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A NOTE ON THE WC-MAPPINGS

Sinopsis

Let X be a topological T_1 space. By wX we denote the Wallman compactification of X. A mapping $f : X \to Y$ is a wc-mapping if f has an unique closed extension $wf : wX \to wY$.

The main purpose of this paper is to give some external and internal characterizations of the wc-mappings. The main theorem establishes that if Y is a Hausdorff Frechet space then f is a wc-mapping iff f is an uw-mapping (Theorem 1.13.)

0. Introduction

In the present paper we deal with T_1 topological spaces and with continuous onto mappings $f : X \to Y$.

By ClA or by $Cl_X A$ the closure of a subset A of a space X is denoted. A space X is quasi-compact [2 : 177] if each open cover of X has a finite subcover.

For each T_1 space X there is the Wallman compactification wX ([1] or [2]). The Wallman compactification wX of a T_1 space X is the union of the space X and a family $F_0(X)$ of all free ultrafilters consisting of the closed subset of X. For each open $U \subseteq X$ we define

$$U^* = U \cup \{\mathcal{F} \in F_0(X) : F \subseteq U \text{ for some } F \in \mathcal{F}\} \subseteq wX. \tag{1}$$

Similarly, for each closed subset F of X we define

$$F_* = F \cup \{\mathcal{F} \in F_0(X) : F \in \mathcal{F}\} \subseteq wX. \tag{2}$$

It is easy to see that [2 : 231]:

$$(U \cup V)^* = U^* \cup V^* \quad \text{and} \quad (U \cap V)^* = U^* \cap V^*, \tag{3}$$

$$(F \cup G)_* = F_* \cup G_* \quad \text{and} \quad (F \cap G)_* = F_* \cap G_*. \tag{4}$$

From (3) and (4) it follovs that the family $\{U^* : U \text{ open in } X\}$ is a base for open sets on wX and $\{F_* : F \text{ closed in } X\}$ is a base for closed sets on wX. By virtue of (1) and (2) it folows that if $F \subset X$ is closed then $Cl_{wX}F = F_*$. Moreover, if U is open and F is closed in X such that $F \subseteq U$ then $F_* \subseteq U^*$ i.e. $Cl_{wX}F \subseteq U^*$. The dense embedding X in wX we denote by $w_X : X \to wX$.

Let $f : X \to Y$ be a continuous mapping. We say that a mapping $f : X \to Y$ is a w-mapping (uw-mapping) if f has a (an unique) continuous extension $wf : wX \to wY$. It is known that there exist the mappings $f : X \to Y$ without w-extensions [3].

1. A characterization of the WC-mappings

The notion of the wc-mappings was introduced by Harris [4].

A mapping $f : X \to Y$ is called a wc-mapping if f is an uw-mapping with closed extension wf [4].

This definition is external since the Wallman compactification is used. The question of an internal characterizations was raised by Harris [3].

The main purpose of this Section is to give an internal characterization of wc-mappings for some classes of spaces. The question of an internal characterization in the general case remains open.

We start with the following lemma.

1.1. LEMMA. An uw-mapping $f : X \to Y$ is a wc-mapping iff $wf(F_*)$ is closed in wY for each closed subset $F \subseteq X$.

Proof. *Necessity.* For each closed $F \subseteq X$ we have that $Cl_{wX}F = F_*$. If $f : X \to Y$ is the wc-mapping, then $wf : wX \to wY$ is closed. Thus, $wf(F_*)$ is closed in wY.

Sufficiency. Suppose that each $wf(F_*)$ is closed an let us prove that wf is closed. Let A be a closed subset of wX. There is a family $\{F_\mu : F_\mu \text{ is closed in } X, \mu \in M\}$ such that $A = \bigcap \{F_{\mu^*}, \mu \in M\}$. Clearly, $wf(A) \subseteq \bigcap \{wf(F_{\mu^*}) : \mu \in M\}$. Let us prove that $wf(A) \supseteq \bigcap \{wf(F_{\mu^*}) : \mu \in M\}$. For each $y \in \bigcap \{wf(F_{\mu^*}) : \mu \in M\}$ we infer that $(wf)^{-1}(y) \cap F_{\mu^*}$ is non-empty. Since $(wf)^{-1}(y)$ is quasi-compact, we have that the intersection $\bigcap \{(wf)^{-1}(y) \cap F_{\mu^*} : \mu \in M\}$ is non-empty. Thus, there is a point $x \in (wf)^{-1}(y)$ such that $y \in \bigcap \{F_{\mu^*} : \mu \in M\} = A$. This means that $y \in wf(A)$. Finally, we have that $wf(A) = \bigcap \{wf(F_{\mu^*}) : \mu \in M\}$. Since each $wf(F_{\mu^*})$ is closed, we infer that wf(A) is closed. The proof is completed.

In the sequel we use the following relations. From the continuity of wf it follows

$$wf(F_*) \subseteq Cl_{wY} wf(F) = Cl_{wY} f(F), \text{ F is closed in X.} \tag{1}$$

On the other hand we have

$$Cl_Y f(F) = Cl_{wY} f(F) \cap Y \subseteq Cl_{wY} f(F). \tag{2}$$

The inclusion $f(F) \subseteq Cl_Y f(F)$ gives

$$Cl_{wY} f(F) \subseteq Cl_{wY} (Cl_Y f(F)). \tag{3}$$

Similarly from (2) we obtain

$$Cl_{wY} (Cl_Y f(F)) \subseteq Cl_{wY} f(F). \tag{4}$$

Finally we have

$$Cl_{wY} f(F) = Cl_{wY}(Cl_Y f(F)). \tag{5}$$

From (1) and the last relation it follows

$$wf(F_*) \subseteq Cl_{wY}(Cl_Y f(F)),\ F \text{ is closed in } X. \tag{6}$$

1.2. LEMMA. An uw-mapping $f: X \to Y$ is a wc-mapping iff for each closed set $F \subseteq X$ it follows $wf(F_*) = Cl_{wY}(Cl_Y f(F)) = (Cl_Y f(F))_*$.

Proof. Apply Lemma 1.1. and the relations (1) – (6).

1.3. COROLLARY. If $f: X \to Y$ is closed then f is a wc-mapping.

Proof. It is sufficient to prove that $wf(F_*) = (f(F))_*$ for each closed $F \subseteq X$. From (1) it follows that $wf(F_*) \subseteq (f(F))_*$. Clearly, $f(F) \subseteq wf(F_*) \subseteq (f(F))_*$. We now use the condition (KC) [5].

(KC) If A is a closed subset of Y and $K \subseteq wY$ is quasi-compact with $A \subseteq K \subseteq Cl_{wY} A$, then K is closed.

If we prove that wY satisfies the condition (KC) then Corollary 1.3. is proved since $wf(F_*)$ is quasi-compact.

1.4. LEMMA. The Wallman compactification wX of a T_1 space X satisfies the condition (KC).

Proof. Let us note that Lemma follows from Theorem 4.1. – 4.5. of [5]. We give the proof based on the properties of the Wallman compactification. Suppose that we have a closed subset A of X and quasi-compact subset K such that $A \subseteq K \subseteq Cl_{wX} A$. If we suppose that K is not closed then there exists a point $y \in Cl_{wX} A \setminus K$. For each point $k \in K$ there is an open set U_k^* [2 : 232] such that $k \in U_k^*$ and $y \not\subset U_k^*$. From the compactness of K it follows that there is a finite subfamily $\{U_{k_1}^*, \ldots, U_{k_n}^*\}$ which covers K. Since $(U_{k_1} \cup \ldots \cup U_{k_n})^* = (U_{k_1}^* \cup \ldots \cup U_{k_n}^*)$ [2 : 231] we infer that $A \subseteq U_{k_1} \cup \ldots \cup U_{k_n}$. This means that $Cl_{wX} A \subseteq (U_{k_1} \cup \ldots \cup U_{k_n})^*$. This is impossible since $y \notin (U_{k_1} \cup \ldots \cup U_{k_n})^*$. The proof is completed.

If $f: X \to Y$ is a mapping between the T_1 spacesX and Y and U, V are open subsets of X, Y respectively, then $U <_f V$ if for each closed $A \subseteq U$ we have $Cl_Y f(A) \subseteq V$. A w-cover of a space is a finite open cover. If ν is a w-cover of Y and μ a w-cover of X, then $\mu <_f \nu$ will be writen if for each $U \in \mu$ there is a $V \in \nu$ with $U <_f V$. The mappings f is a WO-mapping if for each w-cover ν of Y there is a w-cover μ of X such that $\mu <_f \nu$.

1.5. LEMMA. [3]. Every WO-mapping has an unique w-extension and the extension is also WO-mapping.

1.6. LEMMA. [3]. Every wc-mapping is WO-mapping.

1.7. EXAMPLE. [3]. There is a WO-mapping between compact spaces which is not closed i.e. which is not wc-mapping.

1.8. LEMMA. An uw-mapping $f: X \to Y$ is a wc-mapping iff for each closed $F \subseteq X$ and each $y \in Cl_Y f(F)$ there is a $x \in F_*$ such that $wf(x) = y$.

Proof. *Sufficiency.* In this case we have $Cl_Y f(F) \subseteq wf(F_*) \subseteq Cl_{wY} Cl_Y f(F)$ for each closed $F \subseteq X$. By (KC) it follows that $wf(F_*) = Cl_{wY} Cl_Y f(F)$. Apply Lemma 1.2.

Necessity. If $f: X \to Y$ is a wc-mapping then by Lemma 1.2. we have

$wf(F_*) = (Cl_Y f(F))_*$. Since $Cl_Y f(F) \subseteq (Cl_Y f(F))_*$ we infer that for each $y \in Cl_Y f(F)$ there is a point $x \in F_*$ such that $wf(x) = y$. The proof is completed.

1.9. REMARK. If f is closed, then f(F) is closed and $Cl_Y f(F) = f(F)$. Thus by Lemma 1.8. wf is closed since for each $y \in Cl_X f(F) = f(F)$ there is a point $x \in F \subseteq F_*$ such that $f(x) = y$ i.e. $wf(x) = y$.

1.10. LEMMA. A mapping $f : X \rightarrow Y$ is a wc-mapping iff f is a WO-mapping (uw-mapping) with the following property:

(W) For each closed $F \subseteq X$ and each point $x \in Cl_Y f(F) \setminus f(F)$ there is a point $y \in F_* \setminus F$ such that $wf(x) = y$.

Proof. *Necessity.* If f is a wc-mapping then f is the WO-mapping (Lemma 1.6.) and uw-mapping. The condition (W) holds from Lemma 1.8.

Sufficiency. If f is the WO(uw)-mapping, then f has an unique extension wf. Moreover, from the condition (W) and Lemma 1.8. it follows that f is the wc-mapping. The proof is completed.

If the space Y in Lemma 1.10. is regular, then we have

1.11. LEMMA. A mapping $f : X \rightarrow Y$, Y regular, is a wc-mapping iff f is a WO-mapping.

Proof. *Necessity.* If f is the wc-mapping, then f is the WO-mapping, (Lemma 1.6.).

Sufficiency. Let $y \in Cl_Y f(F) \setminus f(F)$ and let $\mathcal{U}_y = \{U_y : y \in U_y \text{ and } U_y \text{ is open}\}$. Consider a centred family $f^{-1}(\mathcal{U}_y) = \{f^{-1}(U_y) : U_y \in \mathcal{U}_y\}$. Clearly, $V_y \cap F \neq \emptyset$ for each $V_y \in f^{-1}(\mathcal{U}_y)$. Let ω be a maximal centred family which contains a family $\{Cl_X V_y : V_y \in f^{-1}(\mathcal{U}_y)\}$. Clearly, $\omega \in F_*$. For each $V \in \omega$ we have $V \cap Cl_X V_y \neq \emptyset$ and $f(V) \cap Cl_Y U_y \neq \emptyset$, where $V_y = f^{-1}(U_y)$. If we suppose that $y \in Cl_Y f(V)$ then there is a O_y such that $y \in O_y \subseteq Cl_Y O_y \subseteq Y \setminus Cl_Y f(V)$. This means that $Cl_X f^{-1}(O_y) \cap V = \emptyset$. This is impossible since $V \cap U_y$ is non-empty for each $U_y \in \mathcal{U}_y$. The proof is completed.

From the proof of the sufficient part of Lemma 1.11. in fact we have.

1.12. COROLLARY. If Y is regular and if $f : X \rightarrow Y$ is an w-mapping, then wf is closed. Moreover, if f is an uw-mapping, then f is a wc-mapping.

Let us recal that the first part of this Corollary was proved in the paper [10].

A topological space X is called a *Fréchet space* if for each $A \subseteq X$ and each $x \in Cl_X A$ there exists a sequence $x_1, x_2, \ldots$ of points of A converging to x [2 : 78].

1.13. THEOREM. Let Y be a Fréchet Hausdorff space. A mapping $f : X \rightarrow Y$ is a (closed) wc-mapping iff f is an (uw-mapping) WO-mapping.

Proof. *Necessity* folows from the definition of a wc-mapping and Lemma 1.6.

Sufficiency. We apply Lemma 1.10. Let F be a closed subset of X and let y be any point of $Cl_Y f(F) \setminus f(F)$. From the assumption that Y is a Fréchet space it follows that there is a sequence $y_1, y_2, \ldots$ of points of f(F) converging to y. Let $x_n \in F$, $n \in N$, such that $f(x_n) = y_n$. Now we prowe that the sets $G_n = \{x_m : m \geq n\}$ are closed, $n \in N$. Suppose that $x \in Cl_X G_n \setminus G_n$. Clearly, $x \in F$ and $f(x) \neq y$. Let U, V be a pair of disjoint open sets about y and f(x). There is a neighbourhood W of x such that $f(W) \subseteq V$. This means that V contains infinitely many points $f(x_n)$. On the other hand the set U contains the points $f(x_n) = y_n$ for all $n \geq n_O$. This is impossible since $U \cap V = \emptyset$. Thus each G_n is closed. Let $\mathcal{A}$ be a maximal centred family which contains a family $\{G_n : n \in N\}$. It is obviosly that wf $(\mathcal{A}) = y$ since $y \in Cl_Y f(A)$ for each $A \in \mathcal{A}$. The proof is completed.

1.14. COROLLARY. Let Y be a first-countable Hausdorff space. Then a mapping $f: X \rightarrow Y$ is a (closed mapping) wc-mapping iff f is (an uw-mapping) a WO-mapping.

A space is said to be an E_1-space [11] if every its point is a countable intersection of closed neighbourhoods of that point. Clearly, each E_1-space is a Hausdorff space.

We close this Section with the following theorem.

1.15. THEOREM. Let Y be a countably compact E_1-space. A mapping $f: X \rightarrow Y$ is a wc-mapping iff f is an uw-mapping. Moreover, every w-mapping $f: X \rightarrow Y$ has a closed extension $wf: wX \rightarrow wY$.

Proof. *Necessity* follows from the definition of the wc-mapping.

Sufficiency. Let F be a closed subset of X and let y be any point of $Cl_Y f(F) \setminus f(F)$. There is a family $\mathcal{U} = \{U_n : n \in N\}$ of open sets U_n such that $y = \bigcap \{Cl_Y U_n : n \in N\}$. There exists $x \in F_* \setminus F$ which contains a family $\{Cl_X f^{-1}(U_n) : n \in N\}$. For each $G \in x$ we have $Cl_Y f(G) \cap Cl_Y U_n \neq \emptyset$, $n \in N$. Since $Cl_Y f(F)$ is countably compact we infer that $\bigcap \{Cl_Y f(F) \cap Cl_Y U_n : n \in N\} = Cl_Y f(F) \cap \{Cl_Y U_n : n \in N\} = Cl_Y f(F) \cap \{y\}$ is non-empty. This means that $y \in Cl_Y f(F)$ for each $G \in x$. Thus $wf(x) = y$. The proof is completed.

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SADRŽAJ

Klasu wc-preslikavanja uveo je Harris u radu [4] i postavio pitanje interne definicije tih preslikavanja tj. definicije u terminima prostora X i Y. To je interesantno zato što su wc-preslikavanja definirana tako da wf : wX → wY bude zatvoreno.

U radu su dane neke externe i interne karakterizacije wc-preslikavanja. Nužni i dovoljni uvjeti da bi f : X → Y bilo wc-preslikavanje dani su za T_2 Fréchetov prostor Y (Teorem 1.13.) i E_1 prebrojivo kompaktan prostor (Teorem 1.15.).

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