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L O C A L C O N N E C T E D N E S S O F I N V E R S E L I M I T

Abstract. The local connectedness of inverse limit spaces was studied in many papers ([2], [5], [7], [8], [11]).

The main purpose of the present paper is to prove the following theorem.

THEOREM 1.8 Let $X=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system such that the projections $f_{\alpha\beta}$ are irreducible fully closed mappings. In order that limX be locally connected it is necessary that each X_α be locally connected and it is sufficient that each X_α be a locally connected space without local cut points.

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0. Introduction

Introduction contains some basic definitions and notations.

0.1. Let Y be a subset of a space X.By Cl_X Y or ClY is denoted the closure of Y in X. The boundary of the subset $A\subseteq X$ is denoted by Fr(A).

0.2. The symbols N and R denote the sets of positive integers and real numbers.

0.3. By $|A|$ the cardinality of A is denoted.

0.4 If A is a well-ordered set, then cf(A) denotes the smallest ordinal number which is cofinal in A.

0.5. The initial ordinal number and its cardinality are denoted by ω_τ and $\aleph_\tau$.

0.6. By $X=\{X_\alpha, f_{\alpha\beta}, A\}$ is denoted an inverse system and by lim X its limit.

0.7. An inverse system $X=\{X_\alpha, f_{\alpha\beta}, A\}$ is σ-directed if A is σ-directed, i.e., for each sequence $a_1, a_2, ..., a_n, ...$, of the members of A there is an $a\in A$ such that $a\geq a_i$, $i=1, 2, ...$.

0.8. A topological space X is called pseudocompact [4:263] if X is completely regular and every real- valued function defined on X is bounded.

0.9. A space X is $\aleph_m$-*compact* if each open cover of X of cardinality $\leq \aleph_m$ has a finite subcover.

1. Local connectedness of the inverse limit space

In the sequel we use the following characterization of local connectedness.

LEMMA 1.1 [*10:II, 242, Teorema 1.*]. *A space X is locally connected if and only if each family $\{A_t: t\in T\}$ of subsets A_t of X has the property*

$$(1) \qquad Fr(\cup\{A_t : t\in T\})\subset Cl(\cup\{FrA_t : t\in T\})$$

REMARK 1.2 a) In the paper [9] it was proved that a non-locally connected space X contains a family $\{A_t:t\in T\}$, $|T|\geq\aleph_0$ of open disjoint subsets A_t of X for which (1) is not satisfied.

b) If X is regular, then the family $\{A_t\}$ may be choosen so that $X - Cl(\bigcup\{A_t:t\in T\})$ is non-empty. Namely, if X is not locally connected at some point p, then there are neighbourhoods U and V such that $x\in ClV\subseteq U$ and V (as a subspace) is not locally connected at p.Now apply a) on the subspace V.

We start with the following theorem.

THEOREM 1.3 *Let* $X=\{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system such that the following condition (FR) is satisfied:*

(FR) For each open subset $U\subseteq limX$ *and each* $\alpha\in A$ *it follows*

$$f_\alpha(FrU)=Fr(f_\alpha(U))$$

If the spaces X_α, $\alpha\in A$, *are locally connected, then limX is locally connected.*

Proof. Suppose that limX is not locally connected. By virtue of Remark 1.2. it follows that there exists an infinite family $\{A_t:t\in T\}$ of open subsets A_t of limX such that there is a point $x\in Fr(\bigcup\{A_t:t\in T\})\setminus Cl(\bigcup\{FrA_t:t\in T\})$. This means that there is an open set $U_\alpha\subseteq X_\alpha$ such that $f_\alpha^{-1}(U_\alpha)$ is a neighbourhood of x and $f_\alpha^{-1}(U_\alpha)\cap(\bigcup\{FrA_t:t\in T\})$ is empty. Clearly, $U_\alpha\cap f_\alpha(\bigcup\{FrA_t:t\in T\})=\emptyset$. By (FR) we infer that $f_\alpha(x)\notin Cl(\bigcup\{Frf_\alpha(A_t):t\in T\})$. On the other hand from $x\in Fr(\bigcup\{A_t:t\in T\})$ and (FR) it follows that $f_\alpha(x)\in Fr(\bigcup\{f_\alpha(A_t): t\in T\})$. This means that the family $\{f_\alpha(A_t): t\in T\}$ has the property $Fr(\cup\{f_\alpha(A_t): t\in T\})\setminus Cl(\bigcup\{Frf_\alpha(A_t):t\in T\})$. By Lemma 1.1. this is impossible since X_α is locally connected. The proof is compplеted.

If M is connected set and p is point of M such that M-p is not connected, then p will be called a cut point of M [20:41]. A point $x\in X$ is said to be a *local cut point* of X if for each neighbourhood U of x there exists a neighbourhod V of x, $x\in V\subseteq U$, such that x is a cut point of V [15]. Each R^n,$n\geq2$, is a space without local cut points. Each point of real line R is a local cut point. The Niemytzki plane is an example of a completely regular not normal space [4:62] without local cut points.

We say that a mapping $f:X\rightarrow Y$ onto Y is *irreducible* if for each non-empty open subset $U\subseteq X$ the set $f^*(U)=\{y:y\in Y, f^{-1}(y)\subseteq U\}$ is non-empty.

The notion of fully closed mapping was introduced by V.V. Fedorčuk in [6].

A mapping $f:X\rightarrow Y$ is said to be *fully closed* if for each $y\in Y$ and each finite open cover $\{U_1, ..., U_s\}$ of $f^{-1}(y)$ the set $\{y\}\cup f^*(U_1)\cup ...\cup f^*(U_s)$ is a neighbourhood of y.

The space obtained by identifying to a point a closed subset A of a space X is denoted by X/A [4:127]. The natural mapping $q:X\rightarrow X/A$ is a simple fully closed mapping. Each fully closed mapping is a limit of an inverse system of simple fully closed mapping [6:Teorema 2.].

LEMMA 1.4 *Each fully closed mapping* $f:X\rightarrow Y$ *is closed.*

Proof. Let y be any point of Y and let U be any open set such that $f^{-1}(y)\subseteq U$. By the definition of fully closed mapping we infer that $V=\{y\}\cup f^*(U)$ is a neighbourhood of y. Now we have $V=f^*(U)$ since $f^{-1}(y)\subseteq U$. This means that V is an open set about y such that $f^{-1}(V)\subseteq U$. By [4:Theorem 1.4.13] it follows that f is closed.

LEMMA 1.5 [*1:356*]. *If* $f:X\rightarrow Y$ *is closed and irreducible and if U is an open subset of X, then* $f(ClU)=Clf^*(U)$.

LEMMA 1.6 [*19:70*]. *Let X be locally connected. If* $f:X\rightarrow Y$ *is closed and onto, then Y is locally connected.*

LEMMA 1.7 *Let $f:X\to Y$ be a fully closed irreducible mapping onto a space Y without local cut points. For each open $U\subseteq X$ one has $f(FrU)=Fr(f^*(U))$.*

Proof. By virtue of Lemmas 1.4 and 1.5 we have $f(FrU)\subseteq Fr(f^*U)$ since f is closed and irreducible mapping. In order to complete the proof it suffices to prove that $f(FrU)\supseteq Fr(f^*U)$. If $y\in Fr(f^*U)=Clf^*(U)\setminus f^*(U)$, then by Lemma 1.5 it follows that there is a $x\in ClU$ such that $y=f(x)$. On the other hand from the relation $y\notin f^*(U)$ it follows that $f^{-1}(y)\not\subseteq U$. If we suppose that the set $f^{-1}(y)\cap FrU$ is empty, then we have that $f^{-1}(y)=(f^{-1}(y)\cap U)\cup(f^{-1}(y)\cap(XClU))$. The sets $f^{-1}(y)\cap U$ and $f^{-1}(y)\cap(X\text{-}ClU)$ are open and disjoint. Now, we have the finite open cover $\{U,V\}$, $V=X\text{-}ClU$, of $f^{-1}(y)$. Thus, $W=\{y\}\cup f^*(U)\cup f^*(V)$ is a neighborhood of y.Moreover $f^*(U)$ and $f^*(V)$ are non-empty since f is irreducible. From the fact that U and V are disjoint it follows that $f^*(U)$ and $f^*(V)$ are disjoint open sets. Thus, the set $W\text{-}\{y\}$ is disconnected. This means that for each neighborhood W_1 of y there exist the open sets $U_1=f^{-1}(W_1)\cap U$ and $V_1=f^{-1}(W_1)\cap(X\text{-}ClU)$ such that $\{U_1, V_1\}$ is a finite open cover of $f^{-1}(y)$. Thus, $W_2=\{y\}\cup f^*(U_1)\cup f^*(V_1)$ is a neighborhood of y contained in W. Moreover, we have $f^*(U_1)\subseteq f^*(U)$ and $f^*(V_1)\subseteq f^*(V)$. This means that $W_2\text{-}\{y\}$ is disconnected. This is impossible since Y has no local cut points. Thus, the set $f^{-1}(y)\cap FrU$ is non-empty. The proof is completed.

The following theorem is the main theorem of this Section.

THEOREM 1.8 *Let $X=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system such that the projections f_α are irreducible fully closed mappings. In order that limX be locally connected it is necessary that each X_α be locally connected and it is sufficient that each X_α be a locally connected spaces without local cut points.*

Proof. *Necessity.* Apply Lemma 1.6.

Sufficiency. Suppose that lim**X** is not locally connected. By Remark 1.2b) it follows that there exists a infinite family $\{A_t:t\in T\}$ of open subsets A_t of lim**X** such that there is a point $x\in Fr(\cup\{A_t:t\in T\})\setminus Cl(\cup\{FrA_t:t\in T\})$. This means that there is an open $U_\alpha\subseteq X_\alpha$ such that $f_\alpha^{-1}(U_\alpha)\cap(\cup\{FrA_t:t\in T\})$ is the empty set. Clearly, $(U_\alpha)\cap f_\alpha(\cup\{FrA_t:t\in T\})$ is also empty set. By 1.7 we have $U_\alpha\cap(\cup\{f_\alpha FrA_t:t\in T\})=\emptyset$ and $U_\alpha\cap(\cup\{Frf_\alpha^*A_t:t\in T\})=\emptyset$. We infer that $f_\alpha(x)\notin Cl(\cup\{Frf_\alpha^*(A_t):t\in T\})$. On the other hand, for each neighbourhood V_α of $f_\alpha(x)$ and for $U=f_\alpha^{-1}(V_\alpha)$, from $x\in Fr(\cup\{A_t:t\in T\})$ it follows that each neighbourhood U of x meets $Int(\cup\{A_t:t\in T\})=\cup\{A_t:t\in T\}$ and $X\setminus Cl(\cup\{A_t:t\in T\})=W$, W is non-empty by 1.2.b). We define $V=U\cap W$ and $V_1=U\cap(\cup\{A_t:t\in T\})$.The sets $f_\alpha^*(V)$ and $f_\alpha^*(V_1)$ are non-empty since f_α is closed and irreducible. Clearly, V_α is a neighbourhood of $f_\alpha(x)$ which contains $f_\alpha^*(V)$ and $f_\alpha^*(V_1)$. Furthermore, $f_\alpha^*(V_1)$ meets some $f_\alpha^*(At)$, i.e., $f_\alpha^*(V_1)$ meets $\cap\{f_\alpha^*(A_t):t\in T\}$. Since $f_\alpha^*(V)$ is non-empty, we infer that V meets $X_\alpha\setminus(\{f_\alpha^*(A_t):t\in T\})$. This means that $f_\alpha(x)\in Fr(\cup\{f_\alpha^*(A_t):t\in T\})$. Finally, $f_\alpha(x)\in Fr(\cup\{f_\alpha^*(A_t):t\in T\})\setminus Cl(\cup\{Fr(f_\alpha^*(A_t)):t\in T\})$. This is impossible since X_α is locally connected. The proof is completed.

THEOREM 1.9 *Let $X=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with irreducible bonding mappings $f_{\alpha\beta}$. Then the projections $f_\alpha:limX\to X_\alpha$, $a\in A$, are irreducible.*

Proof. In order to prove that f_α is irreducible it suffices to prove that for each open non-empty $U\subseteq$lim**X** the set $f_\alpha^*(U)$ is non-empty. Let x be any point of U. By virtue of the definition of a base of lim**X** there is a $\beta\in A$ and open set $U_\beta\subseteq X_\beta$ such that $x\in f_\beta^{-1}(U_\beta)\subseteq U$. Let $\gamma\geq\alpha,\beta$. Then for open set $f_{\beta\gamma}^{-1}(U_\beta)=U_\gamma$ we have $f_\gamma^{-1}(U_\gamma)\subseteq U$. Moreover, the set $f_{\alpha\gamma}^*(U_\gamma)$ is non-empty since $f_{\alpha\gamma}$ is irreducible. Clearly, $f_\alpha^{-1}(f_{\alpha\gamma}^*(U_\gamma))\subseteq U$. This means that $f_\alpha^*(U)$ is non-empty. The proof is completed.

An inverse system$Y=\{Y_\alpha, g_{\alpha\beta}, B\}$ is said to be a *subsystem* of a system $X=\{X_\alpha, f_{\alpha\beta}, A\}$ if $B\subseteq A$, $Y_\alpha\subseteq X_\alpha$ and $g_{\alpha\beta}=f_{\alpha\beta}/Y_\beta$. If B is cofinal in A and $Y_\alpha=X_\alpha$ for each $\alpha\in B$, then lim**Y**= lim**X** [4:140]. Similarly, if B is cofinal in A and $Y_\alpha\subseteq X_\alpha$ for each $\alpha\in B$, then lim**Y** is homeomorphic to a subset of lim**X** [4:138,2.5.8. Theorem.].

We say that an inverse system $\mathbf{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ is an *N-system* if each subsystem $\mathbf{Y}=\{Y_\alpha, g_{\alpha\beta}, B\}$, $Y_\alpha \neq \emptyset$, Y_α closed in X_α, has a non-empty limit $\lim\mathbf{Y}$.

THEOREM 1.10 *Let $X=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with fully closed mappings $f_{\alpha\beta}$. If $\mathbf{X}$ is an N-system, then the projections $f_\alpha:\lim\mathbf{X}\to X_\alpha$, $\alpha\in A$, are fully closed.*

Proof. Let x_α be any point of X_α and let $\{U_1, ..., U_s\}$ be an open finite cover of a set $f_\alpha^{-1}(x_\alpha)$. Let us note that from the fact that $\mathbf{X}$ is N-system it follows that $f_\alpha^{-1}(x_\alpha)$ is non-empty. Namely, a subsystem $\{f_{\alpha\beta}^{-1}(x_\alpha), f_{\alpha\gamma}/f_{\alpha\gamma}^{-1}(x_\alpha), \alpha\leq\beta\leq\gamma\}$has a non-empty limit homeomorphic to $f_\alpha^{-1}(x_\alpha)$. By the definition of a base in $\lim\mathbf{X}$ we infer that for U_i, $1\leq i \leq s$, there is $\alpha(i)\in A$ and an open non-empty set $U_{\alpha(i)}$ in $X_{\alpha(i)}$ such that $f_{\alpha(i)}^{-1}(U_{\alpha(i)})\subseteq U_i$. We may assume that $U_{\alpha(i)}$ is maximal open set such that $f_{\alpha(i)}^{-1}(U_{\alpha(i)})\subseteq U_i$. Let $\beta \geq \alpha(1), ..., \alpha(s)$. There is a maximal open $U_\beta^i \supseteq f_{\alpha(i)\beta}^{-1}(U_{\alpha(i)})$ such that $f_\beta^{-1}(U_\beta^i)\subseteq U_i$. Moreover, $U_i=\cup\{U_\gamma^i:\gamma\geq\beta\}$. Suppose that $Y_\gamma = f_{\alpha\gamma}^{-1}(x_\alpha) - (\cup\{U_\gamma^i:1\leq i\leq s\})$ is non-empty for each $\gamma \geq \alpha, \beta$. Then, we have a subsystem $\mathbf{Y}=\{Y_\delta, f_{\delta\delta'}/Y_{\delta'}, \gamma \geq \delta \geq\delta'\}$. By the assumption that $\mathbf{X}$ is N-system, it follows that $\lim\mathbf{Y}$ is non-empty. Each point $y\in \lim\mathbf{Y}$ we shall identify with a point of $\lim\mathbf{X}$ such that $\lim\mathbf{Y}=\cap\{f_\gamma^{-1}(Y\gamma):\gamma \geq \alpha, \beta\}$. For each $y\in \lim\mathbf{Y}$ we have $y\in f_\alpha^{-1}(x_\alpha)$ and $y\in \lim\mathbf{X} - f_\alpha^{-1}(x_\alpha)$. This is impossible. Thus, there is a $\gamma\in A$ such that Y_γ is empty, i.e., $f_{\alpha\gamma}^{-1}(x_\alpha)=\cup\{U_\gamma^i:1\geq i\geq s\}$. Now, $U_\alpha=\{x_\alpha\}\cup f_{\alpha\gamma}^*(U_\gamma^1)\cup ... \cup f_{\alpha\gamma}^*(U_\gamma^s)$ is a neighbourhood of x_α. From the maximality of U_γ^i it follows that $f_\alpha^*(U_i)=f_{\alpha\gamma}^*(U_\gamma^i)$. Finally, we infer that $U=\{x_\alpha\}\cup f_\alpha^*(U_1)\cup ... \cup f_\alpha^*(U_s)$ is a neighbourhood of x_α, i.e., f_α is fully closed. The proof is completed.

REMARK 1.11 If in 1.10. the mappings $f_{\alpha\beta}$ are fully closed with compact fibers $f_{\alpha\beta}^{-1}(x_\alpha)$(i.e.fully closed and perfect), then see [6].

2. Applications of the main theorem

In this Section we apply Theorem 1.8. on the inverse systems with fully closed irreducible bonding mappings.

THEOREM 2.1 *Let $\mathbf{X}=\{X_n, f_{nm}, N\}$ be an inverse sequence such that the mappings f_{nm} are fully closed irreducible mappings and the spaces X_n are regular countably compact spaces. In order that $\lim\mathbf{X}$ be locally connected it is necessary that each X_n be locally connected and it is sufficient that each X_n be a locally connected space without local cut points.*

Proof. From Theorem 8. of [17] it follows that the projections $f_n:\lim\mathbf{X}\to X_n$, $n\in N$, are closed. Theorem 1.9. establishes that f_n is irreducible. Moreover, $\mathbf{X}$ is an N-system [13]. This means that f_n is fully closed. Theorem 1.8. completes the proof.

We say that a mapping $f:X\to Y$ is *perfect* if f is closed and each fiber $f^{-1}(y)$, $y\in Y$, is compact [4:236]. A mapping f is said to be *fully perfect* if f is perfect and fully closed.

THEOREM 2.2 *Let$X=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with fully perfect irreducible bonding mappings $f_{\alpha\beta}$. In order that limX be locally connected it is necessary that each X_α be locally connected and it is sufficient that each X_α be a locally connected space without local cut points.*

Proof. The projections f_α are fully perfect [6] and irreducible, [Theorem 1.9.]. Theorem 1.8. completes the proof.

COROLLARY 2.3 *Let $\mathbf{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of compact spaces X_α and fully closed irreducible mappings $f_{\alpha\beta}$. In order that $\lim\mathbf{X}$ be locally connected it is necessary that each X_α be locally connected and it is sufficient that each X_α be a locally connected space without local cut points.*

A topological space is a *q-space* [16] if fr each x∈ X there is a sequence $U_1, U_2,...$ of open sets such that $x_i \in U_i$, i∈N, with the property: if $x_n \in U_n$, $x_n \neq X_m$ for m≠n, then there is an accumulation point of $\{x_n: n\in N\}$.

LEMMA 2.4 *(15) .Let f:X→Y be a closed mapping of a normal space X onto T_1 q- space Y, then Fr $f^{-1}(y)$ is countably compact for each y∈ Y.*

COROLLARY 2.5 *If f in Lemma 2.5. is closed and irreducible and if X is T_1, then $f^{-1}(y)$ is countably compact for each y∈ Y.*

Proof. By [1:356, Exercise 112.] we have that $|f^{-1}(y)|=1$ or $Frf^{-1}(y)=f^{-1}(y)$. The proof is completed.

We say that a space X is *iso-compact* if each countably compact closed subspace Y⊆X is compact.

COROLLARY 2.6 *Let f:X→Y be a closed irreducible mapping of a normal T_1 - iso-compact space X onto a T_1 q-space Y, then $f^{-1}(y)$, y∈ Y, is compact i.e., f is perfect and irreducible.*

THEOREM 2.7 *Let* $\mathbf{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system of a T_1 normal iso-compact q-spaces with fully closed irreducible mappings. In order that lim***X** *be locally connected, it is neccessary that each X_α be locally connected and it is sufficient that each X_α be a locally connected space without local cut points.*

Proof. By 2.6. and 1.11., it follows that the projections f_α are fully closed. Apply Theorem 1.8.

COROLLARY 2.8 *Let* $\mathbf{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system of metric spaces X_α and fully closed irreducible mappings $f_{\alpha\beta}$. In order that lim***X** *be locally connected, it is necessary that each X_α be locally connected and it is sufficient that each X_α be a locally connected space without local cut points.*

Proof. A metric space X is a q-space since X is first-countable. A metric space X is iso-compact since a metric countably compact space X is compact [4:320]. Apply Theorem 2.7.

REMARK 2.9 Corollary 2.8. holds if we replace" metric" by" paracompact q-space" or by" first-countable paracompact".

LEMMA 2.10 *[15]. Let f:X→Y be a closed mapping of a normal space X onto a T_1-q- space Y. If $|f^{-1}(y)| \leq \aleph_0$, y∈ Y, then Fr $f^{-1}(y)$ is compact, for each y∈ Y.*

Proof. By 2.4. $Frf^{-1}(y)$ is countably compact. Since each countable countably compact space is compact, we infer that $Frf^{-1}(y)$ is compact, for each y∈ Y.

THEOREM 2.11 *Let* $\mathbf{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system of T_1-normal q-spaces X_a and of fully closed irreducible mappings $f_{\alpha\beta}$ with countable fibers $f_{\alpha\beta}^{-1}(x_\alpha)$, for each $x_\alpha \in X_\alpha$, β≥α. In order that lim***X** *be locally connected, it is necessary that each X_α be locally connected and it is sufficient that each X_α be a locally connected space without local cut points.*

Proof. Apply 2.10. and 2.2.

COROLLARY 2.12 *Let* $\mathbf{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system of T_1 - normal first-countable spaces X_α and of closed irreducible mappings with countable fibers $f_{\alpha\beta}^{-1}(x_\alpha)$. In order that lim***X** *be locally connected it is necessary that each X_α be locally connected and it is sufficient that each X_α be a locally connected space without local cut points.*

3. Concluding remarks

We close this paper with two lemmas.

LEMMA 3.1 *Let X be a normal space and let Y be a locally connected space without local cut points. If f:X→Y is a fully closed irreducible mapping, then f is monotone.*

Proof. Supose that for some point $y\in Y$ the set $f^{-1}(y)$ is not connected. This means that there is a pair of disjoint closed (in $f^{-1}(y)$) sets F_1, F_2 such that $f^{-1}(y)=F1\cup F2$. Clearly, the sets F_1 and F_2 are closed in X. There exist a pair U, V of disjoint open sets in X such that $F1 \subseteq U$ and $F_2 \subseteq V$. Now, we have a finite open cover {U, V} of $f^{-1}(y)$. A set $W=\{y\} \cup f^*(U)\cup f^*(V)$ is a neighbourhood of y. Moreover, $f^*(U)$ and $f^*(V)$ are non-empty since f is irreducible. From the fact that U and V are disjoint, it follows that $f^*(U)$ and $f^*(V)$ are disjoint open sets. Thus, the set $W\setminus\{y\}$ is disconnected. This means that for each neighborhood W_1 of y there exist open sets $U_1= f^{-1}(W_1)\cap U$ and $V_1 = f^{-1}(W_1)\cap V$ such that $\{U_1, V_1\}$ is a finite open cover of $f^{-1}(y)$. Thus, $W_2=\{y\}\cup f^*(U_1)\cup f^*(V_1)$ is a neighborhood of y contained in W. Moreover, we have $f^*(U_1)\subseteq f^*(U)$, $f^*(V_1)\subseteq f^*(V)$. This means that $W_2 - \{y\}$ is disconnected. This is impossible since Y is locally connected and has no cut-points. Thus the set $f^{-1}(y)$ is connected. The proof is completed.

LEMMA 3.2 *Let f:X→Y be a closed monotone irreducible mapping. If U⊆X is open, then f(FrU)=Fr(f*(U)).*

Proof. If $x\in FrU$, then $f(x)\in Clf^*(U)$. It is clear that $f(x)\notin f^*(U)$ since $f(x)\in f^*(U)$ implies $f^{-1}f(x)\subseteq U$, i.e, $x\in U$. This is impossible because $x\in FrU = ClU - U$. Thus, $f(FrU)\subseteq Fr(f^*U)$. In order to complete the proof it suffices to prove that $f(FrU)\supseteq Fr(f^*U)$. If $y\in Fr(f^*U)=Clf^*(U)-f^*(U)$, then by Lemma 1.5., it follows that there is a $x\in ClU$ such that $y=f(x)$. On the other hand, from the relation $y\notin f^*(U)$, it follows that $f^{-1}(y)\not\subseteq U$. If we suppose that the set $f^{-1}(y)\cap FrU$ is empty, then we have $f^{-1}(y)=(f^{-1}(y)\cap U)) \cup(f^{-1}(y)\cap(X-ClU))$. The sets $f^{-1}(y)\cap U)$ and $f^{-1}(y)\cap(X-ClU)$ are open and disjoint in $f^{-1}(y)$. This is impossible since $f^{-1}(y)$ is connected. Thus, $f^{-1}(y)\cap FrU$ is non-empty. The proof is completed.

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Lokalna povezanost inverznog limesa

Sadržaj

U radu je izučavana lokalna povezanost inverznog limesa inverznog sistema $X=\{X_\alpha, f_{\alpha\beta}, A\}$ s potpuno zatvorenim ireducibilnim projekcijama.

Prvi odjeljak sadrži glavni teorem rada, teorem 1.8., koji tvrdi da je limes takvog sistema s prostorima koji su lokalno povezani bez lokalnih prereznih točaka lokalno povezan. U teoremu 1.10. dan je dovoljan uvjet za potpunu zatvorenost projekcija f_α uz potpuno zatvorena vezna preslikavanja.

U drugom odjeljku dane su primjene glavnog teorema na razne inverzne sisteme.

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