_\nu, q'_{\nu\nu'}, N)$. Given $\nu = (\alpha, \beta) \in N$, put $X'_\nu = X_\lambda$ and $Y'_\nu = Y_\mu$; for every related pair $\nu = (\alpha, \beta) \leq (\alpha', \beta') = \nu'$ in N, put $p'_{\nu\nu'} = \widetilde{p}_{\alpha\alpha'}$ and $q'_{\nu\nu'} = \widetilde{q}_{\beta\beta'}$. Now, for every $\nu = (\alpha, \beta) \in N$, put $f_\nu'^n = w_\beta^n \widetilde{p}_{w(\beta)\alpha} : X'_\nu \to Y'_\nu$. Then $(1_N, f_\nu'^n) : \boldsymbol{X}' \to \boldsymbol{Y}'$ is a simple S^*-morphism. Indeed, if $\nu \leq \nu'$, then $\beta \leq \beta'$, Since (w, w_β^n) is simple, there exists $n \in \mathbb{N}$ such that, for every $n' \geq n$,

$$w_\beta^{n'} \widetilde{p}_{w(\beta)w(\beta')} = \widetilde{q}_{\beta\beta'} w_{\beta'}^{n'}.$$

Since $\alpha \geq w(\beta)$, $\alpha' \geq w(\beta')$, $w(\beta') \geq w(\beta)$ and $\alpha' \geq \alpha$, it implies

$$\begin{aligned} f'^{n'}_{\nu} p'_{\nu\nu'} &= w^{n'}_{\beta} \widetilde{p}_{w(\beta)\alpha} \widetilde{p}_{\alpha\alpha'} = w^{n'}_{\beta} \widetilde{p}_{w(\beta)w(\beta')} \widetilde{p}_{w(\beta')\alpha'} \\ &= \widetilde{q}_{\beta\beta'} w^{n'}_{\beta'} \widetilde{p}_{w(\beta')\alpha'} = q'_{\nu\nu'} f'^{n'}_{\nu'}. \end{aligned}$$

Let $s : N \to \Lambda$ be defined by putting $s(\nu) = \alpha$, where $\nu = (\alpha, \beta)$, and let, for every $n \in \mathbb{N}$, $s^n_\nu : \widetilde{X}_\alpha \to X'_\nu = \widetilde{X}_\alpha$ be the identity $1_{\widetilde{X}_\alpha}$ in $\mathcal{C}$ for each $\nu \in N$. In the same way, let $t : N \to M$ be defined by putting $t(\nu) = \beta$, and let, for every n, $t^n_\nu : \widetilde{Y}_\beta \to Y'_\nu = \widetilde{Y}_\beta$ be the identity $1_{\widetilde{Y}_\beta}$ for each ν. It is readily seen that $\boldsymbol{s}^* = [(s, s^n_\nu)] : \widetilde{\boldsymbol{X}} \to \boldsymbol{X}'$ and $\boldsymbol{t}^* = [(t, t^n_\nu)] : \widetilde{\boldsymbol{Y}} \to \boldsymbol{Y}'$ are simple commutative morphisms. Even more, they are induced by morphisms $(s, s_\nu = 1_{\widetilde{X}_\alpha})$ and $(t, t_\nu = 1_{\widetilde{Y}_\beta})$ of *inv*-$\mathcal{C}$ respectively. Notice that, in *pro*-$\mathcal{C}$, $\boldsymbol{s} = [(s, s^n_\nu)] : \widetilde{\boldsymbol{X}} \to \boldsymbol{X}'$ and $\boldsymbol{t} = [(t, t^n_\nu)] : \widetilde{\boldsymbol{Y}} \to \boldsymbol{Y}'$ are isomorphisms. Since $\boldsymbol{s}^* = \underline{J}(\boldsymbol{s})$ and $\boldsymbol{t}^* = \underline{J}(\boldsymbol{t})$, we infer that $\boldsymbol{s}^*$ and $\boldsymbol{t}^*$ are isomorphisms in *pro**-$\mathcal{C}$. Moreover, for every $\nu = (\alpha, \beta) \in N$ and every $n \in \mathbb{N}$,

$$t^n_\nu w^n_{t(\nu)} \widetilde{p}_{wt(\nu)\alpha} = w^n_\beta \widetilde{p}_{w(\beta)\alpha} = f'^n_\nu = f'^n_\nu s^n_\nu,$$

which implies $(t, t^n_\nu)(w, w^n_\beta) \sim (1_N, f'^n_\nu)(s, s^n_\nu)$. Therefore, $\boldsymbol{t}^* \widetilde{\boldsymbol{f}}^* = \boldsymbol{f}'^* \boldsymbol{s}^*$. Finally, put $\boldsymbol{i}^* \equiv \boldsymbol{s}^* \boldsymbol{u}^* : \boldsymbol{X} \to \boldsymbol{X}'$ and $\boldsymbol{j}^* \equiv \boldsymbol{t}^* \boldsymbol{v}^* : \boldsymbol{Y} \to \boldsymbol{Y}'$, which are isomorphisms in *pro**-$\mathcal{C}$. Then

$$\boldsymbol{j}^* \boldsymbol{f}^* = \boldsymbol{t}^* \boldsymbol{v}^* \boldsymbol{f}^* = \boldsymbol{t}^* \widetilde{\boldsymbol{f}}^* \boldsymbol{u}^* = \boldsymbol{f}'^* \boldsymbol{s}^* \boldsymbol{u}^* = \boldsymbol{f}'^* \boldsymbol{i}^*.$$

□

4. The coarse shape category

Let $\mathcal{D}$ be a full (not essential, but a convenient condition) and dense subcategory of $\mathcal{C}$. Let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{p}' : X \to \boldsymbol{X}'$ be $\mathcal{D}$-expansions of the same object X of $\mathcal{C}$, and let $\boldsymbol{q} : Y \to \boldsymbol{Y}$ and $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ be $\mathcal{D}$-expansions of the same object Y of $\mathcal{C}$. Then there exist two natural (unique) isomorphisms $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{X}'$ and $\boldsymbol{j} : \boldsymbol{Y} \to \boldsymbol{Y}'$ in *pro*-$\mathcal{D}$. Consequently, $\boldsymbol{i}^* \equiv \underline{J}(\boldsymbol{i}) : \boldsymbol{X} \to \boldsymbol{X}'$ and $\boldsymbol{j}^* \equiv \underline{J}(\boldsymbol{j}) : \boldsymbol{Y} \to \boldsymbol{Y}'$ are isomorphisms in *pro**-$\mathcal{D}$. A morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *pro**-$\mathcal{D}$ *equivalent to* a morphism $\boldsymbol{f}'^* : \boldsymbol{X}' \to \boldsymbol{Y}'$, denoted by $\boldsymbol{f}^* \sim \boldsymbol{f}'^*$, provided the following diagram in *pro**-$\mathcal{D}$ commutes:

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\boldsymbol{i}^*} & \boldsymbol{X}' \\ \boldsymbol{f}^* \downarrow & & \downarrow \boldsymbol{f}'^* \\ \boldsymbol{Y} & \xrightarrow{\boldsymbol{j}^*} & \boldsymbol{Y}' \end{array}.$$

According to the analogous facts in *pro*-$\mathcal{D}$, and since $\underline{J}$ is a functor, it defines an equivalence relation on the appropriate subclass of $Mor(pro^*\text{-}\mathcal{D})$, such that $\boldsymbol{f}^* \sim \boldsymbol{f}'^*$ and $\boldsymbol{g}^* \sim \boldsymbol{g}'^*$ imply $\boldsymbol{g}^* \boldsymbol{f}^* \sim \boldsymbol{g}'^* \boldsymbol{f}'^*$ whenever it is defined. The equivalence class of $\boldsymbol{f}^*$ is denoted by $\langle \boldsymbol{f}^* \rangle$. Further, given $\boldsymbol{p}$, $\boldsymbol{p}'$, $\boldsymbol{q}$, $\boldsymbol{q}'$ and $\boldsymbol{f}^*$, there exists a unique $\boldsymbol{f}'^*$ $(= \boldsymbol{j}^* \boldsymbol{f}^* (\boldsymbol{i}^*)^{-1})$ such that $\boldsymbol{f}^* \sim \boldsymbol{f}'^*$.

We are now to define the (*abstract*) *coarse shape category* $Sh^*_{(\mathcal{C},\mathcal{D})}$ *for* $(\mathcal{C},\mathcal{D})$ as follows. The objects of $Sh^*_{(\mathcal{C},\mathcal{D})}$ are all the objects of $\mathcal{C}$. A morphism $F^* \in Sh^*_{(\mathcal{C},\mathcal{D})}(X,Y)$ is the *pro**-$\mathcal{D}$ equivalence class $\langle \boldsymbol{f}^* \rangle$ of a morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$, with respect to any choice of a pair of $\mathcal{D}$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$. In other words, a *coarse shape morphism* $F^* : X \to Y$ is given by a diagram

$$\begin{array}{ccc} \boldsymbol{X} & \xleftarrow{\boldsymbol{p}} & X \\ {\scriptstyle \boldsymbol{f}^*} \downarrow & & \downarrow {\scriptstyle F^*} \\ \boldsymbol{Y} & \xleftarrow{\boldsymbol{q}} & Y \end{array}$$

The *composition* of $F^* : X \to Y$, $F^* = \langle \boldsymbol{f}^* \rangle$ and a $G^* : Y \to Z$, $G^* = \langle \boldsymbol{g}^* \rangle$, is defined by the representatives, i.e., $G^*F^* : X \to Z$, $G^*F^* = \langle \boldsymbol{g}^* \boldsymbol{f}^* \rangle$. The *identity coarse shape morphism* on an object X, $1^*_X : X \to X$, is the *pro**-$\mathcal{D}$ equivalence class $\langle \mathbf{1}^*_{\boldsymbol{X}} \rangle$ of the identity morphism $\mathbf{1}^*_{\boldsymbol{X}}$ in *pro**-$\mathcal{D}$. Since

$$Sh^*_{(\mathcal{C},\mathcal{D})}(X,Y) \approx pro^*\text{-}\mathcal{D}(\boldsymbol{X},\boldsymbol{Y})$$

is a set, the coarse shape category $Sh^*_{(\mathcal{C},\mathcal{D})}$ is well defined. One may say that *pro**-$\mathcal{D}$ is the *realizing* category for the coarse shape category $Sh^*_{(\mathcal{C},\mathcal{D})}$.

For every $f : X \to Y$ in $\mathcal{C}$ and every pair of $\mathcal{D}$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$, there exists $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ in *pro**-$\mathcal{D}$, such that the following diagram in *pro**-$\mathcal{C}$ commutes:

$$\begin{array}{ccc} \boldsymbol{X} & \xleftarrow{\boldsymbol{p}} & X \\ {\scriptstyle \boldsymbol{f}^*} \downarrow & & \downarrow {\scriptstyle f} \\ \boldsymbol{Y} & \xleftarrow{\boldsymbol{q}} & Y \end{array} .$$

(Hereby, we consider $\mathcal{C} \subseteq$ *pro*-$\mathcal{C}$ to be subcategories of *pro**-$\mathcal{C}$!) The same f and another pair of $\mathcal{D}$-expansions $\boldsymbol{p}' : X \to \boldsymbol{X}'$, $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ yield $\boldsymbol{f}'^* : \boldsymbol{X}' \to \boldsymbol{Y}'$ in *pro**-$\mathcal{D}$. Then, however, $\boldsymbol{f}^* \sim \boldsymbol{f}'^*$ in *pro**-$\mathcal{D}$ must hold. Thus, every morphism $f \in \mathcal{C}(X,Y)$ yields a *pro**-$\mathcal{D}$ equivalence class $\langle \boldsymbol{f}^* \rangle$, i.e., a coarse shape morphism $F^* \in Sh^*_{(\mathcal{C},\mathcal{D})}(X,Y)$. If one defines $S^*(X) = X$, $X \in Ob\mathcal{C}$, and $S^*(f) = F^* = \langle \boldsymbol{f}^* \rangle$, $f \in Mor\mathcal{C}$, then

$$S^*_{(\mathcal{C},\mathcal{D})} : \mathcal{C} \to Sh^*_{(\mathcal{C},\mathcal{D})}$$

becomes a functor, called the *abstract coarse shape functor*. Comparing to the abstract shape functor, we shall show that the restriction of S^* to $\mathcal{D}$ into the full subcategory of $Sh^*_{(\mathcal{C},\mathcal{D})}$, determined by $Ob\mathcal{D}$, is *not* a category isomorphism (Example 7.4). Nevertheless, we shall prove that P and Q are isomorphic objects of $\mathcal{D}$ if and only if they are isomorphic in $Sh^*_{(\mathcal{C},\mathcal{D})}$, i.e., they are of the same abstract coarse shape (Claim 3 below). Thus, clearly, the abstract coarse shape type classification on $\mathcal{D}$ coincides with the abstract shape type classification. Further, recall that for every $X \in Ob\mathcal{C}$ and every $Q \in Ob\mathcal{D}$, the abstract shape functor induces a bijection

$$S| \cdot : \mathcal{C}(X,Q) \to Sh_{(\mathcal{C},\mathcal{D})}(X,Q).$$

However, we shall see that, in the same circumstances, the abstract coarse shape functor induces an injection

$$S^*|\cdot : \mathcal{C}(X,Q) \to Sh^*_{(\mathcal{C},\mathcal{D})}(X,Q),$$

which, in general, is *not* a surjection (Example 7.4). Finally, the functor $S^*_{(\mathcal{C},\mathcal{D})}$ factorizes as $S^*_{(\mathcal{C},\mathcal{D})} = J_{(\mathcal{C},\mathcal{D})} S_{(\mathcal{C},\mathcal{D})}$, where $S_{(\mathcal{C},\mathcal{D})} : \mathcal{C} \to Sh_{(\mathcal{C},\mathcal{D})}$ is the abstract shape functor, while $J_{(\mathcal{C},\mathcal{D})} : Sh_{(\mathcal{C},\mathcal{D})} \to Sh^*_{\mathcal{C},\mathcal{D})}$ is induced by the "inclusion" functor $\underline{J} \equiv \underline{J}_{\mathcal{D}} : pro\text{-}\mathcal{D} \to pro^*\text{-}\mathcal{D}$. (This implies that the induced function $\mathcal{C}(X,Q) \to Sh^*_{(\mathcal{C},\mathcal{D})}(X,Q)$ is an injection.)

As in the case of the abstract shape, the most interesting example of the above construction is $\mathcal{C} = HTop$ - the homotopy category of topological spaces and $\mathcal{D} = HPol$ - the homotopy category of polyhedra, or $\mathcal{D} = HANR$ - the homotopy category of ANR's for metric spaces. In this case, one speaks about the (ordinary or standard) *coarse shape category*

$$Sh^*_{(HTop,HPol)} \equiv Sh^*(Top) \equiv Sh^*$$

of topological spaces and of (ordinary or standard) *coarse shape functor*

$$S^* : HTop \to Sh^*,$$

which factorizes as $S^* = JS$, where $S : HTop \to Sh$ is the shape functor, and $J : Sh \to Sh^*$ is induced by the "inclusion" functor $\underline{J} \equiv pro\text{-}HPol \to pro^*\text{-}HPol$. The realizing category for Sh^* is the category $pro^*\text{-}HPol$ (or $pro^*\text{-}HANR$). The underlying theory might be called the (ordinary or standard) *coarse shape theory* (for topological spaces). Clearly, on locally nice spaces (polyhedra, CW-complexes, ANR's, ...) the coarse shape type classification coincides with the shape type classification and, consequently, with the homotopy type classification.

Similarly to the case of the shape of compacta, let us consider the homotopy (sub)category of compact metric spaces, $HcM \subseteq HTop$. Since $HcPol \subseteq HcM$ and $HcANR \subseteq HcM$ are "*sequentially*" dense (and homotopically equivalent) subcategories, there exist the coarse shape category of compacta,

$$Sh^*(cM) \equiv Sh^*_{(HcM,HcPol)} \cong Sh^*_{(HcM,HcANR)},$$

and the corresponding (restriction of the) coarse shape functor

$$S^* : HcM \to Sh^*(cM),$$

such that $S^* = JS$, where $S : HcM \to Sh(cM)$ is the shape functor on compacta, and $J : Sh(cM) \to Sh^*(cM)$ is induced by the "inclusion" functor $\underline{J} : tow\text{-}HcPol \to tow^*\text{-}HcPol$ (or $\underline{J} : tow\text{-}HcANR \to tow^*\text{-}HcANR$). The category $Sh^*(cM)$ is a full subcategory of Sh^*. Notice that the realizing category for $Sh^*(cM)$ is the category $tow^*\text{-}HcPol$ as well as the category $tow^*\text{-}HcANR$.

The category tow^*-$HcANR$, being isomorphic to the category $\underline{\mathcal{S}}^*$ restricted to inverse sequences of compact ANR's (Corollary 3.12), classifies (by its isomorphisms) compact ANR inverse sequences strictly coarser than the category tow-$HcANR$ (see [22] and Corollary 5.3 below as well as [16] and our Example 7.2).

Let $\mathcal{D}$ be a full and dense subcategory of $\mathcal{C}$, let $X \in Ob\mathcal{C}$ and let $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a $\mathcal{D}$-expansion of X. Further, let $Q \in Ob\mathcal{D}$ and let a sequence (Φ^n) of morphisms $\Phi^n : X \to Q$, $n \in \mathbb{N}$, in $\mathcal{C}$ be given. We say that (Φ^n) *uniformly factorizes through* $\boldsymbol{p}$ provided there exists a fixed $\lambda \in \Lambda$ such that, for every n, Φ^n factorizes through X_λ. Such a sequence (Φ^n) determines a coarse shape morphism $F^* : X \to Q$. Namely, there is $\lambda \in \Lambda$ such that, for every $n \in \mathbb{N}$, there exists a morphism $f^n : X_\lambda \to Q$ of $\mathcal{D}$ ($\mathcal{D} \subseteq \mathcal{C}$ is full) satisfying $\Phi^n = f^n p_\lambda$. Hence, the sequence (f^n) determines a morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Q} = (Q)$ of pro^*-$\mathcal{D}$. Since $\mathbf{1} : Q \to \boldsymbol{Q}$ is a $\mathcal{D}$-expansion of Q, the morphism $\boldsymbol{f}^*$ determines a coarse shape morphism $F^* = \langle \boldsymbol{f}^* \rangle : X \to Q$. We say that such F^* is *induced by* (Φ^n). Notice that the above construction depends on the index λ.

CLAIM 1. *Let $X \in Ob\mathcal{C}$, let $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a $\mathcal{D}$-expansion of X and let $Q \in Ob\mathcal{D}$. Then every coarse shape morphism $F^* : X \to Q$ is induced by a sequence of morphisms $\Phi^n : X \to Q$ in $\mathcal{C}$, $n \in \mathbb{N}$, such that (Φ^n) uniformly factorizes through $\boldsymbol{p}$.*

PROOF. Let $F^* : X \to Q$ be a coarse shape morphism. For $\mathcal{D}$-expansions $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X}$ and $\mathbf{1} : Q \to \boldsymbol{Q} = (Q)$, there exists a representative $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Q}$ in pro^*-$\mathcal{D}$ of F^*. Consequently, there exists a sequence (f^n) of morphisms $f^n : X_\lambda \to Q$, $n \in \mathbb{N}$, in $\mathcal{D}$ which determines $\boldsymbol{f}^*$. Thus, by putting $\Phi^n = f^n p_\lambda$, $n \in \mathbb{N}$, one obtains the desired sequence (Φ^n). ∎

Let (Φ^n) and (Φ'^n) uniformly factorize through the same $\boldsymbol{p} : X \to \boldsymbol{X}$ (via λ and λ' respectively). Then (Φ^n) and (Φ'^n) is said to be *almost equal* provided there exist $n_0 \in \mathbb{N}$ and $\lambda_0 \geq \lambda, \lambda'$ such that, for every $n \geq n_0$, $f^n p_{\lambda\lambda_0} = f'^n p_{\lambda'\lambda_0}$. Obviously, (Φ^n) and (Φ'^n) are almost equal if and only if $\Phi^n = \Phi'^n$ for almost all n.

CLAIM 2. *Let (Φ^n) and (Φ'^n) uniformly factorize through the same $\boldsymbol{p} : X \to \boldsymbol{X}$. Let $F^* : X \to Q$ and $F'^* : X \to Q$ be induced by (Φ^n) and (Φ'^n) respectively. Then $F^* = F'^*$ if and only if (Φ^n) and (Φ'^n) are almost equal.*

PROOF. Let (Φ^n) and (Φ'^n) uniformly factorize through the same $\boldsymbol{p} : X \to \boldsymbol{X}$, i.e., let there exist $\lambda, \lambda' \in \Lambda$ such that, for every $n \in \mathbb{N}$, $\Phi^n = f^n p_\lambda$ and $\Phi'^n = f'^n p_{\lambda'}$, where $f^n : X_\lambda \to Q$ and $f'^n : X_{\lambda'} \to Q$ are morphisms of $\mathcal{D}$. Let $F^* : X \to Q$ and $F'^* : X \to Q$ be coarse shape morphisms induced by (Φ^n) and (Φ'^n) respectively. Let $\boldsymbol{f}^*, \boldsymbol{f}'^* : \boldsymbol{X} \to \boldsymbol{Q} = (P)$ in pro^*-$\mathcal{D}$ be representatives of F^*, F^* respectively. Now, if $F^* = F'^*$ then $\boldsymbol{f}^* = \boldsymbol{f}'^*$, and $\boldsymbol{f}^*$, $\boldsymbol{f}'^*$ are determined by the sequences (f^n), (f'^n) respectively.

Therefore, there exist $\lambda_0 \geq \lambda, \lambda'$ and $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $f^n p_{\lambda\lambda_0} = f'^n p_{\lambda'\lambda_0}$. This means that (Φ^n) and (Φ'^n) are almost equal.

Conversely, if (Φ^n) and (Φ'^n) are almost equal, then the corresponding sequences (f^n) and (f'^n) induce the same morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Q}$ in pro^*-$\mathcal{D}$. Thus, the sequences (Φ^n) and (Φ'^n) induce the same coarse shape morphism $F^* = \langle \boldsymbol{f}^* \rangle = F'^*$. ∎

Consider now the special case where $X \equiv P \in Ob\mathcal{D}$ too. Then $\mathbf{1} : P \to \boldsymbol{P} = (P)$ and $\mathbf{1} : Q \to \boldsymbol{Q} = (Q)$ are $\mathcal{D}$-expansions. Thus, every coarse shape morphism $F^* : P \to Q$ is induced by a sequence of morphisms $f^n : P \to Q$ in $\mathcal{D}$, $n \in \mathbb{N}$. Furthermore, any two such sequences (f^n), (f'^n) induce the same coarse shape morphism if and only if $f^n = f'^n$ for almost all n. This implies that there is a surjection

$$(\mathcal{C}(P,Q))^{\mathbb{N}} \to Sh^*_{(\mathcal{C},\mathcal{D})}(P,Q)$$

of the set of all sequences of $\mathcal{C}$-morphisms $P \to Q$ onto the set of all coarse shape morphisms $P \to Q$. Finally, one can readily see that if $F^* : P \to Q$ is induced by (f^n) and $G^* : Q \to R$ is induced by (g^n), then the composition $G^*F^* : P \to R$ is induced by $(g^n f^n)$.

Claim 3. *For every pair $P, Q \in Ob\mathcal{D}$, the following assertions are equivalent:*

(i) P and Q are isomorphic objects in $\mathcal{D}$;
(ii) P and Q have the same abstract shape;
(iii) P and Q have the same abstract coarse shape.

Proof. The equivalence (i) ⇔ (ii) is the well known fact. The implication (ii) ⇒ (iii) follows by the functor $J_{(\mathcal{C},\mathcal{D})} : Sh_{(\mathcal{C},\mathcal{D})} \to Sh^*_{(\mathcal{C},\mathcal{D})}$. Let $P, Q \in Ob\mathcal{D}$ have the same coarse shape. Then there exists a pair of coarse shape isomorphisms $F^* : P \to Q$, $G^* : Q \to P$ such that $G^*F^* = 1^*_P$ and $F^*G^* = 1^*_Q$ in $Sh^*_{(\mathcal{C},\mathcal{D})}$. By the above consideration, there exist sequences (f^n) and (g^n) of morphisms $f^n : P \to Q$ and $g^n : Q \to P$ in $\mathcal{D}$, $n \in \mathbb{N}$, which induce F^* and G^* respectively. Furthermore, the sequences $(g^n f^n)$ and $(f^n g^n)$ induce 1^*_P and 1^*_Q. Since the constant sequences (1_P) and (1_Q) also induce 1^*_P and 1^*_Q respectively, Claim 2 implies that $g^n f^n = 1_P$ and $f^n g^n = 1_Q$ for almost all $n \in \mathbb{N}$. Consequently, P and Q are isomorphic objects of $\mathcal{D}$, and thus, (iii) ⇒ (i). ∎

5. An application

In [3], K. Borsuk had defined the relation of quasi-equivalence $\overset{q}{\simeq}$ of compacta in terms of sequences of fundamental sequences between compacta lying in AR-spaces. In order to characterize this relation in a category framework, the second named author adapted in [20] the original definitions in terms of the Mardešić-Segal shape theory [15]. Let us briefly sketch the indispensable definitions and facts from [20].

Let $\boldsymbol{f} = (f, [f_j]), \boldsymbol{f}' = (f', [f'_j]) \in tow\text{-}HcM(\boldsymbol{X}, \boldsymbol{Y})$ and let $s \in \mathbb{N}$. Then $\boldsymbol{f}$ is said to be *s-homotopic to* $\boldsymbol{f}'$, denoted by $\boldsymbol{f} \simeq_s \boldsymbol{f}'$, provided

$$(\forall j \in [1, s]_{\mathbb{N}})(\exists i = i(j) \geq f(j), f'(j))\ [f_j][p_{f(j)i}] = [f'_j][p_{f'(j)i}].$$

Observe that $\boldsymbol{f} \simeq \boldsymbol{f}'$ if and only if $\boldsymbol{f} \simeq_s \boldsymbol{f}'$ for every $s \in \mathbb{N}$, where $\simeq$ is the usual homotopy (equivalence) relation of morphisms of inverse sequences. Then,

(i) for every $s \in \mathbb{N}$, the relation $\simeq_s$ is an equivalence relation on each set $tow\text{-}HcM(\boldsymbol{X}, \boldsymbol{Y})$;
(ii) $(\forall s' \leq s)$ $(\boldsymbol{f} \simeq_s \boldsymbol{f}' \Rightarrow \boldsymbol{f} \simeq_{s'} \boldsymbol{f}')$.

Moreover, for every $s \in \mathbb{N}$, the relation $\simeq_s$ is compatible with respect to the composition to the right, i.e.,

$$(\forall \boldsymbol{h} : \boldsymbol{W} \to \boldsymbol{X})(\boldsymbol{f} \simeq_s \boldsymbol{f}' \Rightarrow \boldsymbol{f}\boldsymbol{h} \simeq_s \boldsymbol{f}'\boldsymbol{h}).$$

On the other side, if $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$, then $\boldsymbol{f} \simeq_s \boldsymbol{f}'$ implies $\boldsymbol{g}\boldsymbol{f} \simeq_{s'} \boldsymbol{g}\boldsymbol{f}'$ whenever $g[[1, s']_{\mathbb{N}}] \subseteq [1, s]_{\mathbb{N}}$.

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be compact ANR inverse sequences. Then $\boldsymbol{X}$ is said to be *quasi-equivalent to* $\boldsymbol{Y}$, denoted by $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$, provided for every $n \in \mathbb{N}$ there exist morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{X}$ such that $\boldsymbol{g}\boldsymbol{f} \simeq_n 1_{\boldsymbol{X}}$ and $\boldsymbol{f}\boldsymbol{g} \simeq_n 1_{\boldsymbol{Y}}$.

This relation $\overset{q}{\simeq}$ is an isomorphism invariant relation. By [20, Theorem 3.1] *if X, Y are compact metrizable spaces and if $\boldsymbol{X}$, $\boldsymbol{Y}$ are arbitrary with them associated compact ANR inverse sequences respectively (via the inverse limits), then*

$$X \overset{q}{\simeq} Y \Leftrightarrow \boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}.$$

Consequently, a compactum X is quasi-equivalent to a compactum Y (in the sense of Borsuk), $X \overset{q}{\simeq} Y$, if and only if, for every $n \in \mathbb{N}$, there exist morphisms $\boldsymbol{f}^n : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}^n : \boldsymbol{Y} \to \boldsymbol{X}$ such that $\boldsymbol{g}^n\boldsymbol{f}^n \simeq_n 1_{\boldsymbol{X}}$ and $\boldsymbol{f}^n\boldsymbol{g}^n \simeq_n 1_{\boldsymbol{Y}}$. One may assume, without loss of generality, that all the morphisms realizing the relations $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ are simple. We may also assume that $n' \geq n$ implies $f^{n'} \geq f^n$ and $g^{n'} \geq g^n$. Further, it is obvious that the defining conditions for $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ can be relaxed to the conditions $\boldsymbol{g}^n\boldsymbol{f}^n \simeq_{s_n} 1_{\boldsymbol{X}}$ and $\boldsymbol{f}^n\boldsymbol{g}^n \simeq_{t_n} 1_{\boldsymbol{Y}}$ respectively, where (s_n) and (t_n) are increasing unbounded sequences in $\{0\} \cup \mathbb{N}$.

In [20] is constructed a certain category $\underline{\mathcal{K}}$ which describes the relation $\overset{q}{\simeq}$ by means of an appropriate relation on the morphisms of $\underline{\mathcal{K}}$. The objects of $\underline{\mathcal{K}}$ are all compact ANR inverse sequences, while

$$\underline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y}) = \{F = (\boldsymbol{f}^n) \mid \boldsymbol{f}^n \in tow\text{-}HcM(\boldsymbol{X}, \boldsymbol{Y}) \text{ simple}, n \in \mathbb{N}\}.$$

The composition in $Mor\underline{\mathcal{K}}$ is the coordinatewise composition, i.e.,

$$GF = (\boldsymbol{g}^n \boldsymbol{f}^n) = ((f^n g^n, [g^n_k f^n_{g^n(k)}])),$$

while the identity morphism on an object $\boldsymbol{X} \in Ob\underline{\mathcal{K}}$ is $1_{\boldsymbol{X}} = (1^n_{\boldsymbol{X}})$, where $1^n_{\boldsymbol{X}} = 1_{\boldsymbol{X}}$ for each $n \in \mathbb{N}$. A morphism $F = (\boldsymbol{f}^n) \in \underline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y})$ is said to be *quasi-homotopic to* a morphism $F' = (\boldsymbol{f}'^n) \in \underline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y})$, denoted by $F \overset{q}{\simeq} F'$, provided there exists an increasing and unbounded sequence (s_n) in $\{0\} \cup \mathbb{N}$ such that $\boldsymbol{f}^n \simeq_{s_n} \boldsymbol{f}'^n$, whenever $s_n > 0$. The quasi-homotopy relation $\overset{q}{\simeq}$ is an equivalence relation on each set $\underline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y})$. It is also natural from the right, i.e.,

$$(\forall H \in \underline{\mathcal{K}}(\boldsymbol{W}, \boldsymbol{X}))(\forall F, F' \in \underline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y}))(F \overset{q}{\simeq} F' \Rightarrow FH \overset{q}{\simeq} F'H).$$

Unfortunately, the quasi-homotopy relation $\overset{q}{\simeq}$ is not natural from the left, so there is no corresponding quotient category. Nevertheless, by [20, Theorem 3.27],

$\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ if and only if there exist morphisms $F \in \underline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y})$ and $G \in \underline{\mathcal{K}}(\boldsymbol{Y}, \boldsymbol{X})$ such that $GF \overset{q}{\simeq} 1_{\boldsymbol{X}}$ and $FG \overset{q}{\simeq} 1_{\boldsymbol{Y}}$.

It was also shown in [20] that for a slight strengthening of the Borsuk quasi-equivalence, reinterpreted as above, there exists a complete category characterization. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences in $HcANR$. Then $\boldsymbol{X}$ is said to be *$\overline{q}$-equivalent to* $\boldsymbol{Y}$, denoted by $\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}$, provided $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ and there exists a pair $F = (\boldsymbol{f}^n)$, $G = (\boldsymbol{g}^n)$ of morphisms realizing this relation in the category $\underline{\mathcal{K}}$ such that, for every $i \in \mathbb{N}$ and every $j \in \mathbb{N}$, the sequences $(f^n(j))$ and $(g^n(i))$ are bounded.

For a pair X, Y of compacta, we define $X \overset{\overline{q}}{\simeq} Y$ provided $\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}$ for some (equivalently, any) pair $\boldsymbol{X}$, $\boldsymbol{Y}$ of the associated compact ANR inverse sequences.

Let $\overline{\mathcal{K}}$ be the subcategory of $\underline{\mathcal{K}}$ consisting of $Ob\overline{\mathcal{K}} = Ob\underline{\mathcal{K}}$ and of $Mor\overline{\mathcal{K}} \subseteq Mor\,\underline{\mathcal{K}}$ such that each $\overline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y}) \subseteq \underline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y})$ consists of all the morphisms $F = (\boldsymbol{f}^n)$, where all $\boldsymbol{f}^n = (f^n, [f^n_j])$ have a unique index function $f = f^n$, $n \in \mathbb{N}$. Such a morphism is denoted by $F = (f, \boldsymbol{f}^n)$. The key fact is that the quasi-homotopy relation $\overset{q}{\simeq}$ is a natural equivalence relation on $Mor\overline{\mathcal{K}}$. Therefore, there exists the corresponding quotient category $\overline{\mathcal{K}}/_{\overset{q}{\simeq}} \equiv \overline{\mathcal{Q}}$. Moreover, the quotient category $\overline{\mathcal{Q}}$ yields the associated category $\mathcal{Q}$ on compacta such that

$$Ob\mathcal{Q} = Ob(cM) \text{ and } \mathcal{Q}(X, Y) \approx \overline{\mathcal{Q}}(\boldsymbol{X}, \boldsymbol{Y}),$$

where $\boldsymbol{X}$, $\boldsymbol{Y}$ are any compact ANR inverse sequences associated with X, Y respectively. (For given pair X, Y, any set $\overline{\mathcal{Q}}(\boldsymbol{X}, \boldsymbol{Y})$ may represent $\mathcal{Q}(X, Y)$.) An important result is the following one ([20, Theorem 6]):

$\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}$ if and only if $\boldsymbol{X} \cong \boldsymbol{Y}$ in $\overline{\mathcal{Q}}$.

Consequently, for every pair X, Y *of compacta,* $X \overset{\overline{q}}{\simeq} Y$ *if and only if* $X \cong Y$ *in* $\mathcal{Q}$. Moreover ([20, Theorem 7]), there exist functors

$$Q : HcM \to \mathcal{Q} \text{ and } \Gamma : Sh(cM) \to \mathcal{Q},$$

which keep the objects fixed and $\Gamma S = Q$, where $S : HcM \to Sh(cM)$ is the ordinary shape functor.

According to [20, Remark 8(b)], the quasi-homotopy relation admits a slight strengthening in the following way. A morphism $F = (f, \boldsymbol{f}^n) \in \overline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y})$ is said to be *uniformly* quasi-homotopic to a morphism $F' = (f', \boldsymbol{f}'^n) \in \overline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y})$, denoted by $F \overset{q^*}{\simeq} F'$, provided $F \overset{q}{\simeq} F'$ and there exists a sequence (i_j) in $\mathbb{N}$, $i_j \geq f(j), f'(j)$, such that

$$(\forall n \in \mathbb{N})(\forall j \in [1, s_n]_{\mathbb{N}})\ [f_j^n][p_{f(j)i_j}] = [f_j'^n][p_{f'(j)i_j}],$$

where (s_n) is a realizing sequence for $F \overset{q}{\simeq} F'$. It is readily seen that $\overset{q^*}{\simeq}$ is a natural equivalence relation on $\overline{\mathcal{K}}$. Thus, there exist the corresponding quotient category

$$\overline{\mathcal{K}}/_{\overset{q^*}{\simeq}} \equiv \overline{\mathcal{Q}}^*$$

and the associated category $\mathcal{Q}^*$ on compacta. Further, there exist functors

$$Q^* : HcM \to \mathcal{Q}^* \text{ and } \Gamma^* : Sh(cM) \to \mathcal{Q}^*,$$

which keep the objects fixed and $\Gamma^* S = Q^*$. Moreover, there exists a functor

$$\Pi : \mathcal{Q}^* \to \mathcal{Q},$$

such that $Q = \Gamma Q^*$ and $\Gamma = \Pi\Gamma^*$.

Let $\boldsymbol{X}$, $\boldsymbol{Y}$ be a pair of compact ANR inverse sequences. Then, $\boldsymbol{X}$ is said to be q^*-*equivalent* to $\boldsymbol{Y}$, denoted by $\boldsymbol{X} \overset{q^*}{\simeq} \boldsymbol{Y}$, provided there exists a pair of morphisms $F : \boldsymbol{X} \to \boldsymbol{Y}$, $G : \boldsymbol{Y} \to \boldsymbol{X}$ in $\overline{\mathcal{K}}$ such that $GF \overset{q^*}{\simeq} 1_{\boldsymbol{X}}$ and $FG \overset{q^*}{\simeq} 1_{\boldsymbol{Y}}$. Clearly, this means $\boldsymbol{X} \cong \boldsymbol{Y}$ in $\overline{\mathcal{Q}}^*$. Let the q^*-*equivalence* of compacta be the induced equivalence relation in the category $\mathcal{Q}^*$.

We want to relate the categories $\overline{\mathcal{Q}}^*$ and $\mathcal{Q}^*$ to our categories tow^*-$HcANR$ and $Sh^*(cM)$ respectively.

THEOREM 5.1. *The category* $\overline{\mathcal{Q}}^*$ *is isomorphic to the subcategory* tow^*_ω-$HcANR \subseteq tow^*$-$HcANR$. *Consequently, the category* $\mathcal{Q}^*$ *is isomorphic to the corresponding subcategory* $Sh^*_\omega(cM)$ *of the coarse shape category* $Sh^*(cM)$ *of compacta.*

PROOF. The object classes of $\overline{\mathcal{Q}}^*$ and of tow^*_ω-$HcANR$ coincide: It is the class of all compact ANR inverse sequences $\boldsymbol{X}$ having the homotopy classes of mappings to be the bonding morphisms. A morphism $\boldsymbol{F} \in \overline{\mathcal{Q}}^*(\boldsymbol{X}, \boldsymbol{Y})$ is the equivalence class $[F]_{\overset{q^*}{\simeq}}$, where $F = (f, \boldsymbol{f}^n) \in \overline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y})$. Observe that $(f, \boldsymbol{f}^n) = (f, [f_j^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ is a commutative S^*-mapping, i.e., $F = (f, [f_j^n])$

is also a morphism in the category $(HcANR^{\mathbb{N}})^*$. Further, the homotopy relation $\overset{q^*}{\simeq}$ guarantees that $F \overset{q^*}{\simeq} F' = (f', \boldsymbol{f}'^n)$ in $\overline{\mathcal{K}}$ implies $(f, [f_j^n]) \simeq (f', [f_j'^n])$ in $(HcANR^{\mathbb{N}})^*$. (Put $\sigma(j) = i_j$ and $\chi(n) = s_n$.) However, the converse holds too. Namely, a commutative S^*-mapping $(f, [f_j^n])$ in $(HcANR^{\mathbb{N}})^*$ is also the morphism $(f, \boldsymbol{f}^n) = F$ in $\overline{\mathcal{Q}}^*$, and $(f, [f_j^n]) \simeq (f', [f_j'^n])$ in $(HcANR^{\mathbb{N}})^*$ implies $(f, \boldsymbol{f}^n) = F \overset{q^*}{\simeq} F' = (f', \boldsymbol{f}'^n)$ in $\overline{\mathcal{K}}$. (Put $i_j = \sigma(j)$ and $s_n = \chi(n)$.) Finally, it is obvious that $(fg, \boldsymbol{g}^n \boldsymbol{f}^n)$ in $\overline{\mathcal{K}}$ becomes $(fg, [g_k^n][f_{g(k)}^n])$ in $(HcANR^{\mathbb{N}})^*$, and that $(1_{\mathbb{N}}, \boldsymbol{1}_{\boldsymbol{X}}^n)$ in $\overline{\mathcal{K}}$ becomes $(1_{\mathbb{N}}, [1_{X_j}^n])$ in $(HcANR^{\mathbb{N}})^*$. Therefore, there exists a functor $\overline{\Phi} : \overline{\mathcal{Q}}^* \to tow_\omega^*\text{-}HcANR$, defined by $\overline{\Phi}(\boldsymbol{X}) = \boldsymbol{X}$ and $\overline{\Phi}(\boldsymbol{F}) = \boldsymbol{f}^*$, where $\boldsymbol{f}^* = [(f, [f_j^n])]$, $(f, [f_j^n]) = (f, \boldsymbol{f}^n) = F$, $[F]_{\overset{q^*}{\simeq}} = \boldsymbol{F}$, which is an isomorphism of the categories. The statement for the categories on compacta follows immediately. □

The next corollary relates the coarse shape classification of compacta to the classification in the subcategory $Sh_\omega^*(cM)$.

Corollary 5.2. *The isomorphism classification in the subcategory $Sh_\omega^*(cM)$ is strictly finer than the isomorphism (coarse shape type) classification in $Sh^*(cM)$, i.e., for every pair X, Y of metric compacta, $Sh_\omega^*(X) = Sh_\omega^*(Y)$ implies $Sh^*(X) = Sh^*(Y)$, while there exists such a pair so that $Sh^*(X) = Sh^*(Y)$ and $Sh_\omega^*(X) \neq Sh_\omega^*(Y)$.*

Proof. It is clear that $Sh_\omega^*(X) = Sh_\omega^*(Y)$ implies $Sh^*(X) = Sh^*(Y)$. By Corollary 3.12, $Sh^*(X) = Sh^*(Y)$ is equivalent to $S^*(X) = S^*(Y)$, and by Theorem 5.1, $Sh_\omega^*(X) = Sh_\omega^*(Y)$ is equivalent to $X \overset{q^*}{\simeq} Y$. Now, if $Sh^*(X) = Sh^*(Y)$ would imply $Sh_\omega^*(X) = Sh_\omega^*(Y)$, then $S^*(X) = S^*(Y)$ would imply $X \overset{q^*}{\simeq} Y$, which contradicts [22, Corollary 7]. □

One can now see that the coarse shape is indeed coarser than the shape:

Corollary 5.3. *For every pair X, Y of metric compacta, $Sh(X) = Sh(Y)$ implies $Sh^*(X) = Sh^*(Y)$, while there exists such a pair so that $Sh^*(X) = Sh^*(Y)$ and $Sh(X) \neq Sh(Y)$.*

Proof. It is clear that $Sh(X) = Sh(Y)$ implies $Sh^*(X) = Sh^*(Y)$ as well as $Sh_\omega^*(X) = Sh_\omega^*(Y)$. If $Sh^*(X) = Sh^*(Y)$ would imply $Sh(X) = Sh(Y)$, then $Sh^*(X) = Sh^*(Y)$ would imply $Sh_\omega^*(X) = Sh_\omega^*(Y)$, which contradicts Corollary 5.2. □

At the end of this section, let us describe a coarse shape morphism $F^* \in Sh^*(X, Y)$, where Y has the homotopy type of a polyhedron (equivalently, an ANR) as well as Y has the coarse shape type a polyhedron (compare Claim 1).

Proposition 5.4. *Let $F^* : X \to Y$ be a coarse shape morphism of a space X to a space Y, where Y has the homotopy type of a polyhedron. Then there exists a representative $\boldsymbol{f}^* \in pro^*\text{-}HPol(\boldsymbol{X},\boldsymbol{Y})$ of $F^* = \langle \boldsymbol{f}^* \rangle$, such that every representative $(f,[f_\mu^n]) \in inv^*\text{-}HPol(\boldsymbol{X},\boldsymbol{Y})$ of $\boldsymbol{f}^* = [(f,[f_\mu^n])]$ consists of a sequence of homotopy class $[f_\lambda^n]$, $n \in \mathbb{N}$, of a single term X_λ of $\boldsymbol{X}$ to the polyhedron. Therefore, F^* is represented by a sequence $([h^n])$ of the homotopy classes $[h^n] : X \to Y$, $n \in \mathbb{N}$, uniformly factorizing through $\boldsymbol{p} : X \to \boldsymbol{X}$.*

Proof. Let $q : Q \to Y$ be a homotopy equivalence, where Q is a polyhedron. Then $\boldsymbol{q} = ([q_\mu] = [q]^{-1}) : Y \to \boldsymbol{Y} = (Q_\mu = Q, q_{\mu\mu'} = 1_Q, M = \{\mu\})$ is a $HPol$-expansion of Y. Let $F^* : X \to Y$ be a coarse shape morphism. Then there exists a morphism $\boldsymbol{f}^* : \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to \boldsymbol{Y}$ of $pro^*\text{-}HPol$ such that $F = \langle \boldsymbol{f}^* \rangle$. Let $(f,[f_\mu^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ in $inv^*\text{-}HPol$ be any representative of $\boldsymbol{f}^*$. Since $\boldsymbol{Y} = (Q)$ is a rudimentary system, the index function f is a constant function of the singleton $M = \{\mu\}$ to $\lambda \in \Lambda$ and, for all $n \in \mathbb{N}$, the homotopy classes $[f_\mu^n]$ are a single homotopy class $[f_\lambda^n] : X_\lambda \to Q$. Finally, put $[h^n] = [q f_\lambda^n p_\lambda]$, $n \in \mathbb{N}$. □

Proposition 5.5. *Let $F^* : X \to Y$ be a coarse shape morphism of a space X to a space Y, where Y has the coarse shape type of a polyhedron. Then there exists a representative $\boldsymbol{f}^* \in pro^*\text{-}HPol(\boldsymbol{X},\boldsymbol{Y})$ of F^* such that, for every representative $(f,[f_\mu^n]) \in inv^*\text{-}HPol(\boldsymbol{X},\boldsymbol{Y})$ of $\boldsymbol{f}^*$, the index function f is λ-constant and all $[f_\mu^n] : X_\lambda \to Y_\mu$ factorize through the polyhedron. Therefore, F^* is represented by a family $([h_\mu^n])$ of the homotopy classes $[h_\mu^n] : X \to Y_\mu$, $n \in \mathbb{N}$, $\mu \in M$, uniformly factorizing through $\boldsymbol{p} : X \to \boldsymbol{X}$ and through the polyhedron.*

Proof. Let $G^* : Q \to Y$ be a coarse shape isomorphism, where Q is a polyhedron. Then there exists a representative $\boldsymbol{g}^* : (Q_\nu = Q) \to \boldsymbol{Y}$ of G^* such that, for every representative $(g,[g_\mu^n])$ of $\boldsymbol{g}^*$, g is the constant function and $[g_\mu^n] : Q \to Y_\mu$, $n \in \mathbb{N}$, $\mu \in M$. Let $F^* : X \to Y$ be a coarse shape morphism. Then $F^* = G^* H^*$, where $H^* = (G^*)^{-1} F^* : X \to Q$. By Proposition 5.4, there exists a representative $\boldsymbol{h}^* : \boldsymbol{X} \to (Q)$ of H^* such that every its representative $(h,[h_\nu^n])$ consists of a sequence of homotopy classes $[h^n] : X_\lambda \to Q$, $n \in \mathbb{N}$, with a fixed λ. Thus, every representative of $\boldsymbol{f}^* \equiv \boldsymbol{g}^* \boldsymbol{h}^* \in F^*$ is a family $([f_\mu^n])$, $n \in \mathbb{N}$, $\mu \in M$, where $f_\mu^n = g_\mu^n h^n : X_\lambda \to Q \to Y_\mu$ for a unique $\lambda \in \Lambda$. Finally, put $h_\mu^n = f_\mu^n p_\lambda : X \to Y_\mu$. □

Remark 5.6. Concerning Proposition 5.4, recall that a shape morphism $F : X \to Y$, $Y \simeq Q$, is represented by a unique homotopy class $[h_\lambda] = [q f_\lambda p_\lambda] : X \to Y$. On the other hand, by Proposition 5.5, a coarse shape morphism $F^* : X \to Y$, $Sh(Y) = Sh(Q)$, is represented by a family $([f_\mu])_{\mu \in M} : X \to \boldsymbol{Y}$, such that every $[f_\mu] : X \to Y_\mu$ factorizes through a unique X_λ and through Q. Further, according to [21, Lemma 3 and Remark 3(a)], since the same shape type implies the same coarse shape type, Proposition 5.5 applies to FANR's.

6. A coarse shape isomorphism

In this section, we are going to establish an analogue of the well known Morita lemma [17], which should characterize a coarse shape isomorphism in an elegant and rather operative manner. According to the "reindexing theorem" (Theorem 3.27) and definition of the abstract coarse shape category $Sh^*_{(\mathcal{C},\mathcal{D})}$, it suffices to characterize an isomorphism $\boldsymbol{f}^* \in pro^*\text{-}\mathcal{D}(\boldsymbol{X},\boldsymbol{Y})$ which admits a level representative $(1_\Lambda, f_\lambda^n) : \boldsymbol{X} \to \boldsymbol{Y}$ in $inv^*\text{-}\mathcal{D}$. In the case of inverse sequences, a strictly increasing simple representative will do. Since the characterization does not depend on the objects of $\mathcal{D}$, we shall consider such $\boldsymbol{f}^*$ in $pro^*\text{-}\mathcal{C}$ as well as in $tow^*\text{-}\mathcal{C}$.

Theorem 6.1. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ be inverse systems in $\mathcal{C}$ over the same index set. Let a morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ in $pro^*\text{-}\mathcal{C}$ admit a level representative $(1_\Lambda, f_\lambda^n)$. Then $\boldsymbol{f}^*$ is an isomorphism if and only if, for every $\lambda \in \Lambda$, there exist $\lambda' \geq \lambda$ and $n \in \mathbb{N}$ such that, for every $n' \geq n$, there exists a morphism $h_\lambda^{n'} : Y_{\lambda'} \to X_\lambda$ in $\mathcal{C}$, so that the following diagram in $\mathcal{C}$ commutes:*

$$\begin{array}{ccc} X_\lambda & \longleftarrow & X_{\lambda'} \\ f_\lambda^{n'} \downarrow & h_\lambda^{n'} \nwarrow & \downarrow f_{\lambda'}^{n'} \\ Y_\lambda & \longleftarrow & Y_{\lambda'} \end{array} . \tag{4}$$

Proof. Let $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ be an isomorphism in $pro^*\text{-}\mathcal{C}$ which admits a level representative $(1_\Lambda, f_\lambda^n)$. Let $\boldsymbol{g}^* = [(g, g_\lambda^n)] : \boldsymbol{Y} \to \boldsymbol{X}$ be the inverse of $\boldsymbol{f}^*$, i.e.,

$$(g, g_\lambda^n)(1_\Lambda, f_\lambda^n) \sim (1_\Lambda, 1_{X_\lambda}) \ \wedge \ (1_\Lambda, f_\lambda^n)(g, g_\lambda^n) \sim (1_\Lambda, 1_{Y_\lambda}).$$

Given any $\lambda \in \Lambda$, choose $\lambda_1', \lambda_2' \in \Lambda$ according to the above equivalence relations. Then there exists $\lambda' \geq \lambda_1', \lambda_2'$. Thus $\lambda' \geq \lambda, g(\lambda)$. Further, choose $n_1, n_2 \in \mathbb{N}$ according to the above equivalence relations and the given λ. Since $(1_\Lambda, f_\lambda^n)$ is an S^*-morphism, for the pair $g(\lambda) \leq \lambda'$, there exists $n_3 \in \mathbb{N}$ such that the appropriate commutativity condition holds. Put $n = \max\{n_1, n_2, n_3\}$. Let us define, for every $n' \geq n$, a morphism $h_\lambda^{n'} : Y_{\lambda'} \to X_\lambda$ in $\mathcal{C}$ by putting

$$h_\lambda^{n'} = g_\lambda^{n'} q_{g(\lambda)\lambda'}.$$

We are proving that diagram (4) commutes. First, according to the second equivalence relation,

$$f_\lambda^{n'} h_\lambda^{n'} = f_\lambda^{n'} g_\lambda^{n'} q_{g(\lambda)\lambda'} = q_{\lambda\lambda'}.$$

Thus, the left (lower) triangle in (4) commutes. Further, since $n' \geqslant n_3$,

$$h_\lambda^{n'} f_{\lambda'}^{n'} = g_\lambda^{n'} q_{g(\lambda)\lambda'} f_{\lambda'}^{n'} = g_\lambda^{n'} f_{g(\lambda)}^{n'} p_{g(\lambda)\lambda'},$$

while, according to the first equivalence relation,

$$g_{\lambda}^{n'} f_{g(\lambda)}^{n'} p_{g(\lambda)\lambda'} = p_{\lambda\lambda'}.$$

Therefore,

$$h_{\lambda}^{n'} f_{\lambda'}^{n'} = p_{\lambda\lambda'},$$

which proves commutativity of the right (upper) triangle in (4).

Conversely, suppose that a morphism $\boldsymbol{f}^* = [(1_\Lambda, f_\lambda^n)] : \boldsymbol{X} \to \boldsymbol{Y}$ in pro^*-$\mathcal{C}$ fulfils the condition of the theorem. Let $g : \Lambda \to \Lambda$ be defined by that condition, i.e., $g(\lambda) = \lambda' \geq \lambda$. Further, given any $\lambda \in \Lambda$, choose $n = n_\lambda \in \mathbb{N}$ by the condition. Let us define, for every $n \in \mathbb{N}$ and every $\lambda \in \Lambda$, a morphism $g_\lambda^n : Y_{g(\lambda)} \to X_\lambda$ in $\mathcal{C}$ by putting

$$g_\lambda^n = \left\{ \begin{array}{c} h_\lambda^{n_\lambda};\ n < n_\lambda \\ h_\lambda^{n};\ n \geqslant n_\lambda \end{array} \right. ,$$

where h_λ^n comes from the condition. We have to prove that $(g, g_\lambda^n) : \boldsymbol{Y} \to \boldsymbol{X}$ is an S^*-morphism. Let a pair $\lambda \leq \lambda'$ be given. Choose $\lambda_0 \geq g(\lambda), g(\lambda')$ and put $\lambda_1 = g(\lambda_0)$. Since $(1_\Lambda, f_\lambda^n)$ is an S^*-morphism, for the pairs $g(\lambda) \leq \lambda_0$ and $g(\lambda') \leq \lambda_0$, there exist $n_1, n_2 \in \mathbb{N}$ such that the appropriate commutativity conditions hold respectively. Put

$$n = \max\{n_\lambda, n_{\lambda'}, n_{\lambda_0}, n_1, n_2\}.$$

For every $n' \geq n$, consider the following diagram:

$$(5)\qquad \begin{array}{ccccccccc}
X_\lambda & \longleftarrow & X_{\lambda'} & \longleftarrow & X_{g(\lambda')} & & & & \\
 & \nwarrow & & \nwarrow & \downarrow & \nwarrow & & & \\
 & & X_{g(\lambda)} & & \longleftarrow & & X_{\lambda_0} & & \\
 & & \downarrow & & Y_{g(\lambda')} & & \downarrow & & \\
 & \nwarrow & & & & \nwarrow & & \nwarrow & \\
 & & Y_{g(\lambda)} & & \longleftarrow & & Y_{\lambda_0} & \longleftarrow & Y_{\lambda_1}
\end{array}$$

We are going to prove, by chasing diagram (5), that

$$(6)\qquad g_\lambda^{n'} q_{g(\lambda)\lambda_1} = p_{\lambda\lambda'} g_{\lambda'}^{n'} q_{g(\lambda')\lambda_1}.$$

Since $n' \geqslant n_{\lambda_0}$, the condition of the theorem implies

$$(7)\qquad g_\lambda^{n'} q_{g(\lambda)\lambda_1} = h_\lambda^{n'} q_{g(\lambda)\lambda_0} f_{\lambda_0}^{n'} h_{\lambda_0}^{n'}.$$

Since $n' \geqslant n_1$,

$$(8)\qquad h_\lambda^{n'} q_{g(\lambda)\lambda_0} f_{\lambda_0}^{n'} h_{\lambda_0}^{n'} = h_\lambda^{n'} f_{g(\lambda)}^{n'} p_{g(\lambda)\lambda_0} h_{\lambda_0}^{n'}.$$

Since $n' \geqslant n_\lambda, n_{\lambda'}$, the condition of the theorem implies

$$(9)\qquad h_\lambda^{n'} f_{g(\lambda)}^{n'} p_{g(\lambda)\lambda_0} h_{\lambda_0}^{n'} = p_{\lambda\lambda_0} h_{\lambda_0}^{n'} = p_{\lambda\lambda'} h_{\lambda'}^{n'} f_{g(\lambda')}^{n'} p_{g(\lambda')\lambda_0} h_{\lambda_0}^{n'}.$$

Since $n' \geqslant n_2$,

$$(10)\qquad p_{\lambda\lambda'} h_{\lambda'}^{n'} f_{g(\lambda')}^{n'} p_{g(\lambda')\lambda_0} h_{\lambda_0}^{n'} = p_{\lambda\lambda'} h_{\lambda'}^{n'} q_{g(\lambda')\lambda_0} f_{\lambda_0}^{n'} h_{\lambda_0}^{n'}.$$

Finally, since $n' \geqslant n_{\lambda_0}$, the condition of the theorem implies

$$p_{\lambda\lambda'} h_{\lambda'}^{n'} q_{g(\lambda')\lambda_0} f_{\lambda_0}^{n'} h_{\lambda_0}^{n'} = p_{\lambda\lambda'} h_{\lambda'}^{n'} q_{g(\lambda')g(\lambda_0)} = p_{\lambda\lambda'} g_{\lambda'}^{n'} q_{g(\lambda')\lambda_1}. \tag{11}$$

Now, by combining (7), (8), (9), (10) and (11), one establishes (6), which proves that (g, g_λ^n) is an S^*-morphism. Moreover, by the condition of the theorem, it is readily seen that, for every $\lambda \in \Lambda$ and every $n' \in \mathbb{N}$, $n' \geqslant n_\lambda$,

$$g_\lambda^{n'} f_{g(\lambda)}^{n'} = h_\lambda^{n'} f_{g(\lambda)}^{n'} = p_{\lambda g(\lambda)} \ \wedge \ f_\lambda^{n'} g_\lambda^{n'} = f_\lambda^{n'} h_\lambda^{n'} = q_{\lambda g(\lambda)}.$$

This shows that

$$(g, g_\lambda^n)(1_\Lambda, f_\lambda^n) \sim (1_\Lambda, 1_{X_\lambda}) \ \wedge \ (1_\Lambda, f_\lambda^n)(g, g_\lambda^n) \sim (1_\Lambda, 1_{Y_\lambda}),$$

which means that $\boldsymbol{g}^* = [(g, g_\lambda^n)] : \boldsymbol{Y} \to \boldsymbol{X}$ is the inverse of $\boldsymbol{f}^*$. Therefore, $\boldsymbol{f}^*$ is an isomorphism in *pro**-$\mathcal{C}$. □

Remark 6.2. (a) Let us consider *pro*-$\mathcal{C}$ to be a subcategory of *pro**-$\mathcal{C}$ (see Proposition 3.24). Then, in *pro*-$\mathcal{C}$, Theorem 5.1 allows to put $n_\lambda = 1$ for every λ. Consequently, for each λ, the sequence (h_λ^n) reduces to a single morphism h_λ. Thus, Theorem 5.1 in the subcategory *pro*-$\mathcal{C}$ becomes the original Morita lemma.

(b) One can easily verify that the condition (of Theorem 5.1) characterizing an isomorphism may be reduced to a cofinal subset $\Lambda' \subseteq \Lambda$. Thus, the following corollary holds.

Corollary 6.3. *If $\boldsymbol{f}^* = [(1_\Lambda, f_\lambda^n)] : \boldsymbol{X} \to \boldsymbol{Y}$ in pro*-$\mathcal{C}$ admits a cofinal subset $\Lambda' \subseteq \Lambda$ such that, for every $\lambda' \in \Lambda'$, there exists $n \in \mathbb{N}$, so that, for every $n' \geq n$, $f_{\lambda'}^{n'}$ is an isomorphism in $\mathcal{C}$, then $\boldsymbol{f}^*$ is an isomorphism.*

According to the proof of Theorem 3.1 and Definition 3.3, one can characterize an S^*-morphism (f, f_j^n) in any category $(\mathcal{C}^{\mathbb{N}})^*$ in the "original" terms of a commutativity radius γ. Further, by the proof of Theorem 3.2 and Definition 3.8, the equivalence relation $(f, f_j^n) \sim (f', f_j'^n)$ in any category $(\mathcal{C}^{\mathbb{N}})^*$ can be characterized in the "original" terms of a shift function σ and a homotopy radius χ. Concerning isomorphisms of inverse sequences, this provides a very useful sufficient condition by means of a simple representative (see Proposition 3.11):

Theorem 6.4. *Let $\boldsymbol{X} = (X_i, p_{ii'})$ and $\boldsymbol{Y} = (Y_j, q_{jj''})$ be inverse sequences in a category $\mathcal{C}$, let $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism in tow*-$\mathcal{C}$ and let (f, f_j^n) be any simple representative of $\boldsymbol{f}^*$ with a commutativity radius γ and f strictly increasing. If for every $n \in \mathbb{N}$ and every $j = 1, \ldots, \gamma(n) - 1$, there exists a morphism $h_j^n : Y_{j+1} \to X_{f(j)}$ in $\mathcal{C}$ such that the diagram*

$$\begin{array}{ccc} X_{f(j)} & \longleftarrow & X_{f(j+1)} \\ f_j^n \downarrow & h_j^n \nwarrow & \downarrow f_{j+1}^n \\ Y_j & \longleftarrow & Y_{j+1} \end{array}$$

commutes, then $\boldsymbol{f}^*$ is an isomorphism in tow*-$\mathcal{C}$.

Conversely, if $\boldsymbol{f}^*$ is an isomorphism in tow*-$\mathcal{C}$, then, for every $j \in \mathbb{N}$, there exist $j' \geq j$ and $n \in \mathbb{N}$ such that, for every $n' \geq n$, there exists a morphism $h_j^{n'} : Y_{j'} \to X_{f(j)}$ in $\mathcal{C}$ so that the following diagram in $\mathcal{C}$ commutes:

$$\begin{array}{ccc} X_{f(j)} & \longleftarrow & X_{f(j')} \\ f_j^{n'} \downarrow & h_j^{n'} \nwarrow & \downarrow f_{j'}^{n'} \\ Y_j & \longleftarrow & Y_{j'} \end{array}$$

PROOF. Since $f : \mathbb{N} \to \mathbb{N}$ is strictly increasing, one can define $g : \mathbb{N} \to \mathbb{N}$ by putting

$$g(i) = \left\{ \begin{array}{c} 1, \quad i \in [1, f(1)]_{\mathbb{N}} \\ j+1, \, i \in [f(j)+1, f(j+1)]_{\mathbb{N}}, \, j \in \mathbb{N} \end{array} \right. .$$

Let $n \in \mathbb{N}$. If $\gamma(n) = 1$, put $g_i^n : Y_{g(i)} \to X_i$, $i \in \mathbb{N}$, to be arbitrary morphisms in $\mathcal{C}$. If $\gamma(n) > 1$, then the compositions of the existing morphisms $h_j^n : Y_{j+1} \to X_{f(j)}$ in $\mathcal{C}$, $j = 1, \ldots, \gamma(n) - 1$, and the appropriate bonding morphisms $p_{ii'}$ determine morphisms $g_i^n : Y_{g(i)} \to X_i$ in $\mathcal{C}$, $i = 1, \ldots, f(\gamma(n) - 1)$. Observe that $g_1^n, \ldots, g_{f(\gamma(n)-1)}^n$ commute with the corresponding bonding morphisms $p_{ii'}$ and $q_{jj'}$. If $i > f(\gamma(n)-1)$, put $g_i^n : Y_{g(i)} \to X_i$ to be arbitrary morphisms in $\mathcal{C}$. Then $(g, g_i^n) : \boldsymbol{Y} \to \boldsymbol{X}$ is an S^*-morphism in *tow**-$\mathcal{C}$ having $\gamma' : \mathbb{N} \to \mathbb{N}$, $\gamma'(n) = f(\gamma(n) - 1)$, as a commutativity radius. (Since γ is increasing and unbounded and since f is strictly increasing, the function γ' is increasing and unbounded.) It is obvious now, by construction, that

$$(*) \qquad (g, g_i^n)(f, f_j^n) \sim (1_{\mathbb{N}}, 1_{X_i}^n) \wedge (f, f_j^n)(g, g_i^n) \sim (1_{\mathbb{N}}, 1_{Y_j}^n).$$

Indeed, for the first relation, the shift function is $\sigma = fg$ and the homotopy radius is $\chi = f(\gamma - 1_{\mathbb{N}})$, while for the second relation, the shift function is $\sigma' = gf$ and the homotopy radius is $\chi' = \gamma - 1_{\mathbb{N}}$. This proves that $\boldsymbol{g}^* \boldsymbol{f}^* = \boldsymbol{1}_{\boldsymbol{X}}^*$ and $\boldsymbol{f}^* \boldsymbol{g}^* = \boldsymbol{1}_{\boldsymbol{Y}}^*$, where $\boldsymbol{g}^* = [(g, g_i^n)]$. Therefore, $\boldsymbol{f}^*$ is an isomorphism in *tow**-$\mathcal{C}$.

Conversely, let $\boldsymbol{g}^* = [(g, g_i^n)] : \boldsymbol{Y} \to \boldsymbol{X}$ be the inverse of $\boldsymbol{f}^*$. Let the equivalence relations $(*)$ be realized via σ, χ and σ', χ' respectively. Then, given $j \in \mathbb{N}$, put $j' = \max\{\sigma(j), \sigma'(j)\}$ and choose $n \in \mathbb{N}$ such that $\chi'(n) \geq j'$ (χ' is unbounded!). Finally, for every $n' \geq n$, put $h_j^{n'} = g_{f(j)}^{n'} q_{gf(j)j'} : Y_{j'} \to X_{f(j)}$. The conclusion follows. ∎

Let us finally prove that the "S^*-condition" characterizes an isomorphic pair of inverse sequences in *tow**-$\mathcal{C}$ [16, Definition 6]).

THEOREM 6.5. *Let $\boldsymbol{X} = (X_i, p_{ii'})$ and $\boldsymbol{Y} = (Y_j, q_{jj'})$ be inverse sequences in a category $\mathcal{C}$. Then $\boldsymbol{X} \cong \boldsymbol{Y}$ in tow*-$\mathcal{C}$ if and only if the following "S^*-condition" is fulfilled:*

$$(\forall j_1 \in \mathbb{N})(\exists i_1 \in \mathbb{N})(\forall i_1' \geq i_1)(\exists j_1' \geq j_1)(\forall j_2 \geq j_1') \;\cdots$$
$$(\forall j_{k+1} \geq j_k')(\exists i_{k+1} \geq i_k')(\forall i_{k+1}' \geq i_{k+1})(\exists j_{k+1}' \geq j_{k+1}) \;\cdots$$

and, for every $n \in \mathbb{N}$, there exist $\mathcal{C}$-morphisms $f_k^n : X_{i_k} \to Y_{j_k}$, $k = 1, \dots, n$, and $g_l^n : Y_{j'_l} \to X_{i'_l}$, $l = 1, \dots, n-1$ and $n > 1$, such that the following diagram commutes:

$$
(12) \qquad \begin{array}{ccccccccccc}
X_{i_1} & \longleftarrow & X_{i'_1} & \longleftarrow & X_{i_2} & \longleftarrow & \cdots & \longleftarrow & X_{i'_{n-1}} & \longleftarrow & X_{i_n} \\
f_1^n \downarrow & & \uparrow g_1^n & & \downarrow f_2^n & & & & \uparrow g_{n-1}^n & & \downarrow f_n^n \\
Y_{j_1} & \longleftarrow & Y_{j'_1} & \longleftarrow & Y_{j_2} & \longleftarrow & \cdots & \longleftarrow & Y_{j'_{n-1}} & \longleftarrow & Y_{j_n}
\end{array} .
$$

Proof. Let $\boldsymbol{X} \cong \boldsymbol{Y}$ in *tow**-$\mathcal{C}$. Then there exist isomorphisms $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}^* : \boldsymbol{Y} \to \boldsymbol{X}$ such that $\boldsymbol{g}^*\boldsymbol{f}^* = \boldsymbol{1}_{\boldsymbol{X}}^*$ and $\boldsymbol{f}^*\boldsymbol{g}^* = \boldsymbol{1}_{\boldsymbol{Y}}^*$ in *tow**-$\mathcal{C}$. By Proposition 3.11, there exist simple representatives $(f', f_j'^n)$ and $(g' g_i'^n)$ of $\boldsymbol{f}^*$ and $\boldsymbol{g}^*$ respectively, such that the index functions f' and g' are strictly increasing. Let the equivalence relations

$$(g', g_i'^n)(f', f_j'^n) \sim (1_{\mathbb{N}}, 1_{X_i}^n) \text{ and } (f', f_j'^n)(g', g_i'^n) \sim (1_{\mathbb{N}}, 1_{Y_j}^n)$$

be realized via (σ, η) and (σ', η') respectively. We are to prove that the "S^*-condition" for $\boldsymbol{X}$ and $\boldsymbol{Y}$ holds. Given $j_1 \in \mathbb{N}$, put $i_1 = f(j_1)$. For every $i'_1 \ge i_1$, put $j'_1 = \max\{\sigma'(j_1), g'(i'_1)\}$. Suppose that, for any $k \in \mathbb{N}$, the indices $i_1, j'_1, \dots, i'_k, j'_k$ are defined. Given $j_{k+1} \ge j'_k$, put $i_{k+1} = \max\{\sigma(i'_k), f'(j_k)\}$, and for every $i'_{k+1} \ge i'_k$, put $j'_{k+1} = \max\{\sigma'(j_k), g'(i'_k)\}$. Observe that, for $(f', f_j'^n)$, there exists an increasing sequence (n_j) in $\mathbb{N}$ such that, for every $j \in \mathbb{N}$,

$$f_{j'}'^n p_{f'(j')f'(j)} = q_{j'j} f_j'^n$$

whenever $j' \le j$ and $n \ge n_j$. (Indeed, if γ is a commutativity radius for $(f', f_j'^n)$, then, for every $j \in \mathbb{N}$, there exists $n \equiv n_j \in \mathbb{N}$ such that $\gamma(n) \ge j$.) In the same way, for $(g', g_i'^n)$, there exists an increasing sequence (n'_i) in $\mathbb{N}$ such that, for every $i \in \mathbb{N}$,

$$g_{i'}'^n q_{g'(i')g'(i)} = p_{i'i} g_i'^n$$

whenever $i' \le i$ and $n \ge n'_i$. Given any $n \in \mathbb{N}$, put

$$\begin{aligned}
m'_n &= \max\{n_{j_n}, n'_{i'_{n-1}}\}, \\
m''_n &= \max\{\eta(i'_l), \eta'(j_k) \mid l = 1, \dots, n-1,\ k = 1, \dots, n\},
\end{aligned}$$

and choose $m \ge m_n = \max\{m'_n, m''_n\}$. Now, for every $k = 1, \dots, n$, put

$$f_k^n = f_{j_k}^m p_{f'(j_k)i_k} : X_{i_k} \to Y_{j_k},$$

and, for every $l = 1, \dots, n-1$ and $n > 1$, put

$$g_l^n = g_{i'_l}'^m q_{g'(i'_l)j'_l} : Y_{j'_l} \to X_{i'_l}.$$

A straightforward verification shows that, for every $k = 1, \dots, n$ and $n > 1$,

$$f_k^n p_{i_k i'_k} g_k^n = q_{j_k j'_k},$$

and, for every $l = 1, \dots, n-1$,

$$g_l^n q_{j'_l j_{l+1}} f_l^n = p_{i'_l i_{l+1}}.$$

Thus, diagram (12) in $\mathcal{C}$ commutes. Conversely, let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences in $\mathcal{C}$ satisfying the "S^*-condition". Then, for $j_1 = 1$ choose i_1, and for $i'_1 = i_1 + 1$ choose $j'_1 \geq j_1$; $\ldots$; for $j_{k+1} = j'_k + 1$ choose $i_{k+1} \geq i'_k$, and for $i'_{k+1} = i_{k+1} + 1$ choose $j'_{k+1} \geq j_{k+1}$; $\ldots$ Further, for every $n \in \mathbb{N}$, there exist morphisms $f^n_k : X_{i_k} \to Y_{j_k}$, $k = 1, \ldots, n$, and $g^n_k : Y_{j'_k} \to X_{i'_k}$, $k = 1, \ldots, n-1$, in $\mathcal{C}$ such that the corresponding diagram (12) commutes. Let us define functions $f', g' : \mathbb{N} \to \mathbb{N}$, by putting $f'(j) = i_k$ whenever $j_{k-1} < j \leq j_k$, $k \in \mathbb{N}$ $(j_0 = 0)$, and $g'(i) = j'_l$ whenever $i'_{l-1} < i \leq i'_l$, $l \in \mathbb{N}$ $(i'_0 = 0)$. Let $n \in \mathbb{N}$. For $j_{k-1} < j \leq j_k$ and $k = 1, \ldots, n$, let

$$f'^n_j = q_{jj_k} f^n_k : X_{f'(j)} \to Y_j,$$

and for every $j > j_n$, let f'^n_j be any $\mathcal{C}$-morphism $X_{f'(j)} \to Y_j$. (The "S^*-condition" assures that the set $\mathcal{C}(X_{f'(j)}, Y_j)$ is not empty!) Further, let the $\mathcal{C}$-morphisms $g'^1_i : Y_{g'(i)} \to X_i$, $i \in \mathbb{N}$, be chosen arbitrarily (the set $\mathcal{C}(Y_{g'(i)}, X_i)$ is not empty), while for $n > 1$, $i'_{l-1} < i \leq i'_l$ and $l = 1, \ldots, n-1$, let

$$g'^n_i = p_{ii'_k} g^n_k : Y_{g'(i)} \to X_i,$$

and for $i > i'_{n-1}$, let g'^n_i be any $\mathcal{C}$-morphism $Y_{g'(i)} \to X_i$ (the set $\mathcal{C}(Y_{g'(i)}, X_i)$ is not empty). It is readily seen that $(f', f'^n_j) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g', g'^n_i) : \boldsymbol{Y} \to \boldsymbol{X}$ are S^*-morphisms, i.e., the morphisms of $(\mathcal{C}^{\mathbb{N}})^*$. It is obvious by construction that $\gamma : \mathbb{N} \to \mathbb{N}$, $\gamma(n) = j_n$, is a commutativity radius for (f', f'^n_j) as well as that $\gamma' : \mathbb{N} \to \mathbb{N}$, $\gamma'(1) = 1$ and $\gamma'(n) = i'_{n-1}$, $n > 1$, is a commutativity radius for (g', g'^n_i). Put $\boldsymbol{f}^* = [(f', f'^n_j)]$ and $\boldsymbol{g}^* = [(g', g'^n_i)]$. Let us define $\sigma : \mathbb{N} \to \mathbb{N}$ by putting $\sigma(i) = i_{l+1}$, whenever $i'_{l-1} < i \leq i'_l$, $l \in \mathbb{N}$, and let us define $\chi : \mathbb{N} \to \mathbb{N} \cup \{0\}$ by putting $\chi(n) = i'_{n-1}$. Then, a straightforward examination shows that σ is a shift function and χ is a homotopy radius for the equivalence of S^*-morphisms $(f'g', g'^n_i f'^n_{g'(i)}), (1_{\mathbb{N}}, 1^n_{X_i}) : \boldsymbol{X} \to \boldsymbol{X}$, i.e.,

$$(g', g'^n_i)(f', f'^n_j) \sim (1_{\mathbb{N}}, 1^n_{X_i}).$$

Thus, $\boldsymbol{g}^* \boldsymbol{f}^* = \boldsymbol{1}^*_{\boldsymbol{X}}$ in *tow**-$\mathcal{C}$. Similarly,

$$(f', f'^n_j)(g', g'^n_i) \sim (1_{\mathbb{N}}, 1^n_{Y_j})$$

holds via (σ', χ'), where $\sigma' : \mathbb{N} \to \mathbb{N}$, $\sigma'(j) = j'_k$, $j_{k-1} < j \leq j_k$, $k \in \mathbb{N}$, and $\chi' : \mathbb{N} \to \mathbb{N} \cup \{0\}$, $\chi'(n) = j_{n-1}$. Thus, $\boldsymbol{f}^* \boldsymbol{g}^* = \boldsymbol{1}^*_{\boldsymbol{Y}}$ in *tow**-$\mathcal{C}$. Hence, $\boldsymbol{X} \cong \boldsymbol{Y}$ in *tow**-$\mathcal{C}$. □

7. The examples

It is clear that isomorphic inverse systems in *pro*-$\mathcal{C}$ are also isomorphic in *pro**-$\mathcal{C}$. We shall now show that the converse does not hold. This will indicate that, beside the shape theory, the coarse shape theory might be a non artificial and useful new one, and therefore, a new geometric (and algebraic - by passing to *pro**-*Grp*) tool for studying and classifying locally bad spaces. The first example is constructed for the pair *pro*-*Grp* $\subseteq$ *pro**-*Grp*.

EXAMPLE 7.1. Let $\boldsymbol{G} = (G_i, p_{ii'})$ and $\boldsymbol{H} = (H_j, q_{jj'})$ be inverse sequences of groups $G_i = H_j = \mathbb{Z}^2$, for all $i, j \in \mathbb{N}$, and homomorphisms $p_{ii'} : \mathbb{Z}^2 \to \mathbb{Z}^2$ and $q_{jj'} : \mathbb{Z}^2 \to \mathbb{Z}^2$ defined via p_{ii+1} and q_{jj+1} respectively as follows:

For every $i \in \mathbb{N}$, p_{ii+1} is given by the integral matrix

$$P_i = \begin{bmatrix} 1 & 0 \\ 0 & -2^{4i} \end{bmatrix};$$

for every $j \in \mathbb{N}$, q_{jj+1} is given by the integral matrix

$$Q_j = \begin{bmatrix} -1 & 0 \\ 2^{2j} & -2^{4j} \end{bmatrix}.$$

Then, $\boldsymbol{G}$ and $\boldsymbol{H}$ are *not* isomorphic in *pro-Grp*, while $\boldsymbol{G} \cong \boldsymbol{H}$ in *pro**-*Grp* (actually, they are isomorphic in the subcategory *tow**-*Grp*).

In order to prove the statement, let us first consider an arbitrary morphism $\boldsymbol{f} : \boldsymbol{G} \to \boldsymbol{H}$ in *pro-Grp*. Let $(f, f_j) : \boldsymbol{G} \to \boldsymbol{H}$ be a representative of $\boldsymbol{f}$. Without loss of generality, we may assume that (f, f_j) is a simple morphism of inverse sequences in *Grp* with the strictly increasing index function $f : \mathbb{N} \to \mathbb{N}$, and that each homomorphism $f_j : G_{f(j)} = \mathbb{Z}^2 \to \mathbb{Z}^2 = H_j$, $j \in \mathbb{N}$, is given by an integral matrix

$$F_j = \begin{bmatrix} \alpha_j & \gamma_j \\ \beta_j & \delta_j \end{bmatrix}.$$

Then, for every $j \in \mathbb{N}$, $q_{jj+1} f_{j+1} = f_j p_{f(j)f(j+1)}$, i.e.,

$$Q_j F_{j+1} = F_j P_{f(j)} \cdots P_{f(j+1)-1}.$$

This means

$$\begin{bmatrix} -1 & 0 \\ 2^{2j} & -2^{4j} \end{bmatrix} \begin{bmatrix} \alpha_{j+1} & \gamma_{j+1} \\ \beta_{j+1} & \delta_{j+1} \end{bmatrix} = \begin{bmatrix} \alpha_j & \gamma_j \\ \beta_j & \delta_j \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sigma_j \end{bmatrix},$$

where

$$\begin{bmatrix} 1 & 0 \\ 0 & \sigma_j \end{bmatrix} = P_{f(j)} \cdots P_{f(j+1)-1},$$

and $\sigma_j = (-1)^{f(j+1)-f(j)} 2^{2k_j}$, $k_j = (f(j+1)-1)f(j+1) - (f(j)-1)f(j)$. It implies that, for every $j \in \mathbb{N}$,

$$-\alpha_{j+1} = \alpha_j, \qquad 2^{2j}\alpha_{j+1} - 2^{4j}\beta_{j+1} = \beta_j.$$

Thus, $\alpha_j = (-1)^j \alpha$ for some $\alpha \in \mathbb{Z}$, while

$$\beta_{j+1} = \frac{(-1)^{j-1} 2^{2j}\alpha - \beta_j}{2^{4j}}.$$

This recursive relation admits the following estimation ($\beta \equiv \beta_1$)

$$\mid \beta_{j+1} \mid \leq \frac{(2^2 + 2^8 + \cdots + 2^{2j^2}) \mid \alpha \mid + \mid \beta \mid}{2^{2j(j+1)}}.$$

Then, clearly,

$$\mid \beta_{j+1} \mid < \frac{\mid \alpha \mid}{2^{2j-1}} + \frac{\mid \beta \mid}{2^{2j(j+1)}},$$

which implies that there exists $j_0 \in \mathbb{N}$, so that, for every $j > j_0$, $\mid \beta_j \mid < 1$. Since each $\beta_j \in \mathbb{Z}$, we infer that $\beta_j = 0$ for all $j > j_0$. Then, by the recursive relation, $\alpha = 0$ must hold. Therefore, $\alpha_j = 0$ for all $j \in \mathbb{N}$. This further implies that $\beta_{j_0} = \cdots = \beta_1 = 0$. Consequently, for every $j \in \mathbb{N}$, the homomorphism f_j is represented by a singular integral matrix

$$F_j = \begin{bmatrix} 0 & \gamma_j \\ 0 & \delta_j \end{bmatrix}.$$

Since all the bonding homomorphisms are represented by regular matrices ($\det P_i \neq 0 \neq \det Q_j$), $\boldsymbol{f}$ cannot be an isomorphism. Thus, $\boldsymbol{G}$ and $\boldsymbol{H}$ are not isomorphic in *pro-Grp*. (Moreover, neither $\boldsymbol{G} \leq \boldsymbol{H}$ nor $\boldsymbol{H} \leq \boldsymbol{G}$ in *pro-Grp* can hold.)

Let us now prove that the inverse sequences $\boldsymbol{G}$ and $\boldsymbol{H}$ are isomorphic in the subcategory *tow***-Grp* $\subseteq$ *pro***-Grp*. Consider a level S^*-morphism $(1_{\mathbb{N}}, f_j^n) : \boldsymbol{G} \to \boldsymbol{H}$ defined by induction as follows:

If $n = 1$, let $f_1^1 : \mathbb{Z}^2 \to \mathbb{Z}^2$ be any homomorphism, and let $f_j^1 = 0 : \mathbb{Z}^2 \to \mathbb{Z}^2$ be the trivial homomorphism for every $j > 1$; assume that, for $n \in \mathbb{N}$ and each $k = 1, \ldots, n$, all homomorphisms f_j^k, $j \in \mathbb{N}$, are defined; let $f_{n+1}^{n+1} = 1_{\mathbb{Z}^2} : \mathbb{Z}^2 \to \mathbb{Z}^2$ be the identity given by the identity matrix

$$F_{n+1}^{n+1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

let f_j^{n+1} be given by an integral matrix F_j^{n+1} defined inductively via the commutativity relation

$$F_j^{n+1} P_j = Q_j F_{j+1}^{n+1},$$

$j = n, \ldots, 1$, and let $f_j^{n+1} = 0$ for all $j > n+1$. One can verify, by a straightforward calculation, that all $F_j^{n+1} \in M_2(\mathbb{Z})$, $j = n, \ldots, 1$, exist. More precisely,

$$F_n^{n+1} = \begin{bmatrix} -1 & 0 \\ 2^{2n} & 1 \end{bmatrix}, \quad F_{n-1}^{n+1} = \begin{bmatrix} 1 & 0 \\ -(2^{2(n-1)} + 2^{6(n-1)+2}) & 1 \end{bmatrix},$$

$$F_{n-2}^{n+1} = \begin{bmatrix} -1 & 0 \\ 2^{2(n-2)} + 2^{6(n-2)+2} + 2^{10(n-2)+8} & 1 \end{bmatrix}, \ldots,$$

$$F_1^{n+1} = \begin{bmatrix} (-1)^n & 0 \\ (-1)^{n-1}(2^2 + 2^8 + \cdots + 2^{2n^2}) & 1 \end{bmatrix}.$$

The commutativity relations assure that $(1_{\mathbb{N}}, f_j^n) : \boldsymbol{G} \to \boldsymbol{H}$ is indeed an S^*-morphism of pro-groups. Let us prove that $\boldsymbol{f}^* = [(1_{\mathbb{N}}, f_j^n)] : \boldsymbol{G} \to \boldsymbol{H}$ is an isomorphism of pro-grops in *pro***-Grp*. According to Theorem 6.1, given any

$j \in \mathbb{N}$, put $j' = j+1$ and $n = j+1$. We have to prove that, for every $n' \geq n$, there exists a homomorphism $h_j^{n'} : \mathbb{Z}^2 \to \mathbb{Z}^2$ such that

$$h_j^{n'} f_{j+1}^{n'} = p_{jj+1} \ \wedge \ f_j^{n'} h_j^{n'} = q_{jj+1}.$$

This means that we have to verify

$$H_j^{n'} F_{j+1}^{n'} = P_j \ \wedge \ F_j^{n'} H_j^{n'} = Q_j,$$

where $H_j^{n'} \in M_2(\mathbb{Z})$ represents $h_j^{n'}$. By Theorem 6.4, it suffices to verify the following condition: For every $n \in \mathbb{N}$ and every $j = 1, \ldots, n$, there exists $H_j^{n+1} \in M_2(\mathbb{Z})$ such that

$$H_j^{n+1} F_{j+1}^{n+1} = P_j \ \wedge \ F_j^{n+1} H_j^{n+1} = Q_j.$$

A straightforward calculation yields

$$H_n^{n+1} = P_n = \begin{bmatrix} 1 & 0 \\ 0 & -2^{4n} \end{bmatrix}, H_{n-1}^{n+1} = \begin{bmatrix} -1 & 0 \\ -2^{6(n-1)+2} & -2^{4(n-1)} \end{bmatrix},$$

$$H_{n-2}^{n+1} = \begin{bmatrix} 1 & 0 \\ -2^{6(n-2)+2} - 2^{10(n-2)+8} & -2^{4(n-2)} \end{bmatrix}, \ldots,$$

$$H_1^{n+1} = \begin{bmatrix} (-1)^{n-1} & 0 \\ -2^8 - 2^{18} - \cdots - 2^{2n^2} & -2^4 \end{bmatrix}.$$

This completes the proof.

The following example is constructed in a quite similar way for the category pair *pro-HTop* $\subseteq$ *pro***-HTop*. First, recall that by a result of W. Scheffer [19], every homotopy class between compact connected abelian groups, $[f] : X \to Y$, contains a continuous homomorphism $h : X \to Y$, $h \in [f]$. Consequently, every homotopy class $[f] : T \to T$ of a 2-torus to itself ($T \approx S^1 \times S^1$, where the multiplicative group S^1 is the standard unit circle in the complex plane) is (uniquely) represented by an integral matrix $F \in M_2(\mathbb{Z})$.

Example 7.2. Let $\boldsymbol{X} = (X_i, [p_{ii'}])$ and $\boldsymbol{Y} = (Y_j, [q_{jj'}])$ be inverse sequences in *HcPol* having each term equal to a 2-torus, i.e., $X_i = Y_j = T$ for all $i, j \in \mathbb{N}$, and with the bonding homotopy classes $[p_{ii'}] : T \to T$ and $[q_{jj'}] : T \to T$ defined via $[p_{ii+1}]$ and $[q_{jj+1}]$ respectively as follows:

For every $i \in \mathbb{N}$, $[p_{ii+1}]$ is given by the integral matrix

$$P_i = \begin{bmatrix} -1 & 0 \\ 0 & 2^{2i} \end{bmatrix};$$

for every $j \in \mathbb{N}$, $[q_{jj+1}]$ is given by the integral matrix

$$Q_j = \begin{bmatrix} 1 & 0 \\ 2^j & -2^{2j} \end{bmatrix}.$$

Then, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are *not* isomorphic in *pro*-HTop, while $\boldsymbol{X} \cong \boldsymbol{Y}$ in *pro***-HTop* (actually, they are isomorphic in the subcategory *tow***-HcPol*).

Similarly to Example 7.1, let us first consider an arbitrary morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in *pro-HTop*. Let $(f, [f_j]) : \boldsymbol{X} \to \boldsymbol{Y}$ be a representative of $\boldsymbol{f}$. We may assume that $(f, [f_j])$ is a simple morphism of inverse sequences in *HcPol* with the strictly increasing index function $f : \mathbb{N} \to \mathbb{N}$, and that each homotopy class $[f_j] : X_{f(j)} = T \to T = Y_j$, $j \in \mathbb{N}$, is given by an integral matrix

$$F_j = \begin{bmatrix} \alpha_j & \gamma_j \\ \beta_j & \delta_j \end{bmatrix}.$$

Then, for every $j \in \mathbb{N}$, $[q_{jj+1}][f_{j+1}] = [f_j][p_{f(j)f(j+1)}]$, i.e.,

$$Q_j F_{j+1} = F_j P_{f(j)} \cdots P_{f(j+1)-1}.$$

This means

$$\begin{bmatrix} 1 & 0 \\ 2^j & -2^{2j} \end{bmatrix} \begin{bmatrix} \alpha_{j+1} & \gamma_{j+1} \\ \beta_{j+1} & \delta_{j+1} \end{bmatrix} = \begin{bmatrix} \alpha_j & \gamma_j \\ \beta_j & \delta_j \end{bmatrix} \begin{bmatrix} (-1)^{f(j+1)-f(j)} & 0 \\ 0 & 2^{k_j} \end{bmatrix},$$

where

$$\begin{bmatrix} (-1)^{f(j+1)-f(j)} & 0 \\ 0 & 2^{k_j} \end{bmatrix} = P_{f(j)} \cdots P_{f(j+1)-1},$$

and $k_j = (f(j+1)-1)f(j+1) - (f(j)-1)f(j)$. Without loss of generality, we may assume that, for every $j \in \mathbb{N}$, $f(j+1) - f(j)$ is an even integer. Hence, for every $j \in \mathbb{N}$,

$$\alpha_{j+1} = \alpha_j, \qquad 2^j \alpha_{j+1} - 2^{2j} \beta_{j+1} = \beta_j,$$

and thus ($\alpha_j \equiv \alpha \in \mathbb{Z}$ for all j),

$$\beta_{j+1} = \frac{2^j \alpha - \beta_j}{2^{2j}}.$$

This recursive relation yields the following estimation ($\beta_1 \equiv \beta \in \mathbb{Z}$)

$$\mid \beta_{j+1} \mid \leq \frac{(2 + 2^4 + \cdots + 2^{j^2}) \mid \alpha \mid + \mid \beta \mid}{2^{j(j+1)}}.$$

Therefore,

$$\mid \beta_{j+1} \mid < \frac{\mid \alpha \mid}{2^{j-1}} + \frac{\mid \beta \mid}{2^{j(j+1)}},$$

which implies that there exists $j_0 \in \mathbb{N}$, so that, for every $j > j_0$, $\mid \beta_j \mid < 1$. Since each $\beta_j \in \mathbb{Z}$, we infer that $\beta_j = 0$ for all $j > j_0$. Then, by the recursive relation, $\alpha = 0$ and, thus, $\alpha_j = 0$ for all $j \in \mathbb{N}$. This further implies that $\beta_{j_0} = \cdots = \beta_1 = 0$. Consequently, for every $j \in \mathbb{N}$, the homotopy class $[f_j]$ is represented by a singular integral matrix

$$F_j = \begin{bmatrix} 0 & \gamma_j \\ 0 & \delta_j \end{bmatrix}.$$

Since all the bonding homotopy classes are represented by regular matrices ($\det P_i \neq 0 \neq \det Q_j$), $\boldsymbol{f}$ cannot be an isomorphism. Thus, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are not isomorphic in *pro-HTop*. (Moreover, neither $\boldsymbol{X} \leq \boldsymbol{Y}$ nor $\boldsymbol{Y} \leq \boldsymbol{X}$ in *pro-HTop* can hold.)

Let us now prove that the inverse sequences $\boldsymbol{X}$ and $\boldsymbol{Y}$ are isomorphic in the subcategory *tow**-*HcPol* ($\subseteq$ *pro**-*HcPol* $\subseteq$ *pro**-*HTop*). Consider a level S^*-morphism $\big(1_{\mathbb{N}}, [f_j^n]\big) : \boldsymbol{X} \to \boldsymbol{Y}$ defined by induction as follows:

If $n = 1$, let $[f_1^1] : T \to T$ be any homotopy class, and let $[f_j^1] = [c] : T \to T$, $j > 1$, be the homotopy class of a constant mapping; assume that, for $n \in \mathbb{N}$ and each $k = 1, \ldots, n$, all homotopy classes $[f_j^k] : T \to T$, $j \in \mathbb{N}$, are defined; let $[f_{n+1}^{n+1}] = [1_T] : T \to T$ be the homotopy class of the identity mapping, which is represented by the identity matrix

$$F_{n+1}^{n+1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

let $[f_j^{n+1}]$ be represented by an integral matrix F_j^{n+1} defined inductively via the commutativity relation

$$F_j^{n+1} P_j = Q_j F_{j+1}^{n+1},$$

$j = n, \ldots, 1$, and let $[f_j^{n+1}] = [c]$ for all $j > n + 1$. One can verify, by a straightforward calculation, that all $F_j^{n+1} \in M_2(\mathbb{Z})$, $j = n, \ldots, 1$, exist. More precisely,

$$F_n^{n+1} = \begin{bmatrix} -1 & 0 \\ -2^n & -1 \end{bmatrix}, \quad F_{n-1}^{n+1} = \begin{bmatrix} 1 & 0 \\ 2^{n-1} - 2^{3(n-1)+1} & 1 \end{bmatrix},$$

$$F_{n-2}^{n+1} = \begin{bmatrix} -1 & 0 \\ -(2^{n-2} - 2^{3(n-2)+1} + 2^{5(n-2)+4}) & -1 \end{bmatrix}, \ldots,$$

$$F_1^{n+1} = \begin{bmatrix} (-1)^n & 0 \\ (-1)^n (2 - 2^4 + \cdots + (-1)^{n-1} 2^{n^2}) & (-1)^n \end{bmatrix}.$$

The commutativity relations assure that $(1_{\mathbb{N}}, [f_j^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ is indeed an S^*-morphism of the inverse sequences. Let us prove that $\boldsymbol{f}^* = [(1_{\mathbb{N}}, [f_j^n])] : \boldsymbol{X} \to \boldsymbol{Y}$ is an isomorphism in *tow**-*HcPol*. According to Theorem 6.1, given any $j \in \mathbb{N}$, put $j' = j + 1$ and $n = j + 1$. We have to prove that, for every $n' \geq n$, there exists a homotopy class $[h_j^{n'}] : T \to T$ such that

$$[h_j^{n'}][f_{j+1}^{n'}] = [p_{jj+1}] \ \wedge \ [f_j^{n'}][h_j^{n'}] = [q_{jj+1}].$$

This means

$$H_j^{n'} F_{j+1}^{n'} = P_j \ \wedge \ F_j^{n'} H_j^{n'} = Q_j,$$

where $H_j^{n'} \in M_2(\mathbb{Z})$ represents $[h_j^{n'}]$. By Theorem 6.4, it suffices to prove that, for every $n \in \mathbb{N}$ and every $j = 1, \ldots, n$, there exists $H_j^{n+1} \in M_2(\mathbb{Z})$ such that

$$H_j^{n+1} F_{j+1}^{n+1} = P_j \ \wedge \ F_j^{n+1} H_j^{n+1} = Q_j.$$

A straightforward calculation yields

$$H_n^{n+1} = P_n = \begin{bmatrix} -1 & 0 \\ 0 & 2^{2n} \end{bmatrix}, \quad H_{n-1}^{n+1} = \begin{bmatrix} 1 & 0 \\ 2^{3(n-1)+1} & -2^{2(n-1)} \end{bmatrix},$$

$$H_{n-2}^{n+1} = \begin{bmatrix} -1 & 0 \\ -(2^{3(n-2)+1} - 2^{5(n-2)+4}) & 2^{2(n-2)} \end{bmatrix}, \dots,$$

$$H_1^{n+1} = \begin{bmatrix} (-1)^n & 0 \\ (-1)^n(2^4 - 2^9 + \cdots + (-1)^n 2^{n^2}) & -2^2 \end{bmatrix}.$$

This completes the proof.

REMARK 7.3. Both examples from above have their roots in the example of Keesling and Mardešić [9, Section 4]. One can also compare [20, Example 5 and Claim 3], [16, Section 4] and [6].

The final example shows that, in general, by the coarse shape functor induced function

$$S^*|\cdot : \mathcal{C}(X,Q) \to Sh^*_{(\mathcal{C},\mathcal{D})}(X,Q)$$

is not surjective (even in the case $X \equiv P \in Ob\mathcal{D}$).

EXAMPLE 7.4. Let $\mathcal{C} = HTop$ and $\mathcal{D} = HPol$. Let $P = \{*\}$ be a singleton and let $Q = \{*\} \sqcup \{*\}$ (disjoint union). Then

$$card(HPol(P,Q)) \equiv card([P,Q]) = card(Sh(P,Q)) = 2,$$

while (see Claim 2 in Section 4)

$$card(Sh^*(P,Q)) = (card([P,Q]))^{\aleph_0} = 2^{\aleph_0}.$$

Consequently, the induced function

$$S^*|\cdot : [P,Q] \to Sh^*(P,Q)$$

cannot be a surjection.

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