GLASNIK MATEMATIČKI
Vol. 42(62)(2007), 195 – 211

THE S_n-EQUIVALENCE OF COMPACTA

Nikica Uglešić and Branko Červar
University of Zadar and University of Split, Croatia

Dedicated to Professor Sibe Mardešić on the occasion of his 80th birthday

Abstract. By reducing the Mardešić S-equivalence to a finite case, i.e., to each $n \in \{0\} \cup \mathbb{N}$ separately, we have derived the notions of S_n-equivalence and S_{n+1}-domination of compacta. The S_n-equivalence for all n coincides with the S-equivalence. Further, the S_{n+1}-equivalence implies S_{n+1}-domination, and the S_{n+1}-domination implies S_n-equivalence. The S_0-equivalence is a trivial equivalence relation, i.e., all non empty compacta are mutually S_0-equivalent. It is proved that the S_1-equivalence is strictly finer than the S_0-equivalence, and that the S_2-equivalence is strictly finer than the S_1-equivalence. Thus, the S-equivalence is strictly finer than the S_1-equivalence. Further, the S_1-equivalence classifies compacta which are homotopy (shape) equivalent to ANR's up to the homotopy (shape) types. The S_2-equivalence class of an FANR coincides with its S-equivalence class as well as with its shape type class. Finally, it is conjectured that, for every n, there exists $n' > n$ such that the $S_{n'}$-equivalence is strictly finer than the S_n-equivalence.

1. Introduction

In the year 1968 the shape theory of (metrizable) compacta was founded by K. Borsuk. The corresponding classification of compacta is strictly coarser than the homotopy type classification, while on the subclass of locally nice spaces (compact ANR's) it coincides with the homotopy type classification. Since 1976 a few new classifications of compacta have been considered. For instance, K. Borsuk [1] introduced the relations of quasi-affinity and quasi-equivalence, while S. Mardešić [6] introduced the S-equivalence relation between compacta. All of them are the shape type invariant relations. These

2000 *Mathematics Subject Classification.* 55P55.
Key words and phrases. Compactum, ANR, shape, S-equivalence.

classifications are strictly coarser than the shape type classification ([1, 3, 5]). Moreover, the quasi-equivalence and S-equivalence classifications coincide with the homotopy type classification on compact ANR's.

The S-equivalence is an equivalence relation on the class of all compacta, which is defined by means of a certain condition depending on every $n \in \mathbb{N}$. Mardešić and the first named author noticed in [8] that it makes sense to consider "the finite parts" of this condition. By following this idea, we have reduced the mentioned condition to the finite cases, i.e., to every $n \in \{0\} \cup \mathbb{N}$ separately. In that way we have derived the notions of S_n-equivalence and S_{n+1}-domination of compacta (Definition 2.3). The S_n-equivalence for all $n \in \{0\} \cup \mathbb{N}$ coincides with the S-equivalence. Further, the S_{n+1}-equivalence implies S_{n+1}-domination, and the S_{n+1}-domination implies S_n-equivalence (Lemma 2.4). The S_0-equivalence is a trivial equivalence relation, i.e., all nonempty compacta are mutually S_0-equivalent. The S_1-equivalence is not trivial (Theorem 2.6), and it is strictly coarser than the S_2-equivalence (Theorem 2.8). Thus, it is strictly coarser than the S-equivalence. The S_1-equivalence restricted to compacta having the homotopy types of ANR's coincides with the homotopy type classification (Theorem 2.10). Similarly, the S_1-equivalence restricted to the class of all FANR's (compacta having the shapes of ANR's, Lemma 2.13) coincides with the shape type classification (Theorem 2.12). A pair of quasi-equivalent compacta [1] is constructed such that they are not S_1-equivalent (Theorem 2.16).

It is noticed that the following properties: connectedness, trivial shape, shape dimension $\leq n$, n-shape connectedness, are invariants of the S_1-domination (Theorem 3.3). Further, the movability and n-movability are invariants of the S_1-equivalence (Theorem 3.4), while the strong movability (being an FANR) is an invariant of the S_2-domination (Theorem 3.5). Moreover, the S_2-equivalence class of an FANR coincides with its S-equivalence class as well as with its shape type class (Corollary 3.6).

At the end, we propose the following two hypotheses ($S(X)$ and $S_n(X)$ denote the S-equivalence class and S_n-equivalence class of X respectively!):

(1) For every $n \in \{0\} \cup \mathbb{N}$, there exists a compactum X such that $S(X) \subsetneqq S_n(X)$;
(2) There exists a compactum X such that, for every $n \in \{0\} \cup \mathbb{N}$, $S(X) \subsetneqq S_n(X)$.

Clearly, if (2) is true then so is (1). The "argument" (Theorems 4.3 and 4.5) supporting both hypotheses is the strong presentiment that the S^*-equivalence [8] should strictly imply the S-equivalence.

2. From the S- to S_n-equivalence

Let $c\mathcal{M}$ denote the class of all compact metrizable spaces (compacta), and let $c\underline{\mathcal{M}}$ denote the class of all inverse sequences over $c\mathcal{M}$. By [6, Definition 1], two inverse sequences $\boldsymbol{X}, \boldsymbol{Y} \in c\underline{\mathcal{M}}$ are said to be S-*equivalent*, denoted by $S(\boldsymbol{Y}) = S(\boldsymbol{X})$, provided, for every $n \in \mathbb{N}$, the following condition is fulfilled:

$$(\forall j_1)(\exists i_1)(\forall i_1' \geq i_1)(\exists j_1' \geq j_1)(\forall j_2 \geq j_1')(\exists i_2 \geq i_1') \cdots$$
$$\cdots (\forall i_{n-1}' \geq i_{n-1})(\exists j_{n-1}' \geq j_{n-1})(\forall j_n \geq j_{n-1}')(\exists i_n \geq i_{n-1}')$$

and there exist mappings $f_k \equiv f_{j_k}^n : X_{i_k} \to Y_{j_k}$, $k = 1, \ldots, n$, and $g_k \equiv g_{i_k'}^n : Y_{j_k'} \to X_{i_k'}$, $k = 1, \ldots, n-1$, making the following diagram

$$(D) \qquad \begin{array}{ccccccccc} X_{i_1} & \leftarrow & X_{i_1'} & \leftarrow & \cdots & \leftarrow & X_{i_{n-1}'} & \leftarrow & X_{i_n} \\ \downarrow f_1 & & \uparrow g_1 & & \cdots & & \uparrow g_{n-1} & & \downarrow f_n \\ Y_{j_1} & \leftarrow & Y_{j_1'} & \leftarrow & \cdots & \leftarrow & Y_{j_{n-1}'} & \leftarrow & Y_{j_n} \end{array}$$

commutative up to homotopy. Two compacta X and Y are said to be S-*equivalent*, denoted by $S(Y) = S(X)$, provided there exists a pair (equivalently, for every pair) of limits $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ of inverse sequences consisting of compact ANR's such that $S(\boldsymbol{Y}) = S(\boldsymbol{X})$ (see [6, Remarks 1 and 2, and Definition 2]). If $\boldsymbol{p} : X \to \boldsymbol{X}$ is the limit, then we also say that $\boldsymbol{X}$ is *associated* with X.

If compacta X and Y have the same shape (type, [7]), $Sh(Y) = Sh(X)$, then $S(Y) = S(X)$. There exist compacta X and Y such that $S(Y) = S(X)$ and $Sh(Y) \neq Sh(X)$ (see [5, Corollary 2], and [3]).

If the choice of indices i_k and j_k' does not depend on a given $n \in \mathbb{N}$ (while the mappings still depend on n, i.e., $f_k \equiv f_{j_k}^n : X_{i_k} \to Y_{j_k}$ and $g_k \equiv g_{i_k'}^n : Y_{j_k'} \to X_{i_k'}$), then the S-equivalence becomes the S^*-*equivalence* (see [8, Definitions 6-9] and [11, Lemmas 4 and 5]). There exists a pair X, Y of compacta such that $S^*(Y) = S^*(X)$ and $Sh(Y) \neq Sh(X)$ (see [8]). However, we have no example yet which could show that the S^*-equivalence is indeed strictly finer than the S-equivalence.

Given an $n \in \mathbb{N}$, let us denote the above condition, relating $\boldsymbol{Y}$ to $\boldsymbol{X}$ by (D_{2n-1}). Further, let us denote by (D_{2n}) the following extension of (D_{2n-1}):

$$(\forall j_1)(\exists i_1)(\forall i_1' \geq i_1)(\exists j_1' \geq j_1) \cdots$$
$$\cdots (\forall j_n \geq j_{n-1}')(\exists i_n \geq i_{n-1}')(\forall i_n' \geq i_n)(\exists j_n' \geq j_n)$$

and there exist mappings

$$f_k \equiv f_{j_k}^n : X_{i_k} \to Y_{j_k},\ g_k \equiv g_{i_k'}^n : Y_{j_k'} \to X_{i_k'},\ k = 1, \ldots, n,$$

making diagram (D), extended by adding one rectangle, commutative up to homotopy.

It is obvious that (relating $\boldsymbol{Y}$ to $\boldsymbol{X}$), for every $m \in \mathbb{N}$,

$$(D_{m+1}) \Rightarrow (D_m).$$

DEFINITION 2.1. *Given any $\boldsymbol{X}, \boldsymbol{Y} \in c\underline{\mathcal{M}}$ and $n \in \{0\} \cup \mathbb{N}$, let $S_n(\boldsymbol{X}, \boldsymbol{Y})$ denote condition (D_{2n+1}) relating $\boldsymbol{Y}$ to $\boldsymbol{X}$. Further, let $S_n^+(\boldsymbol{X}, \boldsymbol{Y})$ denote condition (D_{2n+2}) relating $\boldsymbol{Y}$ to $\boldsymbol{X}$.*

By the above definition the next lemma is obviously true.

Lemma 2.2. *Let $\boldsymbol{X}, \boldsymbol{Y} \in c\underline{\mathcal{M}}$. Then, for every $n \in \mathbb{N} \cup \{0\}$, the implications*

$$S_{n+1}(\boldsymbol{X},\boldsymbol{Y}) \Rightarrow (S_n^+(\boldsymbol{X},\boldsymbol{Y}) \wedge S_n^+(\boldsymbol{Y},\boldsymbol{X})) \text{ and}$$
$$S_n^+(\boldsymbol{X},\boldsymbol{Y}) \Rightarrow (S_n(\boldsymbol{X},\boldsymbol{Y}) \wedge S_n(\boldsymbol{Y},\boldsymbol{X}))$$

hold. Furthermore, the following assertions are equivalent:

(i) $(\forall n \in \{0\} \cup \mathbb{N})\ S_n(\boldsymbol{X},\boldsymbol{Y})$;
(ii) $(\forall n \in \{0\} \cup \mathbb{N})\ S_n(\boldsymbol{Y},\boldsymbol{X})$;
(iii) $(\forall n \in \{0\} \cup \mathbb{N})\ S_n^+(\boldsymbol{X},\boldsymbol{Y})$;
(iv) $(\forall n \in \{0\} \cup \mathbb{N})\ S_n^+(\boldsymbol{Y},\boldsymbol{X})$;
(v) $S(\boldsymbol{Y}) = S(\boldsymbol{X})$.

According to [8, Remark 1], it makes sense to consider conditions (D_m) for a given $m \in \mathbb{N}$ (i.e., conditions $S_n(\boldsymbol{X},\boldsymbol{Y})$ and $S_n^+(\boldsymbol{X},\boldsymbol{Y})$ for a given $n \in \{0\} \cup \mathbb{N}$) separately. To be more precise, we need the following definition:

Definition 2.3. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences of compacta and let $n \in \{0\} \cup \mathbb{N}$. Then $\boldsymbol{Y}$ is said to be S_n-**equivalent to $\boldsymbol{X}$**, denoted by $S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X})$, provided the both conditions $S_n(\boldsymbol{X},\boldsymbol{Y})$ and $S_n(\boldsymbol{Y},\boldsymbol{X})$ are fulfilled. Further, $\boldsymbol{Y}$ is said to be S_{n+1}-**dominated** by $\boldsymbol{X}$, denoted by $S_{n+1}(\boldsymbol{Y}) \leq S_{n+1}(\boldsymbol{X})$, provided condition $S_n^+(\boldsymbol{X},\boldsymbol{Y})$ holds. If X and Y are compacta, then we define $S_n(Y) = S_n(X)$ ($S_{n+1}(Y) \leq S_{n+1}(X)$) provided $S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X})$ ($S_{n+1}(\boldsymbol{Y}) \leq S_{n+1}(\boldsymbol{X})$) for some, equivalently: any, compact ANR inverse sequences $\boldsymbol{X}$, $\boldsymbol{Y}$ associated with X, Y respectively.*

One can easily verify that the part of Definition 2.3 concerning compacta is correct (compare [6, Remark 2 and Definition 2]). Consequently, conditions (D_m), $m \in \mathbb{N}$, as well as $S_n(X,Y)$ and $S_n^+(X,Y)$, $n \in \{0\} \cup \mathbb{N}$, are well defined for ordered pairs of compacta too.

Lemma 2.4. *For each $n \in \{0\} \cup \mathbb{N}$ the following assertions hold:*

(i) *The S_n-equivalence is an equivalence relation on $c\underline{\mathcal{M}}$.*
(ii) *The S_{n+1}-domination is a reflexive and transitive relation on $c\underline{\mathcal{M}}$.*
(iii) $(S_{n+1}(\boldsymbol{Y}) = S_{n+1}(\boldsymbol{X})) \Rightarrow (S_{n+1}(\boldsymbol{Y}) \leq S_{n+1}(\boldsymbol{X}) \wedge S_{n+1}(\boldsymbol{X}) \leq S_{n+1}(\boldsymbol{Y}))$.
(iv) $(S_{n+1}(\boldsymbol{Y}) = S_{n+1}(\boldsymbol{X})) \Rightarrow (S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X}))$.
(v) $(S_{n+1}(\boldsymbol{Y}) \leq S_{n+1}(\boldsymbol{X})) \Rightarrow (S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X}))$.

Further, $S(\boldsymbol{Y}) = S(\boldsymbol{X})$ if and only if, for every $n \in \{0\} \cup \mathbb{N}$, $S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X})$.

Analogous statements hold for compacta.

Proof. The S_n-equivalence is reflexive since condition $S_n(\boldsymbol{X},\boldsymbol{X})$ is obviously fulfilled for every $\boldsymbol{X}$. By fitting together two appropriate diagrams, we infer that $S_n(\boldsymbol{Y},\boldsymbol{Z})$ and $S_n(\boldsymbol{X},\boldsymbol{Y})$ imply $S_n(\boldsymbol{X},\boldsymbol{Z})$. Therefore, the S_n-equivalence is transitive. Finally, it is symmetric by definition. Further, the

S_{n+1}-domination is obviously reflexive. Since $S_n^+(\boldsymbol{X}, \boldsymbol{Y})$ and $S_n^+(\boldsymbol{Y}, \boldsymbol{Z})$ imply $S_n^+(\boldsymbol{X}, \boldsymbol{Z})$, it follows that the S_{n+1}-domination is a transitive relation. The rest of the proof is straightforward by applying the definitions and Lemma 2.2. $\square$

REMARK 2.5. For every $n \in \{0\} \cup \mathbb{N}$ and every compactum X, the class $S_n(X)$ consists of all compacta X' such that $S_n(X, X')$ and $S_n(X', X)$ hold, i.e., $S_n(X') = S_n(X)$. Thus, the notation $S_{n+1}(Y) \leq S_{n+1}(X)$ for the domination might sometimes cause ambiguity. However, by transitivity, $S_{n+1}(Y) \leq S_{n+1}(X)$, $S_{n+1}(Y') \leq S_{n+1}(Y)$ and $S_{n+1}(X) \leq S_{n+1}(X')$ imply $S_{n+1}(Y') \leq S_{n+1}(X')$. Hence, we believe the notation of the S_{n+1}-domination is sufficiently clear.

One should observe that condition $S_1(\emptyset, Y)$ holds for every Y. On the other hand, condition $S_1(Y, \emptyset)$ holds if and only if $Y = \emptyset$. Thus, $S_0(Y) \neq S_0(\emptyset)$ for every $Y \neq \emptyset$. However, for the class of all non empty inverse sequences of compacta (non empty compacta), the S_0-equivalence is the trivial relation. Namely, it is obvious that, for every pair of non empty $\boldsymbol{X}, \boldsymbol{Y} \in c\underline{\mathcal{M}}$ (non empty compacta X, Y), $S_0(\boldsymbol{Y}) = S_0(\boldsymbol{X})$ ($S_0(Y) = S_0(X)$). Therefore, for every non empty compactum X, the class $S_0(X) = c\mathcal{M} \setminus \{\emptyset\}$. Nevertheless, the next theorem, i.e., the example, shows that, for every $n > 0$, the S_n-equivalence is not trivial even for compacta which are closely related homotopically.

THEOREM 2.6. *There exists a pair X, Y of compacta such that Y is homotopy dominated by X, $Y \leq X$, and X is homotopy dominated by Y, $X \leq Y$ (and thus, $S_1(Y) \leq S_1(X)$ and $S_1(X) \leq S_1(Y)$), but $S_n(Y) \neq S_n(X)$ for every $n \in \mathbb{N}$.*

PROOF. One can easily prove that $Y \leq X$ implies $S_0^+(X, Y)$, and thus, $S_1(Y) \leq S_1(X)$ (see the proof of a stronger assertion of Theorem 2.15 below). Therefore, it suffices to construct a pair X, Y of compacta such that $Y \leq X$, $X \leq Y$ and $S_1(Y) \neq S_1(X)$. Hence, the next example gives a proof. $\square$

EXAMPLE 2.7. Let $X = L \times \mathbb{S}^1$, where $L \subseteq \mathbb{R}$ is the image of an injective convergent sequence together with its limit point, while $\mathbb{S}^1$ is the standard 1-sphere. Let $Y = \{*\} \sqcup X$ (disjoint union). Notice that X is a retract of Y, and thus, $X \leq Y$. On the other side, Y is homeomorphic to a retract of X, and thus, $Y \leq X$. However, X and Y are *not* S_1-equivalent.

To prove that $S_1(Y) \neq S_1(X)$, it suffices to show that $S_1(Y, X)$ does *not* hold. Let $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$, where

$$X_i = \bigsqcup_{k=1}^{i} S_k,\ S_k = \mathbb{S}^1,\ i \in \mathbb{N},$$

$$p_{i,i+1} : X_{i+1} \to X_i,$$

$$p_{i,i+1}|S_k = \begin{cases} id : S_k \to S_k, & k \neq i+1 \\ id : S_{i+1} \to S_1, & k = i+1 \end{cases}.$$

Let $\boldsymbol{Y} = (Y_j, q_{jj'}, \mathbb{N})$, where $Y_j = X_j \sqcup \{*\}$ and $q_{j,j+1}$ is the extension of $p_{j,j+1}$ such that $q_{j,j+1}(*) = *$, $j \in \mathbb{N}$. Clearly, $\lim \boldsymbol{X} = X$ and $\lim \boldsymbol{Y} = Y$. It remains to prove that condition $S_1(\boldsymbol{Y}, \boldsymbol{X})$ does *not* hold. Suppose, on the contrary, that

$$(\forall i_1)(\exists j_1)(\forall j_1' \geq j_1)(\exists i_1' \geq i_1)(\forall i_2 \geq i_1')(\exists j_2 \geq j_1')$$

and there exist mappings $g_k' : Y_{j_k} \to X_{i_k}$, $k = 1, 2$, and $f_1' : X_{i_1'} \to Y_{j_1'}$ making the corresponding diagram commutative up to homotopy. Since, for every pair $j \leq j'$, $q_{jj'}(*) = *$ is a (path) component, there exists an $S_k \subseteq X_{i_1'}$ such that $f_1'[S_k] = \{*\}$. Therefore, the restriction $(g_1' q_{j_1 j_1'} f_1')|S_k$ of the composition $g_1' q_{j_1 j_1'} f_1' : X_{i_1'} \to X_{i_1}$ is null homotopic. However, the restriction $p_{i_1 i_1'}|S_k$ of the bonding mapping $p_{i_1 i_1'} : X_{i_1'} \to X_{i_1}$ is the identity mapping on the 1-sphere. This is a contradiction.

The main goal of this work is to prove that the S_1-equivalence and S_2-equivalence are indeed the two different equivalence relations, i.e., that the S_2-equivalence is strictly finer than the S_1-equivalence. Consequently, the S_1-equivalence differs from the S- and S^*-equivalence (see [8, Remark 1]).

THEOREM 2.8. *There exists a pair X, Y of compacta such that $S_1(Y) = S_1(X)$ and $S_2(Y) \nleq S_2(X)$.*

PROOF. The proof follows by Lemma 2.4(v), (iii), and Example 2.9 below (see also [9, Example 4 and Claim 2]). ∎

EXAMPLE 2.9. Let X be the image of an injective convergent sequence in $\mathbb{R}$ together with its limit point. For instance, $X = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$. Let $Y = X \sqcup X$ (disjoint union). Then $S_2(X) \leq S_2(Y)$, while $S_2(Y) \nleq S_2(X)$.

To prove this, let us consider the associated compact ANR-sequences $\boldsymbol{X} = (X_i, p_{ii'})$ and $\boldsymbol{Y} = (Y_j = X_j \sqcup X_j, q_{jj'})$ consisting of finite ANR's, having cardinalities $|X_i| = i$ and $|Y_j| = 2j$, and surjective bonding mappings defined in the obvious way. (The "exploding" point $*$ of X_i yields the point $*$ and exactly one new point of X_{i+1}, while all the other fibres of $p_{i,i+1}$ are singletons. The mapping $q_{j,j+1}$ consists of two copies of $p_{j,j+1}$.) In this case, every homotopy commutative diagram relating $\boldsymbol{X}$ and $\boldsymbol{Y}$ is commutative. We have to show that $S_2(\boldsymbol{X}) \leq S_2(\boldsymbol{Y})$, i.e., that condition $S_1^+(\boldsymbol{Y}, \boldsymbol{X})$ holds. Given an $i_1 \in \mathbb{N}$, put $j_1 = i_1$, and denote

$$X_{i_1} = \{*\} \sqcup A_1,$$

$$Y_{j_1} = \{*\} \sqcup A_1 \sqcup \{*'\} \sqcup A_1'.$$

Hereby, $*'$ and A_1' are the copies of $*$ and A_1 respectively. The same notations of the corresponding subsets should not cause ambiguity. Given a $j_1' \geq j_1$, put $i_1' = 2j_1'$, and denote

$$Y_{j_1'} = \{*\} \sqcup A_1 \sqcup B_2 \sqcup \{*'\} \sqcup A_1' \sqcup B_2',$$

$$X_{i_1'} = \{*\} \sqcup A_1 \sqcup C_1,$$

where $C_1 = B_2 \sqcup \{*'\} \sqcup A_1' \sqcup B_2'$.

Given an $i_2 \geq i_1'$, choose the unique $j_2 \geq j_1'$ such that the numbers of the new points "over $*$" and "over $*'$" equal to the number $|X_{i_2}| - |X_{i_1'}|$ of the new points in X_{i_2}, and denote

$$X_{i_2} = \{*\} \sqcup A_1 \sqcup C_1 \sqcup A_3,$$

$$Y_{j_2} = \{*\} \sqcup A_1 \sqcup B_2 \sqcup A_3 \sqcup \{*'\} \sqcup A_1' \sqcup B_2' \sqcup A_3'.$$

Given a $j_2' \geq j_2$, choose the unique $i_2' \geq i_2$ such that the number of the new points equals to the half of the number of the new points in $Y_{j_2'}$, and denote

$$Y_{j_2'} = \{*\} \sqcup A_1 \sqcup B_2 \sqcup A_3 \sqcup B_4 \sqcup \{*'\} \sqcup A_1' \sqcup B_2' \sqcup A_3' \sqcup B_4',$$

$$X_{i_2} = \{*\} \sqcup A_1 \sqcup C_1 \sqcup A_3 \sqcup B_4.$$

Then there exist surjections $g_1' : Y_{j_1} \to X_{i_1}$, $f_1' : X_{i_1'} \to Y_{j_1'}$ (a bijection), $g_2' : Y_{j_2} \to X_{i_2}$ and an injection $f_2' : X_{i_2'} \to Y_{j_2'}$ making the corresponding diagram commutative (see the picture below).

$$\begin{array}{ccccccc}
\boxed{\begin{matrix} A_1 & A_1' \\ * & *' \end{matrix}} & \leftarrow & \boxed{\begin{matrix} A_1 & A_1' \\ B_2 & B_2' \\ * & *' \end{matrix}} & \leftarrow & \boxed{\begin{matrix} A_1 & A_1' \\ B_2 & B_2' \\ A_3 & A_3' \\ * & *' \end{matrix}} & \leftarrow & \boxed{\begin{matrix} A_1 & A_1' \\ B_2 & B_2' \\ A_3 & A_3' \\ B_4 & B_4' \\ * & *' \end{matrix}} \\
g_1' \downarrow & & f_1' \uparrow & & \downarrow g_2' & & \uparrow f_2' \\
\boxed{\begin{matrix} A_1 \\ * \end{matrix}} & \leftarrow & \boxed{\begin{matrix} A_1 \\ C_1 \\ * \end{matrix}} & \leftarrow & \boxed{\begin{matrix} A_1 \\ C_1 \\ A_3 \\ * \end{matrix}} & \leftarrow & \boxed{\begin{matrix} A_1 \\ C_1 \\ A_3 \\ B_4 \\ * \end{matrix}}
\end{array}$$

More precisely:

$g_1'[A_1] = A_1$, $g_1'[\{*\} \sqcup \{*'\} \sqcup A_1'] = \{*'\}$;
$f_1'|(\{*\} \sqcup A_1) = g_1'^{-1}|(\{*\} \sqcup A_1)$, $f_1'[B_2] = B_2$,
$f_1'(*') = *'$, $f_1'[A_1'] = A_1'$, $f_1'[B_2'] = B_2'$;
g_2' is the inverse of f_1' on the set of all "old" points, while $g_2'[A_3] = A_3$,
$g_2'[A_3'] = \{*'\}$;
similarly, f_2' is the inverse of g_2' wherever it makes sense, while $f_2'[B_4] = B_4$.

Thus, $S_1^+(\boldsymbol{Y}, \boldsymbol{X})$ is fulfilled.

It remains to show that $S_1^+(\boldsymbol{X}, \boldsymbol{Y})$ can not be fulfilled, i.e., that $S_2(\boldsymbol{Y}) \not\leq S_2(\boldsymbol{X})$. Let us analyze condition $S_1(\boldsymbol{X}, \boldsymbol{Y})$. (Notice that conditions $S_1(\boldsymbol{X}, \boldsymbol{Y})$ and $S_1(\boldsymbol{Y}, \boldsymbol{X})$ hold by $S_1^+(\boldsymbol{Y}, \boldsymbol{X})$; see also Lemma 2.4(v).) Given a $j_1 \in \mathbb{N}$, one has to choose an appropriate $i_1 \geq 2j_1$, i.e.,

$$Y_{j_1} = \{*\} \sqcup B_1 \sqcup \{*'\} \sqcup B_1',$$

$$X_{i_1} = \{*\} \sqcup C_1,$$

where $C_1 = B_1 \sqcup \{*'\} \sqcup B_1' \sqcup A_1$ (if $i_1 > 2j_1$ then $A_1 \neq \emptyset$; the same notations of the corresponding subsets should not cause ambiguity).

Given an $i_1' \geq i_1$, one has to choose an appropriate $j_1' \geq i_1'$, i.e.,

$$X_{i_1'} = \{*\} \sqcup C_1 \sqcup A_2,$$

$$Y_{j_1'} = \{*\} \sqcup B_1 \sqcup B_2 \sqcup \{*'\} \sqcup B_1' \sqcup B_2',$$

where $|B_2| = |B_2'| \geq \frac{1}{2}(|A_1| + |A_2|)$.

Given a $j_2 \geq j_1'$, one has to choose an appropriate $i_2 \geq j_2$, i.e.,

$$Y_{j_2} = \{*\} \sqcup B_1 \sqcup B_2 \sqcup B_3 \sqcup \{*'\} \sqcup B_1' \sqcup B_2' \sqcup B_3',$$

$$X_{i_2} = \{*\} \sqcup C_1 \sqcup A_2 \sqcup C_3,$$

where $C_3 = B_3 \sqcup B_3' \sqcup A_3$.

Then there exist surjections $f_1 : X_{i_1} \to Y_{j_1}$, $g_1 : Y_{j_1'} \to X_{i_1'}$ and a function $f_2 : X_{i_2} \to Y_{j_2}$ making the corresponding diagram commutative (see the picture below).

$$\begin{array}{ccccc}
\boxed{\begin{matrix} C_1 \\ * \end{matrix}} & \leftarrow & \boxed{\begin{matrix} C_1 \\ A_2 \\ * \end{matrix}} & \leftarrow & \boxed{\begin{matrix} C_1 \\ A_2 \\ C_3 \\ * \end{matrix}} \\
f_1 \downarrow & & g_1 \uparrow & & \downarrow f_2 \\
\boxed{\begin{matrix} B_1 & B_1' \\ * & *' \end{matrix}} & \leftarrow & \boxed{\begin{matrix} B_1 & B_1' \\ B_2 & B_2' \\ * & *' \end{matrix}}_1 & \leftarrow & \boxed{\begin{matrix} B_1 & B_1' \\ B_2 & B_2' \\ B_3 & B_3' \\ * & *' \end{matrix}}_1
\end{array}$$

More precisely,

$f_1(*) = *$, $f_1[B_1] = B_1$, $f_1(*') = *'$, $f_1[B_1'] = B_1'$, while $f_1|A_1$ may vary;

$g_1|(\{*\} \sqcup B_1 \sqcup \{*'\} \sqcup B_1')$ is the "inverse" of f_1, while

$g_1|(B_2 \sqcup B_2')$ may slightly vary up to the required commutativity:

$f_2|(\{*\} \sqcup C_1 \sqcup A_2)$ is the "inverse" of g_1, while $f_2[C_3] \subseteq \{*\} \sqcup B_3$.

Since all the bonding mappings are surjective, condition $S_1^+(\boldsymbol{X}, \boldsymbol{Y})$ would imply f_2 also to be surjective. However, there does not exist any i_2 which admits a suitable surjection $f_2 : X_{i_2} \to Y_{j_2}$. Namely, if one chooses an i_2 large enough such that there exists a surjection f_2 of X_{i_2} onto Y_{j_2}, then $g_1 q_{j_1' j_2} f_2 \neq p_{i_1' i_2}$. Indeed, the image by $p_{i_1' i_2}$ of the subset of $C_3 \subseteq X_{i_2}$, corresponding to $B_3 \sqcup B_3' \subseteq Y_{j_2}$, is the "exploding" point $* \in X_{i_1'}$, while the image by $g_1 q_{j_1' j_2}$ of B_3' is $\{*'\} \subseteq X_{i_1'}$. The same obstruction remains for every possible choice of the preceding indices i_1 and j_1' and every choice of suitable surjections f_1 and g_1. Therefore, since f_2 can not be surjective, a desired g_2 does not exist. Thus, $S_1^+(\boldsymbol{X}, \boldsymbol{Y})$ can *not* be fulfilled.

We shall now consider the S_n-equivalence on the subclass of all compacta having the homotopy types (shape types) of ANR′s. The next two theorems show that condition (D_3) characterizes the homotopy (shape) types within the class of all compacta which are homotopy (shape) equivalent to ANR's. Therefore, the S_1-equivalence on the considered class coincides with the homotopy (shape) type classification on it.

THEOREM 2.10. *Let X and Y be compacta having the homotopy types of ANR's. Then the following statements are equivalent:*

(i) $S_1(X,Y)$ *or* $S_1(Y,X)$ *is fulfilled;*
(ii) $S_1(Y) = S_1(X)$*;*
(iii) $S_2(Y) \leq S_2(X)$*;*
(iv) $S_2(X) \leq S_2(Y)$*;*
(v) $(\forall n \in \{0\} \cup \mathbb{N})\ S_n(Y) = S_n(X)$*;*
(vi) $S(Y) = S(X)$*;*
(vii) $S^*(Y) = S^*(X)$*;*
(viii) $Sh(Y) = Sh(X)$*;*
(ix) $Y \simeq X$.

PROOF. It suffices to prove that (i) implies (ix). First, consider the case $X \simeq P$ and $Y \simeq Q$, where P and Q compact ANR's. In this case, we may assume that $X = P$ and $Y = Q$. Let us consider the trivial inverse sequences $\boldsymbol{X}$ (each $X_i = X$ and each $p_{ii'}$ is the identity mapping) and $\boldsymbol{Y}$ (each $Y_j = Y$ and each $q_{jj'}$ is the identity mapping) associated with X and Y respectively. By $S_1(\boldsymbol{X},\boldsymbol{Y})$, there exists the following homotopy commutative diagram:

$$\begin{array}{ccccc} X & \overset{1}{\leftarrow} & X & \overset{1}{\leftarrow} & X \\ \downarrow f_1 & & \uparrow g_1 & & \downarrow f_2 \\ Y & \underset{1}{\longleftarrow} & Y & \underset{1}{\longleftarrow} & Y \end{array}$$

Thus, $f_1 g_1 \simeq 1_Y$, $g_1 f_2 \simeq 1_X$ and $f_1 \simeq f_2$, which means $Y \simeq X$. In the same way $S_1(\boldsymbol{Y},\boldsymbol{X})$ implies $Y \simeq X$. Consider now the general case, i.e., $X \simeq P$ and $Y \simeq Q$, where P and Q are ANR's. Let $\boldsymbol{X}$, $\boldsymbol{Y}$ be a pair of compact ANR inverse sequences associated with X, Y respectively. Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be the trivial inverse sequences associated with P and Q respectively. Notice that the limits $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$, $\boldsymbol{1}_P : P \to \boldsymbol{P}$ and $\boldsymbol{1}_Q : Q \to \boldsymbol{Q}$ are also the ANR-rsolutions (see [7, I.6.1]). Consequently (see [7, Theorem I.6.2]), they induce the corresponding HANR-expansions $H\boldsymbol{p} : X \to H\boldsymbol{X}$, $H\boldsymbol{q} : Y \to H\boldsymbol{Y}$, $H\boldsymbol{1}_P : P \to H\boldsymbol{P}$ and $H\boldsymbol{1}_Q : Q \to H\boldsymbol{Q}$ respectively. Since $Sh(P) = Sh(X)$ and $Sh(Y) = Sh(Q)$, the systems $H\boldsymbol{P}$ and $H\boldsymbol{X}$ as well as $H\boldsymbol{Y}$ and $H\boldsymbol{Q}$ are two pairs of isomorphic objects of the (pro)category *pro-$HANR$*. Then, by the Morita lemma ([9, Theorem 1.1]) and by $S_1(\boldsymbol{X},\boldsymbol{Y})$, we can obtain the following homotopy commutative diagram:

$$\begin{tikzcd}
P \arrow[d] & P \arrow[l, "1"'] \arrow[d] & P \arrow[l, "1"'] \arrow[d] \\
X_{i_1} \arrow[d] & X_{i'_1} \arrow[l] \arrow[ul] & X_{i_2} \arrow[l] \arrow[ul] \arrow[d] \\
Y_{j_1} \arrow[d] & Y_{j'_1} \arrow[l] \arrow[u] \arrow[d] & Y_{j_2} \arrow[l] \arrow[d] \\
Q & Q \arrow[l, "1"] \arrow[ul] & Q \arrow[l, "1"] \arrow[ul]
\end{tikzcd}$$

It yields the next homotopy commutative diagram:

$$\begin{tikzcd}
P \arrow[d] & P \arrow[l, "1"'] \arrow[d] \\
Q & Q \arrow[l, "1"] \arrow[ul]
\end{tikzcd}$$

Thus, $Q \simeq P$, and consequently, $Y \simeq Q \simeq P \simeq X$. If $S_1(Y,X)$ holds, the proof is analogous. ∎

REMARK 2.11. Of course, the class $S_1(X)$ of a compact ANR X may contain a compactum Z which does not have the homotopy type of any ANR. For instance, the compact topological sinus curve $Z = A \cup B \subseteq \mathbb{R}^2$, where $A = \{(\xi,\eta) \in \mathbb{R}^2 \mid \eta = \sin\frac{1}{\xi},\ \xi \in \langle 0,1]\}$, $B = \{(0,\eta) \in \mathbb{R}^2 \mid \eta \in [-1,1]\}$, is shape equivalent (and thus, S_1-equivalent) to a point $X = \{*\}$, while Z is not homotopy equivalent to any ANR.

Similarly to Theorem 2.10, the following holds:

THEOREM 2.12. *Let X and Y be FANR's. Then the following statements are equivalent:*

(i) $S_1(X,Y)$ *or* $S_1(Y,X)$ *is fulfilled;*
(ii) $S_1(Y) = S_1(X)$;
(iii) $S_2(Y) \leq S_2(X)$;
(iv) $S_2(X) \leq S_2(Y)$;
(v) $(\forall n \in \{0\} \cup \mathbb{N})\ S_n(Y) = S_n(X)$;
(vi) $S(Y) = S(X)$;
(vii) $S^*(Y) = S^*(X)$;
(viii) $Sh(Y) = Sh(X)$.

To prove the theorem, we need the following lemma (compare [2, Theorem 1.1]).

LEMMA 2.13. *If a compactum is shape dominated by a compact ANR, then it is shape equivalent to an ANR.*

PROOF. Let X be a compactum such that $Sh(X) \leq Sh(P)$, where P is a compact ANR. This means that X is an FANR (see [7, Theorem II.9.14]). If X is connected then, by [4, (6.3) Theorem], $(X, *)$ is pointed FANR for any choice of the base point $*$. By [7, Theorem II.9.15], $(X, *)$ has the (pointed) shape of an ANR $(Q, *)$. Consequently, $Sh(X) = Sh(Q)$. Consider now the simplest nonconnected case, i.e., let X consist of two components, $X = X_1 \sqcup X_2$ (disjoint union). Then an easy analysis shows that $Sh(X) \leq Sh(P)$ implies $P = P_1 \sqcup P_2$, $Sh(X_1) \leq Sh(P_1)$ and $Sh(X_2) \leq Sh(P_2)$, where P_1 and P_2 are the components of P. Clearly, P_1 and P_2 are compact ANR's. Therefore, X_1 and X_2 are connected FANR's. Since, by [4, (6.3) Theorem], each connected FANR is a pointed FANR, we infer that $(X_1, *_1)$ and $(X_2, *_2)$ are pointed connected FANR's (for any choice of the base points). By [7, Theorem II.9.15], there exist ANR's $(Q_1, *_1)$ and $(Q_2, *_2)$ such that $Sh(X_i, *_i) = Sh(Q_i, *_i)$, $i = 1, 2$. Therefore, $Sh(X_i) = Sh(Q_i)$, $i = 1, 2$. Put $Q = Q_1 \sqcup Q_2$, which is an ANR. Then, obviously, $Sh(X_1 \sqcup X_2) = Sh(Q_1 \sqcup Q_2)$, and therefore, $Sh(X) = Sh(X_1 \sqcup X_2) = Sh(Q_1 \sqcup Q_2) = Sh(Q)$. In the general case of a nonconnected compactum, the proof proceeds by induction. (It is a well known fact that every FANR is the disjoint union of at most finitely many FANR continua.) □

REMARK 2.14. (a) According to Lemma 2.13, the central theorem of [2] (Theorem 1.1 in the pointed connected case as well as in the unpointed case) holds in general.

(b) Lemma 2.13 implies that a compactum is stable if and only if it is strongly movable, i.e., it is an FANR. Since the class of all compact ANR's yields a dense (pro-)category for compacta (admitting a representation of the shape theory), it makes sense to define the "new" notion of stability for compacta by asking that a stable compactum has to be shape equivalent to a *compact* ANR. In this case, stability would strictly imply being an FANR (strong movability). Namely, there exist compacta shape dominated by compact ANR's but not shape equivalent to compact ANR's (see [2]).

PROOF OF THEOREM 2.12. It suffices to prove that (i) implies (viii). Assume that condition $S_1(X, Y)$ is fulfilled. First, consider the case $Sh(X) = Sh(P)$ and $Sh(Y) = Sh(Q)$, where P and Q are compact ANR's. Then, clearly, condition $S_1(P, Q)$ is fulfilled. By Theorem 2.10, $Sh(Q) = Sh(P)$ (equivalently, $Q \simeq P$), and therefore, $Sh(Y) = Sh(X)$. Consider now the general case. Then, by Lemma 2.13, $Sh(X) = Sh(P)$ and $Sh(Y) = Sh(Q)$, where P and Q are ANR's. Let $\boldsymbol{X}$, $\boldsymbol{Y}$ be a pair of compact ANR inverse sequences associated with X, Y respectively. Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be the trivial inverse sequences associated with P and Q respectively. Now, by repeating the appropriate part of the proof of Theorem 2.10 from above, we infer that $Q \simeq P$. Therefore, $Sh(Y) = Sh(Q) = Sh(P) = Sh(X)$.

If $S_1(Y, X)$ holds, the proof is analogous. □

THEOREM 2.15. *Let X and Y be compacta such that Y is shape dominated by X, $Sh(Y) \leq Sh(X)$. Then $S_1(Y) \leq S_1(X)$.*

PROOF. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be any compact ANR inverse sequences associated with X and Y respectively. Then $Sh(\boldsymbol{Y}) \leq Sh(\boldsymbol{X})$, i.e., $\boldsymbol{Y} \leq \boldsymbol{X}$ in the corresponding procategory. This means that there exists a pair of maps of inverse sequences, $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{X}$, such that $\boldsymbol{fg} \simeq \boldsymbol{1_Y}$. Without loss of generality, one may assume that $\boldsymbol{f}$ and $\boldsymbol{g}$ are special and with the strictly increasing index functions. By $\boldsymbol{fg} \simeq \boldsymbol{1_Y}$, for every $j \in \mathbb{N}$, $f_i g_{f(j)} \simeq q_{jgf(j)}$. Given a j_1, put $i_1 = f(j_1)$, and for every $i_1' \geq i_1$ put $j_1' = g(i_1')$. Then $j_1' \geq gf(j_1) \geq j_1$. Put $f_1 = f_{j_1} : X_{i_1} \to Y_{j_1}$ and $g_1 = g_{i_1'} : Y_{j_1'} \to X_{i_1'}$. Since $f_{j_1} g_{f(j_1)} \simeq q_{j_1 gf(j_1)}$ and $p_{f(j_1)j_1'} g_{i_1'} \simeq g_{f(j_1)} q_{gf(j_1)g(i_1')}$, the diagram

$$\begin{array}{ccccc} X_{i_1} & \leftarrow & \cdots & \leftarrow & X_{i_1'} \\ \downarrow f_1 & \nwarrow g_{f(j_1)} & & & \uparrow g_1 \\ Y_{j_1} & \leftarrow & Y_{gf(j_1)} & \leftarrow & Y_{j_1'} \end{array}$$

commutes up to homotopy. Therefore, condition $S_0^+(\boldsymbol{X}, \boldsymbol{Y})$ holds. This means $S_1(\boldsymbol{Y}) \leq S_1(\boldsymbol{X})$, i.e., $S_1(Y) \leq S_1(X)$. □

The first named author proved ([10, Examples 4 and 5]) that the Borsuk quasi-equivalence [1] and the S-equivalence [6] are mutually independent relations. In addition, the next theorem shows that the Borsuk quasi-equivalence and the S_n-equivalence, $n > 0$, are mutually independent relations. Furthermore, beside Theorem 2.6, it also shows that the implication of Lemma 2.4(iii) is strict.

THEOREM 2.16. *There exists a pair X, Y of quasi-equivalent compacta, $Y \overset{q}{\simeq} X$, such that condition $S_1(X, Y)$ is fulfilled (which implies $S_1(Y) \leq S_1(X)$ and $S_1(X) \leq S_1(Y)$), while $S_1(Y) \neq S_1(X)$.*

PROOF. The assertion is a consequence of the next example. □

EXAMPLE 2.17. Let X be the same as in Example 2.9, i.e., $X = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$. Let Y be the Cantor set. By [1, (6.3) Theorem], X and Y are quasi-equivalent. We claim that X and Y are not S_1-equivalent, though $S_1(Y) \leq S_1(X)$ and $S_1(X) \leq S_1(Y)$ hold.

We first prove that condition $S_1(X, Y)$ is fulfilled. Let us consider the associated compact ANR-sequences $\boldsymbol{X} = (X_i, p_{ii'})$ and $\boldsymbol{Y} = (Y_j, q_{jj'})$ consisting of finite ANR's, $|X_i| = i$ and $|Y_j| = 2^{j-1}$, and surjective bonding mappings defined in the obvious way. (The "exploding" point $*$ of X_i yields the point $*$ and exactly one new point of X_{i+1}, while all the other fibres of $p_{i,i+1}$ are singletons. Every fiber of $q_{j,j+1}$ consists of two points.) In this case, every homotopy commutative diagram relating $\boldsymbol{X}$ and $\boldsymbol{Y}$ is commutative. Given a

$j_1 \in \mathbb{N}$, put $i_1 = 2^{j_1-1}$, and denote

$$Y_{j_1} = B_1,$$
$$X_{i_1} = B = \{*\} \sqcup (B \setminus \{*\}).$$

Given an $i_1' \geq i_1$, put $j_1' = [\log_2(i_1' - i_1)] + j_1 + 1$ (i.e., j_1' is the minimal integer such that $2^{j_1'-j_1} \geq i_1' - i_1$), and denote

$$X_{i_1'} = \{*\} \sqcup (B \setminus \{*\}) \sqcup A,$$

$$Y_{j_1'} = 2^{j_1'-j_1} B$$

(which suggests that $Y_{j_1'}$ consists of $2^{j_1'-j_1}$ disjoint copies of B and implies that $|Y_{j_1'}| \geq |X_{i_1'}|$). Given a $j_2 \geq j_1'$, choose any $i_2 \geq i_1'$, and denote

$$Y_{j_2} = 2^{j_2-j_1'}(2^{j_1'-j_1} B) = Y_{j_1'} = 2^{j_2-j_1} B,$$

$$X_{i_2} = \{*\} \sqcup (B \setminus \{*\}) \sqcup A \sqcup A'.$$

Then there exist a bijection $f_1 : X_{i_1} \to Y_{j_1}$, a surjection $g_1 : Y_{j_1'} \to X_{i_1'}$ and a function $f_2 : X_{i_2} \to Y_{j_2}$ making the corresponding diagram commutative (see the picture below; the notations are quite similar to those of Example 2.9).

$$\begin{array}{ccccc}
\boxed{\begin{array}{c}B\setminus\{*\}\\ *\end{array}} & \leftarrow & \boxed{\begin{array}{c}B\setminus\{*\}\\ A\\ *\end{array}} & \leftarrow & \boxed{\begin{array}{c}B\setminus\{*\}\\ A\\ A'\\ *\end{array}}\\
f_1\downarrow & & g_1\uparrow & & \downarrow f_2\\
\boxed{B} & \leftarrow & \boxed{2^{j_1'-j_1}B} & \leftarrow & \boxed{2^{j_2-j_1}B}
\end{array}$$

More precisely, starting with a bijection f_1, a desired surjection g_1 can be defined by means of the inverse of f_1 on every copy of $B \setminus \{f_1(*)\}$, while the subset of all other points g_1 has to send *onto* the subset $A \sqcup \{*\}$. Finally, a desired function f_2 can be easily defined according to commutativity of the right rectangle. Therefore, condition $S_1(\boldsymbol{X}, \boldsymbol{Y})$, i.e., $S_1(X, Y)$ is fulfilled. By Lemma 2.2, it implies $S_0^+(X, Y)$ and $S_0^+(Y, X)$, and thus, $S_1(Y) \leq S_1(X)$ and $S_1(X) \leq S_1(Y)$. (Notice that $S_1(X) \leq S_1(Y)$ also holds by Theorem 2.8, since X is a retract of Y, and thus, $Sh(X) \leq Sh(Y)$)

Let us now prove that $S_1(Y) \neq S_1(X)$. It suffices to show that condition $S_1(\boldsymbol{Y}, \boldsymbol{X})$ can not be fulfilled. Consider a diagram (see the picture below) realizing condition $S_0^+(\boldsymbol{Y}, \boldsymbol{X})$,

$$\begin{array}{ccc}
\boxed{\{*\} \sqcup A \sqcup B} & \leftarrow & \boxed{2^{j_1'-j_1}(\{*\} \sqcup A \sqcup B)}\\
g_1'\downarrow & & f_1'\uparrow\\
\boxed{\begin{array}{c}A\\ *\end{array}} & \leftarrow & \boxed{\begin{array}{c}A\\ A'\\ *\end{array}}
\end{array}$$

where $X_{i_1} = \{*\} \sqcup A$, $Y_{j_1} = \{*\} \sqcup A \sqcup B$, $Y_{j_1'} = 2^{j_1'-j_1}(\{*\} \sqcup A \sqcup B)$, $X_{i_1'} = \{*\} \sqcup A \sqcup A'$.

Notice that g_1' must be a surjection. Furthermore, any commutative extension of the above diagram to the right (including a new mapping g_2') asks for f_1' also to be a surjection. Namely, all the bonding mappings are surjective. Now, one should observe that, in general, f_1' can not be surjective. Indeed, an easy analysis shows that if one chooses an i_1' large enough such that it admits a surjection f_1' of $X_{i_1'}$ onto $Y_{j_1'}$ (A' via i_1' can supply as many new points as one needs), then $g_1' q_{j_1 j_1'} f_1' \neq p_{i_1 i_1'}$. Consequently, a desired g_2' does not exist, i.e., condition $S_1(\boldsymbol{Y}, \boldsymbol{X})$ can *not* be fulfilled.

3. The applications

By [12] and [2, Theorem 1.1], there are FANR's (compacta shape equivalent to ANR's, see Lemma 2.13) which are not shape equivalent to compact ANR's. According to Theorem 2.12, the next corollary arises.

Corollary 3.1. *If an FANR is not shape equivalent to any compact ANR, then it is not S_1-equivalent to any compact ANR.*

Proof. Let Y be an FANR, i.e., $Sh(Y) \neq Sh(P)$ for every compact ANR P. Then $S_1(Y) \neq S_1(P)$ for every compact ANR P. Indeed, if there would exist a compact P such that $S_1(Y) = S_1(P)$, then by Theorem 2.12, one would have $Sh(Y) = Sh(P)$, which is a contradiction. □

An immediate consequence of Corollary 3.1 is the next corollary:

Corollary 3.2. *Let X and Y be FANR's. If X is S_1-equivalent (shape equivalent) to a compact ANR, and Y is not shape equivalent (S_1-equivalent) to any compact ANR, then $S_1(Y) \neq S_1(X)$.*

According to [8, Remark 1] and our definitions and notations, our Theorems 3.3, 3.4 and 3.5 below improve [6, Theorems 4, 5, 6, 7 and 7′] as well as [8, Theorems 3, 4] (see also the proofs of the mentioned theorems).

Theorem 3.3. *Let X and Y be compacta such that $S_1(Y) \leq S_1(X)$, i.e., let condition $S_0^+(X,Y)$ be fulfilled. Then the following assertions hold:*

(i) If X is connected, then so is Y;
(ii) If $Sh(X) = 0$, then also $Sh(Y) = 0$;
(iii) If the fundamental dimension $Fd(X) \leq n$, then also $Fd(Y) \leq n$;
(iv) If X is n-shape connected, then so is Y.

Proof. In the proof of [6, Theorem 4] only condition $S_0^+(X,Y)$ of $S(Y) = S(X)$ is used. □

Theorem 3.4. *Let X and Y be compacta such that condition $S_1(X,Y)$ is fulfilled. If X is movable (n-movable), then so is Y.*

PROOF. In the proof of [6, Theorem 5] only condition $S_1(X,Y)$ of $S(Y) = S(X)$ is used. $\square$

The next theorem improves [6, Theorems 5, 7 and 7′].

THEOREM 3.5. *Let X and Y be compacta such that $S_2(Y) \leq S_2(X)$, i.e., let condition $S_1^+(X,Y)$ be fulfilled. If X is an FANR, then so is Y and $Sh(Y) = Sh(X)$.*

PROOF. First of all, by Lemma 2.13, one should notice that in [6, Theorem 7′], the assumption "if X is a pointed FANR" may be weakened to "if X is an FANR". Namely, in its proof only the fact that X has the shape of an ANR is used. Further, in the proof of [6,Theorem 6] (and, consequently, [6, Theorems 7 and 7′]), only condition $S_1^+(X,Y)$ of $S(Y) = S(X)$ is applied. The conclusion follows. $\square$

COROLLARY 3.6. *The shape class of an FANR is determined by its S_2-domination. Therefore, if X is an FANR, then $S_2(X) = S(X) = S^*(X) = Sh(X)$.*

PROOF. It suffices to prove that $S_2(X) \subseteq Sh(X)$. Let $Y \in S_2(X)$, i.e., $S_2(Y) = S_2(X)$. Then, by Lemma 2.4 (iii) and Theorem 3.5, $Sh(Y) = Sh(X)$, i.e., $Y \in Sh(X)$. $\square$

PROBLEM 3.1. *Does there exist a compact ANR (an FANR) X such that $S_1(X) \setminus S_2(X) \neq \emptyset$ (equivalently, $S_1(X) \setminus Sh(X) \neq \emptyset$)?*

REMARK 3.7. Concerning the problem, consider an FANR X which is shape equivalent to a compact ANR P. Then $S_1(X) \setminus S_2(X) = S_1(P) \setminus S_2(P)$. Therefore, in this case, the problem reduces to compact ANR's. Let $Y \in S_1(P)$. By Theorem 3.4, Y is movable, which one can clearly see from the diagram below.

$$\begin{array}{ccccc} P & \xleftarrow{1} & P & \xleftarrow{1} & P \\ \downarrow f_1 & & \uparrow g_1 & & \downarrow f_2 \\ Y_{j_1} & \longleftarrow & Y_{j_1'} & \overset{r}{\rightleftarrows} & Y_{j_2} \end{array}$$

Namely, $S_1(P,Y)$ yields a desired $r = f_2 g_1 : Y_{j_1'} \to Y_{j_2}$. Further, Y is semi-stable (see [10, Definition 3 and Lemma 4]), which is the complementary part of the strong movability. This one can see from the diagram below.

$$\begin{array}{ccccc} Y_{j_1} & \overset{r'}{\rightleftarrows} & Y_{j_1'} & \longleftarrow & Y_{j_2} \\ \downarrow g_1' & & \uparrow f_1' & & \downarrow g_2' \\ P & \xleftarrow{1} & P & \xleftarrow{1} & P \end{array}$$

Namely, $S_1(Y,P)$ yields a desired $r' = f_1' g_1' : Y_{j_1} \to Y_{j_1'}$. However, we can *not* closely enough relate r and r' (by a homotopy), and thus, we may *not* conclude that Y is strongly movable (i.e., an FANR). In other words, although movability and semi-stability are the S_1-invariants, we do not know whether the strong movability (movability and semi-stability with the same mappings) is an S_1-invariant. (By Theorem 3.5, it is an invariant of the S_2-domination.)

4. Two conjectures

Let us denote the compactum X of Example 2.9 by L, and the Cantor set by C. Then, by [1, (6.3) Theorem] and our Examples 2.9 and 2.17 as well as by [10, Remark 11], the above examples imply the following results:

Conclusion 4.1. *$L \sqcup L \overset{q}{\simeq} L \overset{q}{\simeq} C$, $S(L \sqcup L) \neq S(L) \neq S(C)$ and $S(L \sqcup L) \neq S(C)$.*

The same holds for the S^*-equivalence. Further, we have shown that $S_1(L \sqcup L) = S_1(L) \neq S_1(C)$ and $S_2(L \sqcup L) \neq S_2(L)$.

More precisely,

$S_1(L, C)$ holds (which implies $S_1(C) \leq S_1(L)$ and $S_1(L) \leq S_1(C)$);

$S_1(C, L)$ does not *hold (which implies $S_1(C) \neq S_1(L)$);*

$S_1^+(L \sqcup L, L)$ holds (which means $S_2(L) \leq S_2(L \sqcup L)$ and implies $S_1(L \sqcup L) = S_1(L)$);

$S_1^+(L, L \sqcup L)$ does not *hold (which means $S_2(L \sqcup L) \nleq S_2(L)$ and implies $S_2(L \sqcup L) \neq S_2(L)$).*

According to these facts, one is tempted to state the following hypothesis:

Conjecture 4.2. *For every $n \in \{0\} \cup \mathbb{N}$ there exists a compactum X such that $S(X) \subsetneqq S_n(X)$.*

Theorem 4.3. *If Conjecture 4.2 is false, then the S-equivalence reduces to a unique S_n-equivalence, $n \geq 2$. Consequently, the S-equivalence and S^*-equivalence would coincide.*

Proof. Clearly, for every $n \in \{0\} \cup \mathbb{N}$ and every compactum X, $S(X) \subseteq S_{n+1}(X) \subseteq S_n(X)$ holds by definitions. Thus, if Conjecture 4.2 is false, there exists an $n \in \{0\} \cup \mathbb{N}$ such that, for every compactum X and every $n' \geq n$, $S(X) = S_{n'}(X) = S_n(X)$. Theorem 2.8 implies that $n \geq 2$. Consequently, the second claim follows by [11, Lemma 4]. □

On the other hand, if Conjecture 4.2 is true then the following stronger hypothesis makes sense:

Conjecture 4.4. *There exists a compactum X such that, for every $n \in \{0\} \cup \mathbb{N}$, $S(X) \subsetneqq S_n(X)$. Equivalently, there exist a compactum X and a strictly increasing sequence (n_k) in $\{0\} \cup \mathbb{N}$, $n_1 = 0$, such that, for every $k \in \mathbb{N}$, $S(X) \subsetneqq S_{n_{k+1}}(X) = \cdots = S_{n_k+1}(X) \subsetneqq S_{n_k}(X)$.*

Clearly, if Conjecture 4.4 is true then so is Conjecture 4.2.

THEOREM 4.5. *If Conjecture* 4.4 *is false, then each S-equivalence class is an S_n-equivalence class. Consequently, the S-equivalence and S^*-equivalence would coincide.*

PROOF. If Conjecture 4.4 is false then, for every compactum X, there exists an $n_X \in \{0\} \cup \mathbb{N}$ such that $(\forall n \geq n_X)$ $S(X) = S_n(X) = S_{n_X}(X)$. Thus, the second assertion follows by [11, Lemma 4]. $\square$

REFERENCES

[1] K. Borsuk, *Some quantitative properties of shapes*, Fund. Math. **93** (1976), 197-212.
[2] D. A. Edwards and R. Geoghegan, *Shapes of complexes, ends of manifolds, homotopy limits and the Wall obstruction*, Ann. of Math. (2) **101** (1975), 521-535, Correction. Ibid. **104** (1976), 389.
[3] K. R. Goodearl and T. B. Rushing, *Direct limit groups and the Keesling-Mardešić shape fibration*, Pacific J. Math. **86** (1980), 471-476.
[4] H. M. Hastings and A. Heller, *Splitting homotopy idempotents*, in: Shape theory and geometric topology (Dubrovnik, 1981), Lecture Notes in Math. **870**, Springer, Berlin-New York, 1981, 23-36.
[5] J. Keesling and S. Mardešić, *A shape fibration with fibres of different shape*, Pacific J. Math. **84** (1979), 319-331.
[6] S. Mardešić, *Comparing fibres in a shape fibration*, Glasnik Mat. Ser. III **13**(**33**) (1978), 317-333.
[7] S. Mardešić and J. Segal, Shape Theory, North-Holland Publishing Co., Amsterdam-New York-Oxford 1982.
[8] S. Mardešić and N. Uglešić, *A category whose isomorphisms induce an equivalence relation coarser than shape*, Topology Appl. **153** (2005), 448-463.
[9] K. Morita, *The Hurewicz and the Whitehead theorems in shape theory*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **12** (1974), 246-258.
[10] N. Uglešić, *A note on the Borsuk quasi-equivalence*, submitted.
[11] N. Uglešić and B. Červar, *A subshape spectrum for compacta*, Glasnik Mat. Ser. III **40(60)** (2005), 351-388.
[12] C. T. C. Wall, *Finiteness conditions for CW-complexes*, Ann. of Math. (2) **81** (1965), 55-69.

N. Uglešić
University of Zadar
Studentski dom, F. Tuđmana 24 D
23000 Zadar
Croatia
E-mail: `nuglesic@unizd.hr`

B. Červar
Department of Mathematics
University of Split
Teslina ulica 12/III
21000 Split
Croatia
E-mail: `brankoch@pmfst.hr`

Received: 19.4.2006.
Revised: 7.2.2007.