THE STABLE SHAPE OF COMPACT SPACES WITH COUNTABLE COHOMOLOGY GROUPS

Sławomir Nowak

University of Warsaw, Poland

Dedicated to Professor Sibe Mardešić on the occasion of his 80th birthday

ABSTRACT. Let X be a compact Hausdorff space and $q \in \mathbb{Z}$ be an integer such that the integral cohomology groups $H^n(X; \mathbb{Z})$ are countable for $n < q$ and the stable cohomotopy groups $\pi_s^n(X)$ of X are countable for $n \geq q$. Then there exists a compact metrizable compact space Y with the same stable shape as X.

The stable shape category ShStab ([Mi, Mi-Se$_1$, Mi-Se$_2$, N$_1$, N$_2$]) can be obtained as a modification of the classical shape category ([Bo, Ma-Se]) when we replace the homotopy classes of maps by the stable homotopy classes of maps. Since the Čech cohomology functors (defined on the homotopy category of compact Hausdorff spaces) factorize via ShStab, it is especially useful when studying cohomological properties of a space. From this point of view it is important to know when a given compact space X can be replaced (up to stable shape) by a simpler one. In particular in [N$_2$] it has been shown that X has the stable shape of a compact metrizable space Y if and only if the stable cohomotopy groups $\pi_s^n(X)$ are countable for $n \in \mathbb{Z}$.

Here we prove that such a compact metric space Y exists iff there is an index p such that $H^n(X; \mathbb{Z})$ are countable for $n \leq p$ and $\pi_s^n(X)$ are countable for $k > p$.

The Cartesian product of uncountably many copies of the Adams-Kahn-Taylor continuum [Ma-Se, p. 284] is an acyclic and non-movable compact space which does not have the stable shape of any metric space.

In the movable case [N$_2$, Theorem B] vanishing the groups $H^n(X; \mathbb{Z})$ for $n > 0$ (or respectively for almost all n) implies that X has the stable shape

of a 0-dimensional space (or that the k-fold suspension of X has the stable shape of a finite dimensional space).

Hence it seems natural to presume that a compact movable space X has the stable shape of a metric space iff the groups $H^n(X; \mathbb{Z})$ are countable for every n, but the question whether it is true or not is still open.

1. Preliminaries

A spectrum $\mathbb{E} = \{E_n, \varepsilon_n\}$ is a sequence of spaces and maps $\varepsilon_n : \mathbf{S}(E_n) \to E_{n+1}$. It is a CW spectrum if E_n are CW complexes and ε_n is a cellular inclusion.

If $\mathbb{E} = \{E_n, \varepsilon_n\}$ and $\mathbb{F} = \{F_n, \varepsilon_n'\}$ are arbitrary spectra, then a Whitehead map (W-map) $\mathbf{f} : \mathbb{E} \to \mathbb{F}$ is a sequence of maps $f_n : E_n \to F_n$ such that the diagram

$$(1.1) \qquad \begin{array}{ccc} \mathbf{S}(E_n) & \xrightarrow{\ \varepsilon_n\ } & E_{n+1} \\ {\scriptstyle \mathbf{S}(f_n)}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \mathbf{S}(F_n) & \xrightarrow{\ \varepsilon_n'\ } & F_{n+1} \end{array}$$

commutes up to homotopy (compare [K-K-S, p. 213]).

We say that $\mathbf{f}$ is a function of spectra if the diagram (1.1) commutes strictly.

As for prerequisites, the reader is expected to be familiar with the basic concepts of the stable homotopy theory ([A, Ma, Sw]). In particular, a knowledge of the notions of maps, morphisms, cells and r-dimensional skeletons of CW spectra is assumed and we use also the homology and homotopy groups of CW spectra (see [Ma] or [Sw, Chapters 8-14]).

For arbitrary spectrum $\mathbb{E} = \{E_n, \varepsilon_n\}$ we may define the n-dimensional homotopy group $\pi_n(\mathbb{E})$ as the direct limit of the direct sequence $\{\ldots \to \pi_{n+k}(E_k) \to \pi_{n+k+1}(E_{k+1}) \to \ldots\}$, where the middle arrow is the composition of the homorphism $\pi_{n+k}(E_{n+k}) \to \pi_{n+k+1}(\mathbf{S}(E_{n+k}))$ induced by the suspension and the homorphism $\pi_{n+k+1}(\varepsilon_{n+k}) : \pi_{n+k+1}(\mathbf{S}(E_{n+k})) \to \pi_{n+k+1}(E_{n+k+1})$.

REMARK 1.1. If $\mathbb{E}$ is a CW spectrum and $\mathbf{f} : \mathbb{E} \to \mathbb{F}$ is a W-map, then there exists ([Sw, Lemma 10.4] and [K-K-S, p. 213]) a function of spectra $\mathbf{g} : \mathbb{E} \to \mathbb{F}$ such that $\mathbf{f}$ and $\mathbf{g}$ are homotopic as W-maps.

Each W-map $\mathbf{f} : \mathbb{E} \to \mathbb{F}$ induces homorphism $\pi_n(\mathbf{f}) : \pi_n(\mathbb{E}) \to \pi_n(\mathbb{F})$ of the homotopy groups ([Sw, Ma]) of spectra. If $\mathbf{f}$ and $\mathbf{g}$ are homotopic, then $\pi_n(\mathbf{f}) = \pi_n(\mathbf{g})$.

A pair $(\mathbb{E}, \boldsymbol{\alpha})$, where $\boldsymbol{\alpha} : \mathbb{E}_0 \to \mathbb{F}$ is a function defined on a cofinal subspectrum $\mathbb{E}_0$ of a CW spectrum $\mathbb{E}$, is called a CW substitute of a spectrum $\mathbb{F}$ iff $\pi_n(\boldsymbol{\alpha})$ is an isomorphism for every index n. When no confusion can arise

we ignore the second element of the pair $(\mathbb{E}, \boldsymbol{\alpha})$ and we say that $\mathbb{E}$ is a CW substitute of $\mathbb{F}$.

REMARK 1.2. For every spectrum there exists a CW substitute [K-K-S, Proposition 2.3].

Suppose that X is a compact Hausdorff space. By $\mathbb{F}(X) = \{(\mathrm{Map}(X, S^n), \lambda_n)\}$ we denote the function spectrum of X, where $\lambda_n : \mathbf{S}(\mathrm{Map}(X, S^n)) \to \mathrm{Map}(X, S^{n+1})$ is defined by the formula $\lambda_n(f \wedge t)(x) = f(x) \wedge t$ for every $f \wedge t \in \mathrm{Map}(X, S^n) \wedge S^1 = \mathbf{S}(\mathrm{Map}(X, S^n))$.

PROPOSITION 1.3 ([N_2, Corollary 6.1.2]). *For every compact Hausdorff space X and every CW substitute $\mathbb{W}_X$ of $\mathbb{F}(X)$ we have $\pi_s^{-n}(X) \cong \pi_n(\mathbb{W}_X)$ and $H^{-n}(X; \mathbb{Z}) \cong H_n(\mathbb{W}_X; \mathbb{Z})$, where $n \in \mathbb{Z}$.*

2. SPECTRA WITH COUNTABLE SKELETONS

We adhere to the convention that the r-dimensional skeleton (r-skeleton) $E^{(r)}$ of a CW complex E (or r-skeleton $\mathbb{E}^{(r)}$ of a CW spectrum $\mathbb{E}$) is also defined for the case when $r = \infty$ and $E^{(\infty)} = E$ (or $\mathbb{E}^{(\infty)} = \mathbb{E}$).

LEMMA 2.1. *Let $q \in \mathbb{Z}$ or $q = \infty$ and $\mathbf{f} = \{f_n\} : \mathbb{E} = \{E_n, \varepsilon_n\} \to \mathbb{F}$ be a W-map such that the following conditions are satisfied:*

(1) *$\mathbb{F}$ is a CW spectrum,*
(2) *$f_n : E_n \to F_n$ is a homotopy equivalence,*
(3) *E_n is a CW complex such that the r-skeleton $E_n^{(r)}$ of E_n is a countable CW complex, where $r = \max(n + q, 0)$,*
(4) *$\varepsilon_n : \mathbf{S}(E_n) \to E_{n+1}$ is a cellular map.*

Then there exists a CW spectrum $\mathbb{G} = \{G_n\}$ and a function of spectra $\mathbf{h} = \{h_n\} : \mathbb{G} \to \mathbb{F}$, which induces an isomorphism of spectra. The q-skeleton of $\mathbb{G}$ is a countable spectrum.

PROOF. It is known [Sw, Proposition 8.3] that for every spectrum $\mathbb{E}$ satisfying the condition (4) there exists a function of spectra $\mathbf{g} = \{g_n\} : \mathbb{G} \to \mathbb{F}$, where $g_n : G_n \to E_n$ is a homotopy equivalence. The nth term G_n of G is obtained as the telescope of the sequence $\mathbf{S}^n(E_0) \xrightarrow{\mathbf{S}^n(\varepsilon_0)} \mathbf{S}^{n-1}(E_1) \xrightarrow{\mathbf{S}^{n-1}(\varepsilon_0)} \mathbf{S}^{n-2}(E_2) \to \ldots \mathbf{S}(E_{n-1}) \xrightarrow{\varepsilon_n} E_n$, i.e., the union of the reduced mapping cylinders M_i of $\mathbf{S}^{n-i}(\varepsilon_i)$, where the top of M_i is identified with the bottom of M_{i+1} and $i \geq 1$. The construction guarantees that the $(q+n)$-skeleton of G_n and the q-skeleton of $\mathbb{G}$ are countable. $\square$

LEMMA 2.2. *Let $q \in \mathbb{Z}$ or $q = \infty$ and suppose that $\mathbb{E}$ is a CW spectrum such that $\pi_k(\mathbb{E})$ is a countable group for $k < q + 1$. Then there exist a CW spectrum $\mathbb{G}$ and an isomorphism of spectra $\mathbf{f} : \mathbb{E} \to \mathbb{G}$ such that the q-skeleton $\mathbb{G}^{(q)}$ of $\mathbb{G}$ is a countable spectrum.*

PROOF. We may assume [A, p. 150] that $\mathbb{E}$ is an Ω_0 spectrum. Hence $\pi_n(E_r) = \pi_{n-r}(\mathbb{E})$ for $n \geq 1$. By [W, Theorem C] there exists a CW complex F_n and a cellular homotopy equivalence $f_n : F_n \to E_n$ such that F_n has a countable $(n+q)$-skeleton.

Let $\varepsilon_n = g_{n+1}^{-1}\mathbf{S}(f_n) : \mathbf{S}(F_n) \to F_{n+1}$, where $g_{n+1}^{-1} : E_{n+1} \to F_{n+1}$ is a homotopy inverse of f_{n+1}. Then (F_n, ε_n) is a spectrum and $\{f_n\} : \mathbb{E} \to (F_n, \varepsilon_n)$ is a W-map. Lemma 2.2 is a consequence of Lemma 2.1. $\square$

LEMMA 2.3. *For every CW spectrum $\mathbb{W}$ with countable q-skeleton there exists a CW spectrum $\mathbb{V}$ and a morphism of spectra $\mathbf{f} : \mathbb{W} \to \mathbb{V}$ such that*

(1) $\pi_n(\mathbf{f}) : \pi_n(\mathbb{W}) \to \pi_n(\mathbb{V})$ *is an isomorphism for $n \leq q$,*
(2) $\pi_n(\mathbb{V}) = 0$ *for $n > q$,*
(3) $\mathbb{V}$ *is countable.*

PROOF. For every CW spectrum $\mathbb{W}$ one can find ([Ma, p. 42, the proof of Proposition 8]) a CW spectrum $\mathbb{W}[-\infty, q]$ and a morphism of spectra $\mathbf{g} : \mathbb{W} \to \mathbb{W}[-\infty, q]$ such that $\pi_n(\mathbf{g}) : \pi_n(\mathbb{W}) \to \pi_n(\mathbb{W}[-\infty, q])$ is an isomorphism for $n \leq q$ and $\pi_n(\mathbb{W}[-\infty, q]) = 0$ for $n > q$.

This means that the homotopy group $\pi_n(\mathbb{W}[-\infty, q])$ of $\mathbb{W}[-\infty, q]$ is countable for every n and by Lemma 2.2 there is an isomorphism $\mathbf{h} : \mathbb{W}[-\infty, q] \to \mathbb{V}$, where $\mathbb{V}$ is a countable CW spectrum. The morphism $\mathbf{f} = \mathbf{hg} : \mathbb{W} \to \mathbb{V}$ satisfies the required conditions. $\square$

3. THE MAIN THEOREM

THEOREM 3.1. *Suppose that X is a compact Hausdorff space and p is an integer such that the following conditions are satisfied:*

(1) $\pi_s^n(X)$ *is countable for every $n \geq p$,*
(2) $H^n(X; \mathbb{Z})$ *is countable for every $n < p$.*

Then there exists a compactum Y with the same stable shape as X.

PROOF. We know that

(3.1) $\pi_s^n(X)$ is a countable Abelian group

for $n \geq p = -q$ and $H^n(X; \mathbb{Z})$ is a countable Abelian group for $n < p = -q$.

It is known (see [N$_2$]) that if the condition (3.1) is satisfied for every n then there is a metric compact space Y with $\mathrm{ShStab}(X) = \mathrm{ShStab}(Y)$.

In order to prove theorem it suffices to show that the condition (3.1) is satisfied for $n \geq p - 1$. Indeed, for each $n < p$ we can repeat the proceeding argument how many times as it is necessary and we can prove that $\pi_s^n(X)$ is countable, which gives that (3.1) is satisfied for every n.

Let $\boldsymbol{\alpha} : \mathbb{W} \to \mathbb{F}(X)$ be a CW substitute of $\mathbb{F}(X)$.

Then

(3.2) $H_n(\mathbb{W}; \mathbb{Z})$ is countable for $n > q$ and $\pi_n(\mathbb{W})$ is countable for $n \leq q$.

By Lemma 2.1 we may assume that the q-skeleton of $\mathbb{W}$ is countable and that $H_n(\mathbb{W}; \mathbb{Z})$ is a countable Abelian group for $n \leq q$. From Lemma 2.3 we conclude that there is a countable CW spectrum $\mathbb{V}$ and a morphism of spectra $\mathbf{f} : \mathbb{W} \to \mathbb{V}$ such that $\pi_n(\mathbf{f}) : \pi_n(\mathbb{W}) \to \pi_n(\mathbb{V})$ is an isomorphism for $n \leq q$ and $\pi_n(\mathbb{V}) = 0$ for $n > q$. Hence

(3.3) $\qquad \pi_n(\mathbb{V})$ and $H_n(\mathbb{V}; \mathbb{Z})$ are countable for every n.

We have the following long exact sequences

(3.4) $\quad \ldots \pi_{n+1}(\mathbb{V}) \to \pi_{n+1}(\mathbb{E}) \to \pi_n(\mathbb{W}) \to \pi_n(\mathbb{V}) \to \pi_n(\mathbb{E}) \to \pi_{n-1}(\mathbb{W}) \to \ldots$

and

(3.5) $\quad \ldots H_{n+1}(\mathbb{V}; \mathbb{Z}) \to H_{n+1}(\mathbb{E}; \mathbb{Z}) \to H_n(\mathbb{W}; \mathbb{Z})$
$$\to H_n(\mathbb{V}; \mathbb{Z}) \to H_n(\mathbb{E}; \mathbb{Z}) \to H_{n-1}(\mathbb{W}; \mathbb{Z}) \to \ldots,$$

where $\mathbb{E}$ denotes the mapping cone of $\mathbf{f} : \mathbb{W} \to \mathbb{V}$.

The exactness of (3.4) implies that $\pi_{n+1}(\mathbb{E})$ and $\pi_n(\mathbb{W})$ are isomorphic for $n > q$ and $\pi_n(\mathbb{E}) = 0$ for $n \leq q + 1$. Therefore the spectrum $\mathbb{E}$ is bounded below. By the Hurewicz Theorem for spectra we conclude that $\pi_{q+2}(\mathbb{E})$ and $H_{q+2}(\mathbb{E}; \mathbb{Z})$ are isomorphic. From (3.2) and (3.3) it follows that the groups $H_{q+1}(\mathbb{W}; \mathbb{Z})$ and $H_{q+2}(\mathbb{V}; \mathbb{Z})$ are countable.

If we have an exact sequence $G_1 \to G \to G_2$ of Abelian groups, then the countability of G_1 and G_2 implies the countability of G. Therefore the exactness (3.5) implies that $H_{q+2}(\mathbb{E}; \mathbb{Z}) \cong \pi_{q+2}(\mathbb{E}) \cong \pi_{q+1}(\mathbb{W}) \cong \pi_s^{-q-1}(X) = \pi_s^{p-1}(X)$ are countable Abelian groups and the condition (3.1) is satisfied for $n \geq p - 1$. $\qquad \square$

CONJECTURE 3.2. *A movable compact Hausdorff space X has the stable shape of a compact metric space iff the cohomology groups $H^n(X; \mathbb{Z})$ are countable for $n \geq 0$.*

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S. Nowak
Institute of Mathematics
University of Warsaw
ul. Banacha 2, 02-097 Warszawa
Poland
E-mail: snowak@mimuw.edu.pl