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BILJEŠKA O PROJEKCIJAMA APROKSIMATIVNOG LIMESA

A NOTE ON THE PROJECTIONS OF AN APPROXIMATE LIMIT

The main purpose of this paper is to prove that if $\mathbf{X} = \{X_a, p_{ab}, A\}$ *is an approximate inverse system of locally compact topologically complete ($|A|$) - compact) spaces and perfect bonding mappings, then the projections are perfect. An example which shows that the local compactness cannot be omitted is also given.*
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Introduction

A space means a Tychonoff space and *a mapping* means a continuous (not necessarily surjective) mapping. The cardinality of a set X will be denoted by $|X|$.

Cov(X) is the set of all normal coverings of a topological space X. For other details see [1].

In this paper we study the approximate inverse system in the sense of S. Mardešić [6].

DEFINITION 1.1. *An approximate inverse system is a collection* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *where* $(A,\leq)$ *is a directd preordered set,* X_a, $a\in A$, *is a to pological space and* $p_{ab}:X_b \to X_a$, $a \leq b$, *are mappings such that* $p_{aa} = id$ *and the following condition (A2) is satisfied:*

(A2) For each $a\in A$ *and each normal cover* $\mathcal{U} \in Cov(X_a)$ *there is an index* $b\geq a$ *such that* $(p_{ac}p_{cd}, p_{ad}) \prec \mathcal{U}$, *whenever* $a\leq b\leq c\leq d$.

Other basic notions, including approximate mapping, the limit of an approximate inverse system and approximate resolution are defined as in [6] and [7].

The inverse systems in the sens [3, p. 135.] we call *a usual inverse systems.*

The main theorem

A topological space (X,t) is said to be *topologically complete* (od Dieudonné complete [3, p. 568.] iff there is a uniformity $\mathcal{U}$ such that (X,$\mathcal{U}$) is complete and topology t is the $\mathcal{U}$-uniform topology. A topological space X is topologically complete iff X is a Tychonoff space and the universal uniformity Cov(X) [3, pp. 536, 568] on the spece X is complete. Each paracompact space is topologically complete. Topological completness is hereditary with respect to closed subsets and is multiplicative. The limit lim**X** of an approximate system **X** of topologically complete spaces is topologically complete [7, Theorem (1.17)].

A space X is called *m-compact* [10, p. 177.] (where m is a infinite cardinal number) provided either of the following conditions holds: (i) for every filter-base F on X, if$|F|$ $\leq$m, then $ad_x F = \bigcap\{ClF : F \in F\} \neq \emptyset$,or (ii) every open cover $\mathcal{U}, |\mathcal{U}| \leq$m, has a finite subcover. $\aleph_0$-compact space are called *countably compact*. It is obvious that each m-compact space is countably compact.

A continuous mapping f : X → Y is *perfect* if X is a Hausdorff space, f is a closed mapping and all fibers $f^{-1}(y)$ are compact subsets of X [3, p. 236].

Let X be a locally compact space. By αX the Alexandroff compactification [3, p.222] of X is denoted. The remainder αX\X by ω_x is denoted. Thus, $\alpha X = X \bigcup \{\omega_x\}$.

Let X and Y be locally compact spaces: If f : X → Y is a surjective perfect mapping, then there exists and extension αf: αX → αY of f such that $\alpha f(\omega_x) = \omega_y$ [2, p. 160].

Let f,g:X → Y be mappings between Tychonoff spaces. If cX and cY are arbitrary Tychonoff extensions of X and Y such that cf and cg exist, and if f and g are $\mathcal{V}|$Y-near, then cf and cg are $\mathcal{U}$-near, where $\mathcal{U}$, $\mathcal{V} \in$ Cov(Y), st$\mathcal{V}<\mathcal{U}$ (Lemma 2.8 [4]). If **X** = $\{X_a, p_{ab}, A\}$ is an approximate inverse system of Tychonoff spaces X_a such that cp_{ab} exist, then c**X**= $\{cX_a, cp_{ab}, A\}$ is an approximate inverse system. The proof is the same as the proof of Lemma 2.9 of [4]. Let **X** = $\{X_a, p_{ab}, A\}$ be an approximate inverse system of locally compact noncompact spaces and surjective perfect mappings p_{ab}. If αX_a is the Alexandroff compactification of X_a with one-point remainder ω_a, then for each, a,b∈A there is the extension αp_{ab} of p_{ab} such that $\alpha p_{ab}(\omega_b) = \omega_a$. Thus, we have an approximate inverse system α**X**= $\{\alpha X_a, \alpha p_{ab}, A\}$.

In the sequel the projections of lim**X** into X_a by p_a are denoted. Similarly, the projections of limα**X** into αX_a by P_a are denoted.

The following theorem is the main theorem of this section.

THEOREM 2.1 *Let X = $\{X_a, p_{ab}, A\}$ be an approximate inverse system of non-empty locally compact noncomapct topologically complete (|A| - compact) speces and surjective perfect bonding mappings. Then the projections pa : X → X_a, a∈A, are surjective and perfect and X = lim**X** is a non-empty topologically complete (|A| - compact) locally compact space. Moreover, αlim**X** is homeomorphic to limα**X**.*

Proof. The proof is broken into several steps.

Step 1. We consider the approximate system α**X** = $\{\alpha X_a, \alpha p_{ab}, A\}$. The bonding mappings $\alpha p_{ab} : \alpha X_b \to \alpha X_a$ are onto mappings such that $\alpha p_{ab}(\omega_b) = \omega_a$. This means that the projections P_a : limα**X** → α$\mathbf{X}_a$ are onto [7, Corollary 4.5.].

Step 2. *The projections p_a, a∈A, are onto.* Let x_a be any point of X_a. By virtue of Step 1. there exists a point x∈Y = limα**X** such that $P_a(x) = x_a$. From the definition of the thread it follows that there exists a b≥a such that for c≥ b $\alpha p_{ac}P_c(x)$ is in X_a since X_a is open in αX_a. We infer that $P_c(x) \in X_c$ since $\alpha p_{ac}(\omega_c)=\omega_a$. Let C = {c : c≥b}. Then C is cofinal in A and $\{P_c(x) : c\in C\}$ is a thread in the approximate inverse system $\{X_c,p_{cd}, C\}$. If the spaces X_b, b∈A, are topologically complete, then by virtue of Theorem 1.19 of [7] the thread $\{P_c(x) : c\in C\}$ induces a thread in **X** = $\{X_a,p_{ab}, A\}$. If the spaces X^b, b∈A, are |A| - compact, then for each b∈A, $\mathcal{N}= \{p_{bc}(P_c(x)) : c\in C\}$ is a Cauchy net in X_b [7, p. 597]. There exists a cluster point $x_b \in X_b$ of the net $\mathcal{N}$ since X_b is |A| - compact. On the other hand, $\mathcal{N}$ is convergent in αX_b. It follows that $\mathcal{N}$ converges to x_b. In both cases we infer that x is a thread in **X** = $\{X_a,p_{ab}, A\}$. Since this is true for each x with $P_a(x) = x_a$, we infer that $P_a^{-1}(x_a)$ is a compact non-empty subset of X. Clearly, $P_a^{-1}(x_a) = p_a^{-1}(x_a)$. This means that p_a, a∈A, are surjective.

Step 3. *The projections p_a, a∈A, are perfect.* From Step 2. it follows that $P_a^{-1}(X_a)$ = X = lim**X** is non-empty. Since $p_a=P_a|$lim**X** and P_a is closed, it follows from [3], Proposition 2.1.4] that p_a is closed and, consequently, perfect.

Step 4. Local compactness of lim**X** follows from Theorem 3.7.24 of [3]. From [7, Theorem (1.17)] it follows that lim**X** is topologically complete. Similarly, by a straightforward modification of the proof of Theorem 3.7.2 [3] we obtain that lim**X** is |A| - compact.

Step 5. *Let us prove that α*lim**X** *is homeomorphic to* lim α**X**. It is clear that lim α**X** is a compactification of lim**X** since the projections p_a are onto. Let us prove that the remainder Z = lim α**X**\lim**X** is non-empty and contains a single point. By virtue of $p_{ab}(\omega_b) = \omega_a$ it follows that $\omega = (\omega_a : a\in A)$ is a point of Z, i.e., Z is non-empty. Suppose that there exists a point $z\in Z\backslash\{\omega\}$. There exists an a∈A such that $p_a(z) \neq p_a(\omega) = \omega_a$. We infer that $p_a(z)$ is in X_a. This is imposible since $P_a^{-1}(X_a)$ = lim**X** (see Step 2). It is known that αlim**X** is the smallest element of the family *C* (lim**X**) of all compactifications of lim**X** [3, p. 222, Theorem 3.5.11]. This means that there is a continuous mapping f: limα**X** → αlim**X** such that f(ω) =Ω and f(x) = x for each x∈lim**X** [3, p. 222], where Ω is the remainder αlim**X**lim\lim**X**. It is clear that f is a homeomorphism. The proof is completed.

If **X**= $\{X_a,p_{ab},A\}$ is a usual inverse system of T_2-spaces and perfect bonding mappings, then the projections are perfect [9, Theorem 7]. The following example shows that the local compactness cannot be omitted in Theorem 2.1.

EXAMPLE 2.2. Let $\mathbb{R}^2$ be the Euclidean plane endowed with the ordinary rectangular coordinate system Oxy. We define the space X as the union of the subsets $I_1,I_2,\ldots,I_n,\ldots I_\infty$, such that, for each n∈$\mathbb{N}$, n ≥1,

$$I_n=\left\{(x,1-\frac{1}{n}) : 0\le x\le\frac{1}{n}, x \text{ is a rational number}\right\}$$

and

$$I_\infty = \{(0,1)\}.$$

It is clear that X is not locally compact. For each $n \in \mathbb{N}$ we define the homeomorphism $h_n : X \to X$ as follows. If $(x, y) \in I_{n+1} \cup \ldots \cup I_\infty$, then $h_n(x,y) = (x,y)$. For $(x,y) = (x, 1 - 1/m) \in I_m$, $1 < m \leq n$, let $h_n(x,y) = (z, 1 - 1/(m - 1)) \in I_{m-1}$ such that the point (0,1, (x, 1 - 1/m) and (z, 1 - 1(m - 1)) are collinear. This means that

$$z : x = \left(1 - \frac{1}{m-1}\right) : \left(1 - \frac{1}{m}\right)$$

or

$$z = \frac{m(m-2)}{(m-1)^2} x.$$

Thus

$$h_n(x,y) = \left(\frac{m(m-2)}{m-1^2} x, 1 - 1/(m-1)\right).$$

Finally, let $h_n(x,0) = ((n - 1)x/n, 1 - 1/n)$, for $(x, 0) \in I_1$.

Let $\mathbf{X} = \{X_n, p_{nm}, \mathbb{N}\}$ be an approximate inverse sequence such that $X_n = X$ for each $n \in \mathbb{N}$. The bonding mapping $p_{nm}: X_m \to X_n$ is defined by $p_{nm}((x,y)) = (x,y)$ for each $m \geq n > 1$ and $n = m = 1$. The mappings p_{1n}, $n \in \mathbb{N}$, $n > 1$, are defined such that $p_{1n}(x, y) = h_n(x,y)$ for each $(x,y) \in X_n$.

Let us prove that **X** satisfies (A2). It suffices to prove that (A2) is satisfied for each normal cover $\mathcal{U}$ of X_1. Let U be a member of $\mathcal{U}$ which contains the point (0,1). There exists a ε- ball B ((0,1), ε) about the point (0,1) such that $B((0,1), \varepsilon) \subseteq U$. The following claim is obvious.

Claim. *There exists a $n_0 \in \mathbb{N}$ such that $I_n \subseteq B((0,1), \varepsilon)$ for each $n \geq n_0$.*

Nov, we prove that (A2) is satisfied for $\mathcal{U}$. We consider the following cases.

1. Let $(x,0) \in I_1 \subseteq X_m$ and let $n_0 \leq m \leq n$. Then $p_{1n}(x,0) \in I_n$ and $P_{mn}(x,0) = (x,0) \in X_m$. Hence $p_{1m}p_{mn}(x,0) = p_{1m}(x,0) \in I_m$. We infer that $p_{1n}(x,0)$ and $p_{1m}p_{mn}(x,0)$ are in $B((0,1), \varepsilon) \subseteq U$ since I_m and I_n are subsets of $B((0,1), \varepsilon)$ for $m, n \geq n_0$ (see Claim). Thus, $p_{1n}(x,0)$ and $p_{1m}p_{mn}(x,0)$ are in some member of $\mathcal{U}$.
2. Let (x,1 - 1/k) be a point of I_k, $1 < k \leq m$. Nov, $p_{1n}(x,1 - 1/k) = \frac{m(m-2)}{(m-1)^2} x, 1 - 1/(k-1))$ and $p_{1m}p_{mn}(x,1 - 1/k) = p_{1m}(x,1 - 1/k) = \frac{m(m-2)}{(m-1)^2}$ x, 1 – 1/(k - 1)) = $p_{1n}(x,1 - 1/k)$. Each member of $\mathcal{U}$ which contains $p_{1m}p_{mn}(x,1-1/k)$ contains p_{1n} (x,1 - 1/k).
3. If $(x,1 - 1/k) \in I_k$, $m < k \leq n$, then $p_{1n}(x,1 - 1/k) = (h_n(x),1 - 1/(k - 1))$ and $p_{1m}p_{mn}(x,1-1/k) = p_{1m}(x,1 - 1/k) =$ (hm(x),1- 1/k). This means that $p_{1n}(x,1- 1/k) \in I_{k-1} \subseteq B((0,1), \varepsilon)$ and $p_{1m}p_{mn}(x,1 - 1/k) \in I_k \subseteq B((0,1), \varepsilon)$, i.e., the points $p_{1n}(x,1 - 1/k)$ and $p_{1m}p_{mn}$(x,1 1/k) are in $U \in \mathcal{U}$.
4. If $(x1 - 1/k) \in I_k$, $k > n$, then $p_{1n}(x,1 - 1/k) = (x,1 - 1/k)$ and $p_{1m}p_{mn}(x,1 - 1/k) = (x,1 - 1/k)$. Thus, $p_{1m}p_{mn}(x,1 - 1/k) = p_{1n}(x,1 - 1/k)$.

5. Finally, if $(x,1) \in I_\infty$ then we have again that $p_{1m}\, p_{mn}\,(x,1) = p_{1n}(x,1)$.

All these imply that (A2) is satisfied for $\mathcal{U}$.

The limit of the sequence **X** *is the space* X. This follows from the fact that the sequence **X** has the subsequence $\{X_n, p_{nm}, 1<n \leq m< \infty\}$ which is a usual inverse sequence with limit homeomorphic to X. Now, applying [7, Theorem (1.19)] we conclude that lim**X** is homeomorphic to X. This means that for each point $z = (z_n) \in \lim\mathbf{X}$ we have $z_n = (x,y)$, $n \geq 1$, where (x,y) is some point of X. Moreover, $p_{mn}(z_n) = z_m$ for $m,n >1$. This means that only $z_1 = \lim\{p_{1n}(z_n) : n >1\}$. This is true iff $z = (x,0) \in \lim \mathbf{X}$. One can readilly see that if $0 \leq x \leq 1$, then $\{z_n : n >1\}$ converges to $(0,1) \in X_1$. We infer that $p_1^{-1}(0,1) = \{(0,1)\} \bigcup \{(x,0) : 0 \leq x \leq 1$, x is a rational number$\}$. This means that $p_1^{-1}(0,1)$ is not compact since $\{(x,0) : 0 \leq x \leq 1\}$ contains only the rational numbers. Thus, p_1 is not perfect.

QUESTION. Is it true that topological completeness (|A| - compactness) cannot be omitted in Theorem 2.1?

LEMMA 2.3. *Let* $X = \{X_a, p_{ab}, A\}$ *be an approximate inverse system as in Theorem 2.1. Then for each pair F,G of disjoint compact subsets of* $X = \lim\mathbf{X}$ *there exists an* $a \in A$ *such that* $p_b(F) \bigcap p_b(G) = \emptyset$ *for each* $b \geq a$.

Proof. Consider the approximate inverse system $\alpha\mathbf{X} = \{\alpha X_a, \alpha p_{ab}, A\}$. The sets F and G are closed in $\lim\alpha\mathbf{X} = \alpha\lim\mathbf{X}$. By virtue of [4, Lemma 2.17] there exists an $a \in A$ such that $P_b(F) \bigcap P_b(G) = \emptyset$, $b \geq a$. It follows that $p_b(F) \bigcap p_b(G) = \emptyset$. The proof is completed.

We close this section by the following theorem.

THEOREM 2.4 *Let* $\mathbf{X} = \{X_n, p_{mn}, \mathbb{N}\}$ *be an approximate inverse sequence of separable locally compact metric spaces with surjective perfect bonding mapping. Then there exist:*

a) *a cofinal subset* $M = \{n_i : i \in \mathbb{N}\}$ of $\mathbb{N}$,

b) *an usual inverse sequence* $\mathbf{Z} = \{Z_i, q_{ij}, M\}$ *such that* $Z_i = X_{ni}$, *and* $q_{ij} = p_{n_i n_{i+1}} p_{n_{i+1} n_{i+2}} \cdots p_{n_{j-1} n_j}$ *for each* $i,j \in \mathbb{N}$,

c) *a homeomorphism* $h : \lim\mathbf{X} \to \lim \mathbf{Z}$.

Proof. Now, $\mathbf{X} = \{\alpha X_n, \alpha p_{nm}, \mathbb{N}\}$ is an approximate inverse sequence of compact metric spaces since $w(\alpha X_n) = w(X_n)$ [3, p.222, Theorem 3.5.11]. By virtue of Theorem 2.11 [5] there exists:

A) a cofinal subset $M = \{n_i : i \in \mathbb{N}\}$ of $\mathbb{N}$,

B) an usual inverse sequence $\mathbf{Y} = \{\alpha Y_i, Q_{ij}, M\}$ such that $Y_i = X_{ni}$ and $Q_{ij} = \alpha p_{n_i n_{i+1}} \alpha p_{n_{i+1} n_{i+2}} \cdots a p_{n_{j-1} n_j}$ for each $i,j \in \mathbb{N}$,

C) a homeomorphism $H : \lim\alpha\mathbf{X} \to \lim\mathbf{Y}$.

such that if $x = (x_n) \in \lim\alpha\mathbf{X}$, then $H(x) = (y_{n_i}) \in \lim\mathbf{Y}$ with $y_{n_i} = \lim \{Q_{ik} p_{n_k}(x) : k \in \mathbb{N}\}$. If $x = (\omega_n)$, then $y_{n_i} = \lim \{Q_{ik} p_{n_k}(\omega_{n_k}) : k \in \mathbb{N}\} = \lim \omega_{n_i} : k \in \mathbb{N}\} = \omega_{n_i}$ since $Q_{ik} p_{n_k}(\omega_{n_k}) = \omega_{n_i}$. Thus H: $\lim\alpha\mathbf{X} \to \mathit{lim}\mathbf{Y}$ is the homeomorphism which maps the remainder of $\alpha\lim\mathbf{X}$ on the remainder of $\alpha\lim\mathbf{Y}$. Let $h = H|\lim\mathbf{X}$. Then $h : \lim\mathbf{X} \to \lim\mathbf{Z}$ is a homeomorphism. The proof is completed.

Applications

We start with the following lemma.

LEMMA 3.1. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of normal spaces with limit X. Let F be a closed subset of X_{a_0} and let U be any open neighbourhood of F. Then there exists :*

a) *a normal cover $\mathcal{U}_1$ of X_{a_0} such that st $\mathcal{U}_1 < \mathcal{U}_0 = \{U, X\backslash F\}$,*

b) *a set C of the pairs (a, $\mathcal{V}$) where $\mathcal{V} \in Cov(X_{a0})$ and $st^2\mathcal{V} < U_1$,*

c) *a family $\mathcal{W} = \{W_c : c \in C\}$ of subset of X which is directed by inclusion such that*

d) $p_{a_0}^{-1}(F) = \bigcap\{W_c : c \in C\}$.

Proof. It is clear that $\mathcal{U}_0$ is a normal cover of X_{a0} since it is a finite cover of the normal space X_{a0} [3, p. 379]. It follows that $\mathcal{U}_1$ exists. We define C as the set of all pairs $C = (a, \mathcal{V})$, where $\mathcal{V} \in Cov(X_{a0})$ $st^2\, \mathcal{V} < \mathcal{U}_1$ and *a* satisfies the equation

$$(p_{a_0a_1}, p_{a_1a_2}, p_{a_0a_2}) < \mathcal{V} \quad \text{for } a_2 > a_1 > a. \tag{1}$$

We define Z_c by

$$Zc = Cl\left[p_{a_0a}^{-1}(st(F, st\mathcal{V}))\right] \subseteq X_a\,,$$

and W_c by

$$Wc = p_a^{-1}(Z_c).$$

Now, we shall prove that the family $\mathcal{W} = \{Wc : c \in C\}$ is directed by inclusion, i.e. that for each finite family $\{W_{ci}, i = 1, ..., n\}$ of the members of $\mathcal{W}$, where $c_i = (a_i, \mathcal{V}_i)$, there exists a member $W_c \in \mathcal{W}$ such that $W_c \subseteq W_{ci}$, i = 1, ..., n. Let $\mathcal{V} = \wedge\{\mathcal{V}_i : 1 = 1, ..., n\}$ be the intersection cover [1, p. 13], i.e. the family of all $V_1 \cap ... \cap V_n$, where $V_i \in \mathcal{V}_i$, i = 1, ..., n. Let $\mathcal{V}'$ be a normal cover of X_{a0} such that st $\mathcal{V}' < \mathcal{V}$. Consider the set D of all pairs $d = (a', \mathcal{V}') \in C$. We claim that $p_{a'}(W_d) \subseteq Z_{ci}$. Let x be any point of $p_{a'}^{-1}(p_{a0a'}^{-1}(st(F, st\mathcal{V}')))$. This means that $p_{a'}(x)\, p_{a_0a'}^{-1}(st(F, st\mathcal{V}'))$. We infer that there exists a $V' \in st\mathcal{V}'$such that $p_{a0a'}p_{a'}(x) \in \mathcal{V}'$ and $F \cap V' \neq \emptyset$. Since st $\mathcal{V}' < \mathcal{V}_i$, for each i = 1, ..., n, we infere that there exist a $V_1^{\,i} \in \mathcal{V}_i$, i = 1, ..., n, such that $V' \subseteq V_1^{\,i}$, i = 1, ..., n.

Hence, $p_{a0a'}p_{a'}(x) \in \mathcal{V}_1^i$, $F \cap V_1^i \neq \emptyset$, i = 1, ..., n. It is clear that $V_1 = V_a^i \cap ... \cap V_1^n$ is a member of $\mathcal{V}$. Since a'>a, it folows from (1) that there exists a $V_2^i \in \mathcal{V}$ such that $p_{a0a'}p_{a'}(x)$, $p_{a0a_i}p_{a_ia'}p_{a'}(x)$ V_2^i, i = 1, ..., n. We infer that $p_{a0a_i}p_{a_ia'}p_{a'}(x) \in st(F, st\,\mathcal{V}_i)$. This means that

$$p_{a_ia'}p_{a'}(x) \in p_{a_0a_i}^{-1}(st(F, st\mathcal{V}_i)), \tag{2}$$

i.e.,

$$p_{a_ia'}(p_{a_0a'}^{-1}(st(F, st\mathcal{V}')) \in p_{a_0a'}^{-1}(st(F, st\mathcal{V}_i), \ i = 1, ..., n. \tag{3}$$

Let d= (a', $\mathcal{V}$') ∈D be such that (AS) [7, p. 592] is satisfied for each a_i and each normal cover $p_{a_0a_i}^{-1}(\mathcal{V}_i)$, i=1, ..., n. This means that

$$(p_{a_i}, p_{a_ib}p_b) < p_{a_0a_i}^{-1}(\mathcal{V}_i), \ i = 1, ..., n, \ b \geq a'. \tag{4}$$

From (2) it follows that there exists a pair $V_i^1, V_i^2 \in p_{a_0a_i}^{-1}\mathcal{V}_i$ such that $p_{a_ia'}p_{a'}(x) \in V_i^2$, $V_i^2 \cap V_i^1 \neq \emptyset$ $p_{a_0a_i}^{-1}(F) \cap V_i^1$. Then from (4) it follows that there exists a $V_i^3 \in p_{a_0a_i}^{-1}\mathcal{V}_i$ such that $p_{a_ia'}p_{a'}(x)$, $p_i(x) \in V_i^3$ and V_i^3 intersects V_i^2, i = 1, ..., n. We infer that $p_{a_i}(x) \in p_{a_0a_i}^{-1}(st(F, st\,\mathcal{V}_i))$, i = 1, ..., n. This means that $x \in p_{a_i}^{-1}(p_{a_0a_i}^{-1}(st(F, st\,\mathcal{V}_i)))$. From the continuity of the projections p_a it follows $W_d \subseteq \bigcap\{W_{ci} : i = 1, ..., n\}$. Finally, let us prove d). If $x \in \bigcap\{Wc : c \in C\}$, then $pc(x) \in Z_c$. This means that $p_{a0a}p_a(x) \in st(F, st\mathcal{V})$, where c = (a, $\mathcal{V}$). From (4) it follows that $p_{a0}(x) \in st(f, stV)$. Since this is true for each normal cover, we infer that $p_{a0}(x) \in F$, i.e., $x \in p_{a_0}^{-1}(F)$. Thus, $p_{a_0}^{-1}(F) \supseteq \bigcap\{W_c : c \in C\}$. Conversely, if $x \in p_{a_0}^{-1}(F)$, then $p_{a0}(x) \in st(F, st\mathcal{V})$ for each $\mathcal{V}$. By (4) it follows that $p_a(x) \in Z_c$, for each c=(a,st$\mathcal{V}$). This means that $x \in \bigcap\{W_c : c \in C\}$, i.e., $p_{a_0}^{-1}(F) \subseteq \bigcap\{Wc : c \in C\}$. Finally, we have $p_{a_0}^{-1}(F) = \bigcap\{W_c : c \in C\}$, as desired. The proofs is completed.

THEOREM 3.2 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of locally compact noncompact topologically complete (|A| - compact) spaces and surjective perfect bonding mappings. If all spaces* X_a *are locally connected and if all bonding mappings* p_{ab} *are monotone, then the projections* $p_a : X \to X_a$, $a \in A$, *are monotone and* $X = lim\mathbf{X}$ *is a locally connected space.*

Proof. Let $a_0 \in A$ and let x_{a0} be any point of X_{a0}. Let us prove that $p_{a_0}^{-1}(x_{a0})$is connected. Let F = $\{x_{a0}\}$. Applyng Lemma 3.1 we see that $p_{a_0}^{-1}(x_{a0}) = \bigcap\{W_c : c \in C\}$. By virtue of the local connectedness of X_{a0} we may assume that each st(F, st*V*) is connected. This means that each Z_c is connected since every p_{a0a} is perfect and monotone. Now, suppose that $p_{a_0}^{-1}(x_{a0}) = \bigcap\{Wc : cC\}$ is disconnected. This means that there exists disjoint closed sets G, H such that $p_{a_0}^{-1}(x_{a0}) = G \cup H$. By virtue of Theorem 2.1 the set $p_{a_0}^{-1}(x_{a0})$ is compact. We infer that G and H are compact. Thus, there exist the open subsets U, V of X such that $G \subseteq U$, $H \subseteq V$ and $ClU \cap ClV = \emptyset$ because of local compactness of X. Moreover, we may assume that ClU and ClV are compact. Since

$G \cup H = \bigcap\{W_c : c \in C\}$ and the family $\{Wc : c \in C\}$ is directed by inclusion, it follows that there exists a $c_1 \in C$ such that $W_c \subseteq U \cup V$ and $W_c \cap U \neq \varnothing W_c \cap V$ for each $c \in C$ with $W_c \subseteq W_{c1}$. By virtue of Lemma 2.3 there exists an $a \in A$ such that $p_b(\text{Cl}U) \cap p_b(\text{Cl}V) = \varnothing$ for each $b \geq a$. Let b and V be such that $c = (b, V) \geq c_1$. Then $p_b(\text{Cl}U)$ and $p_b(\text{Cl}V)$ are disjoint closed subsets of X_b which contains Z_c and both intersects Z_c. This is impossible since Z_c is connected. Hence, $p_{a_0}^{-1}(x_{a_0})$ is connected. At the end of the proof, let us prove that X is locally connected. Let x be any point of X and let U be a neighbourhood of X. By virtue of the definition of a base in X, there exists an $a \in A$ and an open set U_a containing $p_a(x)$ such that $x \in p_a^{-1}(U_a) \subseteq U$. By virtue of the local connectednes of X_a we may assume that U_a is connected. This means that $p_a^{-1}(U_a)$ is connected [3, p. 441, Theorem 6.1.29] since p_a is perfect and monotone. Since, $x \in p_a^{-1}(U_a)$, we infer that X is locally connected. The proof is completed.

The following example shows that the local connectedness of the spaces X_a in Theorem 3.2 cannot be omitted.

EXAMPLE 3.3 Let X be the space as in the example 2.2 and let Y be the closure of X in $\mathbb{R}^2$. This space is locally compact, topologically complete but not locally connected at the point (0,1). Set $Y_n = Y$ and $\mathbf{Y} = \{Y_n, q_{mn}, \mathbb{N}\}$, where q_{mn} are defined by the same equations as p_{mn} in the example 2.2. As in 2.2. we infer that $q_1^{-1}(0,1) = \{(0,1)\} \cup \{(x,0) : 0 \leq x \leq 1\}$. Thus, $q_1^{-1}(0,1)$ is not connected and q_1 is not monotone.

REFERENCES

[1] Alò R.A. and Shapiro H.L., *Normal topological spaces,* Cambridge Univ. Press, 1974.
[2] Burbaki N., *Obščaja topologija (osnovnye struktury)*, Nauka, Moskva, 1968.
[3] Engelking R., *General Topology*, PWN, Warszawa, 1977.
[4] Lončar I., *Approximate limits of m-compact spaces*, Glasnik Matematički 30 (50) (1995), 73 - 84.
[5] Lončar I., *A note on approximate limits*, Zbornik radova Fakulteta organizacije i informatike 19(1995), 1 - 21.
[6] Mardešić S., *On approximate inverse systems and resolutions*, Fund. Math 142(1993), 241-255.
[7] Mardešić S. and Watanabe T., *Approximate resolutions of spaces and mappings*, Glasnik Mat. 24(4) (1989), 587-637.
[8] Matijević V., *A note on limit and resolution of (nongauged) approximate inverse systems*, Glasnik Matematički 28 (48) (1993), 111 - 122.
[9] Puzio E., *Limit mappings and projections of inverse systems*, Fund. math. 80(1973), 57 - 73.
[10] Stephensen R.M., *Products of initially m-compact spaces*, Transactions of the Amer. Math. Soc. 196(1974), 177-189.

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SAŽETAK

Ako je $\mathbf{X} = \{X_a, p_{ab}, A\}$ običan inverzni sistem T_2 - prostora sa savršenim veznim preslikavanjima, tada su i projekcije $p_a : \lim \mathbf{X} \to X_a$ savršene [9, Theorem 7]. Primjerom 2.2 dokazuje se da to nije istina za aproksimativne inverzne sisteme. Glavni teorem rada (Theorem 2.1) tvrdi da su projekcije p_a savršene ako su prostori X_a lokalno kompaktni topološki kompletni (A - kompaktni) a vezna preslikavanja savršena. Spomenuti primjer 2.2 pokazuje da se lokalna kompaktnost u teoremu 2.1 ne može izostaviti. Autoru nije poznato da li se topološka kompletnost (|A| - kompaktnost) može izostaviti.

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