CROATICA CHEMICA ACTA CCACAA **66** (1) 59–80 (1993)

ISSN 0011–1643
CCA—2118

Original Scientific Paper

Enumeration of Directed Column-Convex Animals with a Given Perimeter and Area

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Received May 5, 1992

We give exact formulae for the number of directed column-convex animals with a given bond perimeter and area. The proof is based upon bijections and combinatorics related to the algebraic language theory.

INTRODUCTION

An *animal* is a set of points of $\mathbf{Z} \times \mathbf{Z}$ such that every point of the animal can be reached from another point by a sequence of points of the animal such that any two consecutive points in the sequence are connected by a unit step in the lattice plane. Animals are counted up to a translation.

Finding the exact formula for the number a_n of animals having the n point is important in statistical physics and chemistry. It is also a major problem in combinatorics. In physics, it is connected to the percolation theory, see for example Ref. 16. In combinatorics, animals are called *polyominoes*. These objects are obtained from animals by substituting to each point of the animal a unit square with vertices at integer points (see Figure 1). They have been intensively studied. See, for example, Golomb,[8] Klarner,[11,12] Bender,[1] and Klarner, Rivest.[13]

An animal is a *directed* one if it contains the point (0,0), called the *source point*, such that any other point in the animal can be reached from the source point by a sequence of points of the animal such that any two consecutive points in the sequence are connected by a north or east step in the lattice plane. In physics, this model is related to the *directed percolation* problem,[16] some lattice gas model and the Lee-Yang edge singularity problem; see a survey in Viennot.[20] Recently, exact results were given by Nadal, Derrida and Vannimenus,[14] Hakim, Nadal[10] and Dhar.[6,7] A combinatorial proof for these results and some new ones were given by Gouyou-Beauchamps, Viennot,[9] and Viennot.[20]

On the other hand, enumerative results were found for some classes of polyominoes: the convex and column-convex polyominoes.[5,3] A *column* (resp. *row*) of a

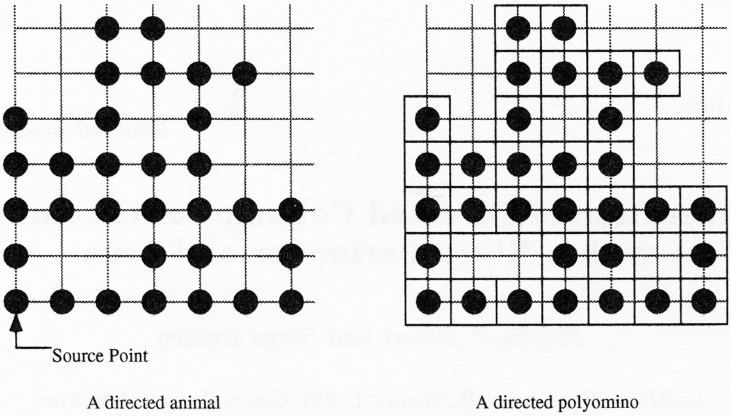


Figure 1. Correspondance between directed animals and polyominoes

polyomino is any infinite vertical (resp. horizontal) strip of unit squares. A *column-* (resp. *row-*) *convex* polyomino is a polyomino such that all its columns (resp. *rows*) are connected. A *convex* polyomino is both row and column convex. The enumeration is done according to the three following parameters:

– the *bond perimeter*, that is the length of the border of the polyomino A, $p(\mathrm{A})$,

– the *site perimeter*, that is the number of unit squares of the outside along the boundary of the polyomino A, $s(\mathrm{A})$,

– the *area*, that is the number of unit squares of the polyomino, $r(\mathrm{A})$.

The two perimeters are of such interest in physics[15] because it seems that they are in the same »*class of universality*« that is

$$p(\mathrm{A}) \approx (\alpha_p)^n n^{-\theta_p} \text{ and } s(\mathrm{A}) \approx (\alpha_s)^n n^{-\theta_s}$$

with $\theta_p = \theta_s$.

In Ref. 5, Delest and Viennot give an exact formula for convex polyominoes according to the bond perimeter. They use the algebraic language methodology which is an old idea introduced by M. P. Schützenberger.[17,18] Let A_n be a class of combinatorial objects enumerated by the integer a_n. The method makes use of three steps. The first one is to find a bijection between A_n and the words of an algebraic language L over an alphabet X. The second one is to give a non-ambigous grammar generating L. Classically, from the non-ambigous grammar, one can associate that a proper algebraic system is noncommutative power series. The unique solution of this system has a component which is the noncommutative generating function $\boldsymbol{L} = \sum_{w \in L} w$. At last, sending all variables x of X onto one variable t, the series $\boldsymbol{L}$ becomes the generating function $f(t) = \sum_{n \geq 0} a_n t^n$, which is the solution of an algebraic system in one variable t. The interest in such a method is in the fact that, after the coding, the computations are straightforward. For a survey on the results in this field see Ref. 19. Other results are in the same vein. They concern the column-convex polyominoes according to the bond perimeter and area[3] and parallelogram polyominoes according to the double distribution bond and site perimeter.[4]

In this paper, we are interesting in the study of directed column-convex polyominoes (*i.e.* polyominoes which are obtained from directed animals and are column-convex) with a given site and bond perimeter or area. We name them, for short, dcc-polyominoes. A dcc-polyomino being directed, the site perimeter is directed and is the number of unit squares of the outside along the boundary which are in the north or in the east of a point of the polyomino (see Figure 2). This directed site perimeter is the one considered by physicists in the directed percolation model.

After definitions and notations, we define a special class of dcc-polyominoes which are named *stair polyominoes* and are enumerated by the Catalan numbers. In sections 3 and 4, we give a bijection between dcc-polyominoes and some words of a language close to the Dyck language. In sections 5, 6 and 7, we give the exact formula for enumerating them according to the site and bond perimeter. At last, in section 8, we prove, using a very simple bijection, that the number of animals with an area n is exactly the Fibonacci number F_{2n}.

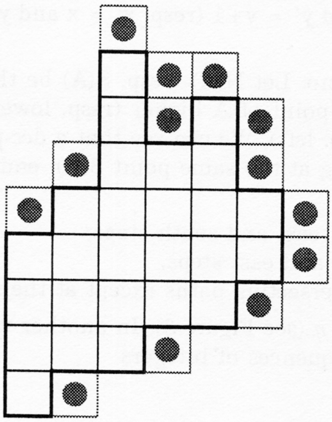


Figure 2. A dcc-polyomino.

1 – DEFINITIONS AND NOTATIONS

Let X be an alphabet, we denote by X^* the free monoid generated by X, that is the set of words written with (*i.e.* finite sequences composed of letters from X. The *empty word* is denoted by ε. The number of occurrences of the letter x in the word w is denoted by $|w|_x$, the length (number of letters) of w by $|w|$.

Let K $\langle\langle X\rangle\rangle$ (resp. K[X] be the algebra of non-commutative (resp. cummutative) power series with variables from X and coefficients in the ring K. We denote by α the canonical morphism which makes the variables commuting. For any language L in X^*, we denote by $\boldsymbol{L}$ the generating function

$$\boldsymbol{L} = \sum_{w \in L} W$$

which is an element of $\mathbf{Z}\langle\langle X\rangle\rangle$.

The *Dyck language* D is the set of words over $\{x, \bar{x}\}$ satisfying the two following conditions:

(i) for any left factor f such that $w = fg$, $|f|_x \geq |f|_{\bar{x}}$

(ii) $|w|_x = |w|_{\bar{x}}$.

The number of Dyck words of length 2n is the classical n^{th} Catalan number:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

A *path* ω is a sequence $\omega = (s_0, s_1, \cdots, s_n)$ of integer coordinate points in the Cartesian plane such that s_i and s_{i+1} are neighbours. The point s_0 (resp. s_n) is the *starting* (resp. *final*) point. The length of ω is n. Each pair (s_i, s_{i+1}) is an *elementary step* of the path. The elementary step is called *North* (resp. *South*, respt. *East*) if $s_i = (x,y)$, $s_{i+1} = (x',y')$ with $x' = x$ and $y' = y+1$ (resp. $x' = x$ and $y' = y-1$, resp. $x' = x+1$ and $y' = y$).

Let A be a dcc-polyomino. Let N(A) (resp. S(A) be the point of the perimeter of A, which has no perimeter point of A higher (resp. lower) than it and no perimeter point of A on its right (resp. left). We can see that a dcc-polyomino is defined by two paths $\omega(A)$ and $\eta(A)$ starting at the same point S(A), ending at the same point N(A), such that:

(i) $\omega(A)$ has only north, east and south steps,

(ii) $\eta(A)$ has only north and east steps,

(iii) ω and η are nonintersecting paths except at the starting and final points.

We suppose that ω is above η (see Figure 3). In another way, the same dcc-polyomino A can be defined by two sequences of integers

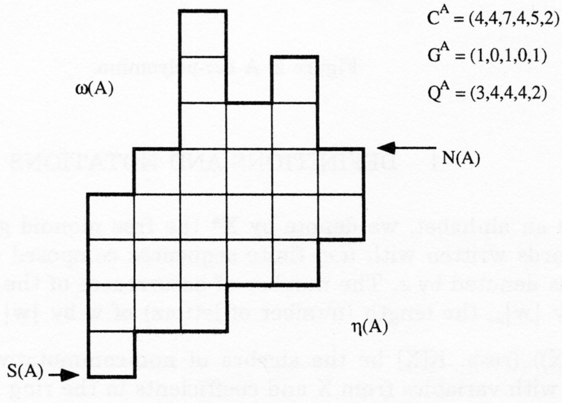


Figure 3. Definition of a dcc-polyomino.

$$C^A = (c_1^A, \cdots, c_k^A) \text{ and } G^A = (g_1^A, \cdots, g_{k-1}^A),$$

with the convention that if $k = 1$, then G^A is empty (we will denote $G^A = \varepsilon$). These two sequences are defined by the following conditions:

for every $i \in [1,k]$, c_i^A is the number of cells of the ith column of the dcc–polyomino A, (1)

for every $i \in [1,k-1]$, g_i^A encodes the way of gluing the column i to the column i+1, namely, starting with the column i+1 glued on the right of the i^{th} column so that their two southmost east steps are on the same horizontal line, then the column i+1 is moved g_i^A steps north. In other words, g_i^A is the difference between the altitude of the southmost east step in the column i+1 and the southmost east step in the column i. (2)

See Figure 3. These two sequences satisfy:

for every $i \in [1,k]$, $c_i^A>0$;

for every $i \in [1,k-1]$, $g_i^A \geq 0$ and $g_i^A < c_i^A$. (3)

In the sections 6 and 7, we will use *Good's formula* (Lagrange inversion with two variables) in the following form. Let φ and ψ be two power series from $\mathbf{C}\langle\langle\{x,y\}\rangle\rangle$, let f and g be two other power series from $\mathbf{C}\langle\langle\{u,v\}\rangle\rangle$, so that

$$f = u\,\varphi(f,g),$$
$$g = v\,\psi(f,g)$$

the coefficient of $u^m v^n$ in f is given by

$$(f,u^m v^n) = \frac{1}{mn} \sum_{k=1}^{n} \sum_{j=0}^{m-1} k(\varphi^m, x^j y^k)(\psi^n, x^{m-1-j} y^{n-k})$$

with $mn \neq 0$ due to the form of f in the previous system of equations. The reader can find a demonstration of this formula using the language theory in Ref. 2.

2 – BIJECTION BETWEEN DYCK WORDS AND STAIR POLYOMINOES

In this section, we prove that a bijection exists between the Dyck words of length 2n and special dcc-polyominoes, which are named *stair polyominoes*, with n columns.

A stair polyomino is a dcc-polyomino, such that the two sequences $C^A = (c_1^A, \cdots, c_k^A)$ and $G^A = (g_1^A, \cdots, g_{k-1}^A)$ satisfy:

for every $i \in [1,k-1]$, $g_i^A = 0$ (4)

for every $i \in [1,k-1]$, $c_{i+1}^A \geq c_i^A - 1$ and $c_k^A = 1$. (5)

See, for example, the polyomino in Figure 4. For each stair polyomino A, we define recursively the Dyck word $w = \mu(A)$ using the following algorithm:

– if $c_i^A = 1$ and $k=1$ then $\mu(A) = x\,\bar{x}$,

– if $c_i^A = 1$ and $k > 1$ then $\mu(A) = x\,\bar{x}\,\mu(A')$ and A' is a stair polyomino defined by the sequence $C^{A'}$ with $c_j^{A'} = c_{j+1}^{A}$ for every $j \in [1,k-1]$ (A') is the polyomino obtained from A by deleting its first column),

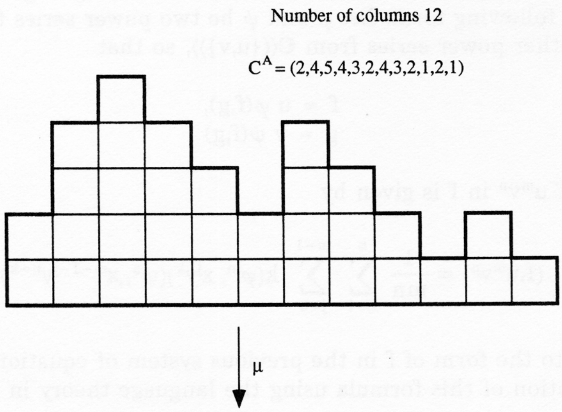
– else, let ι be the smallest i so that $c_i^A = 1$, then $\mu(A) = x\,\mu(A')\,\bar{x}\,\mu(A'')$ with A' a stair polyomino defined by

$$\text{for every } j \in [1,\iota-1],\ c_j^{A'} = c_j^A - 1,$$

and with A'' a stair polyomino defined by

$$C^{A''} = (c_{\iota+1}^A, \cdots, c_k^A).$$

Clearly, the number of columns in A is the number of letters x in $\mu(A)$. The word $\mu(A)$ is a Dyck word; it comes from the fact that when we produce an x in the coding, we associate an $\bar{x}$ in the word $\mu(A)$. We give an example of this coding in Figure 4.



$$\mu(A) = x\ x\,\bar{x}\,x\,x\,x\,\bar{x}\,x\,x\,\bar{x}\,\bar{x}\,\bar{x}\,\bar{x}\,x\,x\,x\,\bar{x}\,\bar{x}\,\bar{x}\,\bar{x}\,x\,x\,\bar{x}\,\bar{x}$$

Figure 4. An example of bijection μ

LEMMA 1. *Map μ is a bijection between Dyck words of length 2n and stair polyominoes having n columns.*

In order to prove it, we construct the reciprocal bijection μ' of μ. Let w be a Dyck word, then we construct a sequence of integers associated to w in the following way:

– the number of components of $\mu'(w)$ is $|w|_{\bar{x}}$,

– if $w \neq \varepsilon$, then let $w = u_i\,\bar{x}\,v_i$ such that $|u_i|_{\bar{x}} = i - 1$.
then the ith component of $\mu'(w)$ is given by $|u_i|_x - i + 1$.

Clearly, $\mu'(w)$ verifies condition (5). Let $C^A = \mu'(w)$ and G^A be given by a sequence of zeroes of length $|C^A| - 1$. Then the two sequences, C^A and G^A, define a stair polyomino A.

REMARK 2. This bijection comes from a classical one between the Dyck words and Lukasiewicz words (for the definition of this language see, for example, Ref. 2.)

Thus, μ' is the reciprocal bijection of μ. We deduce from the lemma 1 the

COROLLARY 3. *The number of stair polyominoes having n columns is the Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

3 – A CODING FOR DIRECTED COLUMN-CONVEX POLYOMINOES

In this section, we give a coding for the dcc-polyominoes preserving the three parameters: the number of columns, bond perimeter and directed site perimeter.

To the two sequences C^A and G^A, defining a dcc-polyomino A with k columns, we associate a sequence of integers Q^A giving the number of cells by which a column is glued to the following. The sequence Q^A is defined by:

If A has one column, then $Q^A = \varepsilon$,
else for every $i \in [1,k-1]$, if $g_i^A < c_i^A - c_{i+1}^A$, then $q_i^A = c_{i+1}^A$,
else $q_i^A = c_i^A - g_i^A$ (6)

See in Figure 5 an example of construction for the sequence Q^A. Note that the bond perimeter of the dcc-polyomino A is given by

$$p(A) = \sum_{i=1}^{k} (2 + 2c_i^A - q_{i-1}^A - q_i^A) \tag{7}$$

with the convention $q_0^A = q_k^A = 0$. The directed site perimeter of A is given by

$$s(A) = c_k^A + 1 + \sum_{i=1}^{k-1} s_i^A \tag{8}$$

with if $g_i^A \geq c_i^A - c_{i+1}^A$ then $s_i^A = c_i^A - q_i^A + 1$ else $s_i^A = c_i^A - q_i^A$.

For the coding we need the notion of a marked dcc-polyomino, which is a dcc-polyomino with one of the last column cells marked. We consider that this cell has no cell on its right. thus, the site perimeter of a marked dcc-polyomino is one less than the same dcc-polyomino without the mark. Nevertheless, the bond perimeter is the same.

DEFINITION 4. *A marked dcc-polyomino* $\mathcal{A}$ *is a pair* (A, κ_A) *where* A *is a dcc-polyomino and* κ_A *is an integer in* $[1, \kappa-1]$ *where* κ *is the last integer in the sequence* C^A (i.e. *the number of cells in the last column of* A).

For a marked dcc-polyomino $\mathcal{A} = (A, \kappa_A)$, we have $s(\mathcal{A}) = s(A) - 1$ and $p(\mathcal{A}) = p(A)$. Let Y be the alphabet $\{x, \bar{x}, y, \bar{y}\}$.

DEFINITION 5. *The language* L *over* Y *is defined by the system of equations:*

$$\boldsymbol{L} = y\bar{y} + x\boldsymbol{L}\bar{y} + y\bar{x}\boldsymbol{L} + x\boldsymbol{L}\bar{x}\boldsymbol{L} + x\boldsymbol{M}\bar{x}y\,\bar{x}\boldsymbol{L} + x\boldsymbol{M}\bar{x}y\bar{y}\,, \tag{9}$$

$$\boldsymbol{M} = y\bar{x} + x\boldsymbol{M}\bar{y} + x\boldsymbol{L}\bar{x} + y\bar{x}\boldsymbol{M} + x\boldsymbol{L}\bar{x}\boldsymbol{M} + x\boldsymbol{M}\bar{x}y\bar{x}\boldsymbol{M} + x\boldsymbol{M}\bar{x}y\bar{x}. \tag{10}$$

REMARK 6. The words of language L (resp. M) will give a coding for the dcc-polyominoes (resp. marked dcc-polyominoes).

We give below the definition of the bijection φ. Let A be a dcc-polyomino.

case 1

If A has only one column with p cells, $p \geq 1$ then $\varphi(A) = x^{p-1}y\,\bar{y}^p$ (see Figure 6.1).

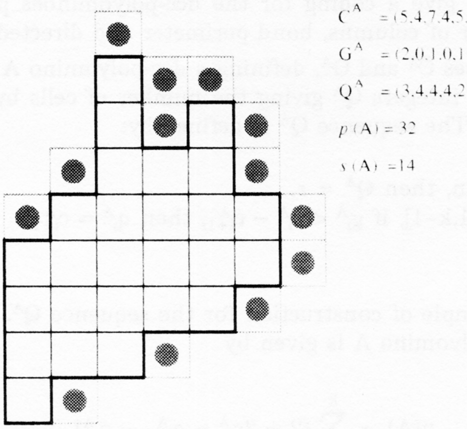


Figure 5. The sequences defining a dcc-polyomino.

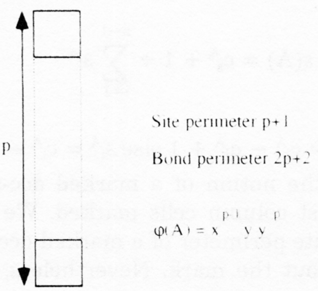


Figure 6.1. An example for case 1 of φ.

case 2

If $k \geq 2$ and for every $i \in [1,k-1]$, $q_i^A \geq 2$, then $\varphi(A) = x\varphi(A')\bar{y}$ with A' a dcc-polyomino defined by $C^{A'}$ so that for every $i \in [1,k]$, $c_i^{A'} = c_i^A - 1$ and $G^{A'} = G^A$ (see Figure 6.2).

case 3

If $k \geq 2$ and $c_1^A = 1$, then $\varphi(A) = y\bar{x}\varphi(A')$ with A' a dcc-polyomino defined by $C^{A'} = (c_2^A, \cdots, c_k^A)$ and $G^{A'} = (g_2^A, \cdots, g_{k-1}^A)$ (or $G^{A'} = \varepsilon$ if $k = 2$) (see Figure 6.3).

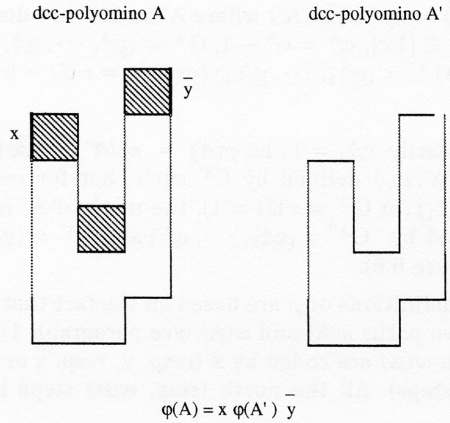


Figure 6.2. An example for case 2 of φ.

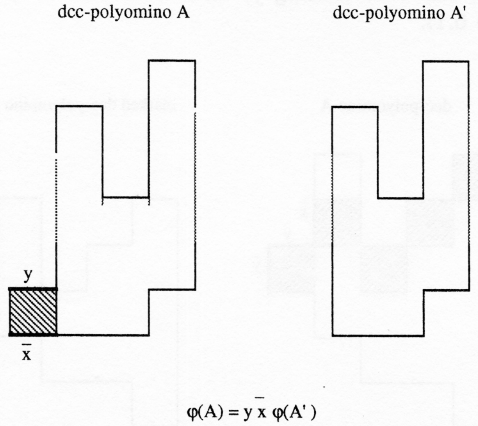


Figure 6.3. An example for case 3 of φ.

case 4

If $k \geq 2$ and $c_k^A = 1$, then $\varphi(A) = x\varphi(\mathcal{A}')\,\bar{x}\,y\,\bar{y}$ with $\mathcal{A}' = (A', \kappa_{A'})$ the marked dcc-polyomino defined by $C^{A'}$ so that for every $i \in [1,k-1]$, $c_i^{A'} = c_i^A - 1$, $G^{A'} =$ $= (g_1^A, \cdots, g_{k-2}^A)$ or $G^{A'} = \varepsilon$ if $k=2$), and the mark of A' is $\kappa_{A'} = g_{k-1}^A + 1$ (see Figure 6.4).

case 5

Let ι be the smallest i, such that $q_\iota = 1$.

case 5a

If $c_{\iota+1}^A \neq 1$, then $\varphi(A) = x\varphi(A')\bar{x}\varphi(A'')$ where A' and A'' are dcc-polyominos defined by $C^{A'}$ so that for every $i \in [1,c]$, $c_i^{A'} = c_i^A - 1$, $G^{A'} = (g_\iota^A, \cdots, g_{\iota-1}^A)$ (or $G^{A'} = \varepsilon$ if $\iota = 1$), $C^{A''} = (c_{\iota+1}^A, \cdots, c_k^A)$ and $G^{A''} = (g_{\iota+1}^A, \cdots, g_{k-1}^A)$ (or $G^{A''} = \varepsilon$ if $\iota = k-1$) (see Figure 6.5).

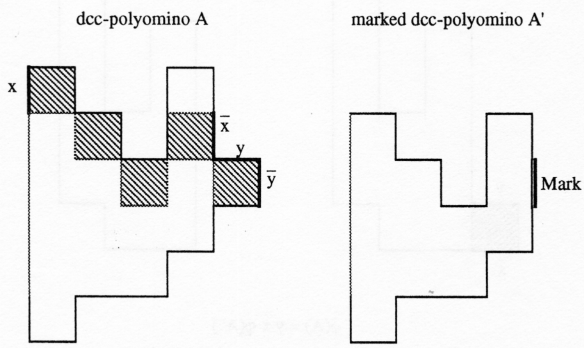
case 5b

Else we have necessarily $c_{\iota+1}^A = 1$, let $\varphi(A) = x\varphi(\mathcal{A}')\,\bar{x}\,y\,\bar{x}\varphi(A'')$ with $\mathcal{A}'$ is the marked dcc-polyomino $(A', \kappa_{A'})$ defined by $C^{A'}$ such that for every $i \in [1, \iota]$, $c_i^{A'} = c_i^{A'} - 1$, $G^{A'} = (g_1^A, \cdots, g_{\iota-1}^A)$ (or $G^{A'} = \varepsilon$ if $\iota = 1$), the mark of A' is $\kappa_{A'} = g_{\iota+1}^A$ and A'' is the dcc-polyomino defined by $C^{A''} = (c_{\iota+2}^{A'}, \cdots, c_k^{A'})$ and $G^{A''} = (g_{\iota+2}, \cdots, g_{k-1}^A)$ (or $G^{A''} = \varepsilon$ if $\iota = k-1$) (see Figure 6.6).

REMARK 7. All the definitions or φ are based on the fact that every dcc-polyomino can also be defined by two paths $\eta(A)$ and $\omega(A)$ (see paragraph 1). All the north (resp. east, resp. south) steps in $\omega(A)$ are coded by x (resp. y, resp. $\bar{y}$ or $\bar{x}$ if it is the last one in a sequence of south steps). All the north (resp. east) steps in $\eta(A)$ are coded by $\bar{y}$ (resp. $\bar{x}$).

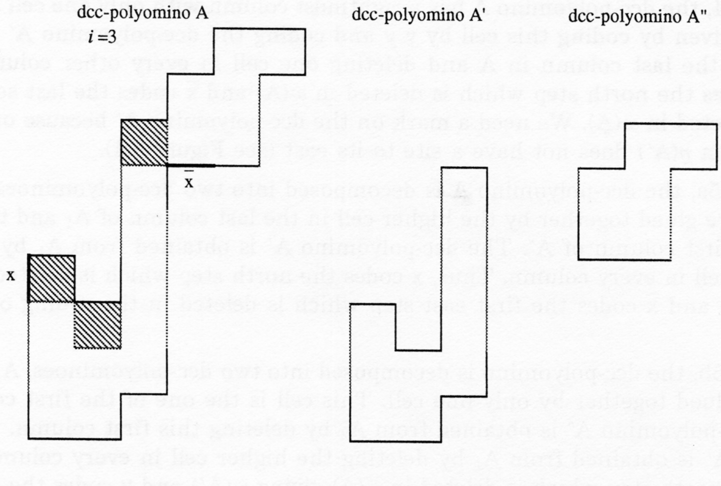
We explain now the construction of $\varphi(A)$ directly on the picture of a dcc-polyomino A using remark 7.

In case 1, there is only one column in A and in the site perimeter we want to count the cells which are to the east (coding $\bar{y}$) and to the north (coding y) of the dcc-polyomino (see Figure 6.1).



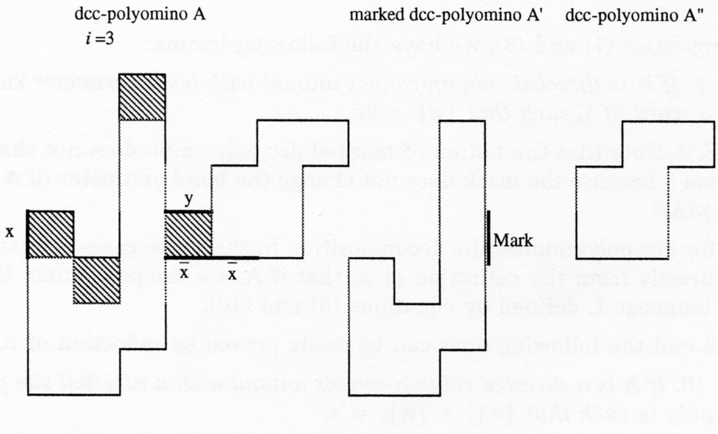
$\varphi(A) = x\,\varphi(A')\,\bar{x}\,y\,\bar{y}$

Figure 6.4. An example for case 4 of φ.



$$\varphi(A) = x\ \varphi(A')\ \overline{x}\ \varphi(A'')$$

Figure 6.5. An example for case 5a of φ.



$$\varphi(A) = x\ \varphi(A')\ \overline{x}\ y\ \overline{x}\ \varphi(A'')$$

Figure 6.6. An example for case 5b of φ.

Case 2 says that the dcc-polyomino A′ is obtained by deleting the higher cell in every column of A. Thus, x codes the north step which is deleted in ω(A) giving ω(A′) and $\overline{\text{y}}$ codes the last north step which is suppressed in η(A) giving η(A′) (see Figure 6.2).

In case 3, we say that if the dcc-polyomino A has a leftmost column with only one cell, then its code is obtained by coding this cell by y $\overline{\text{x}}$ and coding the dcc-polyomino A′, which is obtained by deleting the first column in A (see Figure 6.3).

In case 4, the dcc-polyomino A has a rightmost column with only one cell and then its code is given by coding this cell by y $\bar{y}$ and coding the dcc-polyomino A′ obtained by deleting the last column in A and deleting one cell in every other column of A. Thus, x codes the north step which is deleted in $\omega(A)$ and $\bar{x}$ codes the last south step which is deleted in $\omega(A)$. We need a mark on the dcc-polyomino A′ because one of the north step on $\eta(A')$ does not have a site to its east (see Figure 6.4).

In case 5a, the dcc-polyomino A is decomposed into two dcc-polyominoes, A_1 and A″, which are glued together by the higher cell in the last column of A_1 and the lower cell in the first column of A″. The dcc-polyomino A′ is obtained from A_1 by deleting the higher cell in every column. Thus, x codes the north step which is deleted in $\omega(A)$ giving $\omega(A')$ and $\bar{x}$ codes the first east step which is deleted in the coding of A″ (see Figure 6.5).

In case 5b, the dcc-polyomino is decomposed into two dcc-polyominoes, A_1 and A_2, which are glued together by only one cell. This cell is the one of the first column of A_2. The dcc-polyomino A″ is obtained from A_2 by deleting this first column. The dcc-polyomino A′ is obtained from A_1 by deleting the higher cell in every column. Thus, x codes the north step which is deleted in $\omega(A)$ giving $\omega(A')$ and y codes the east step which is deleted by the central cell. The two letters $\bar{x}$ code the two east steps of $\eta(A)$ which are deleted by the central cell and the coding of A″ (see Figure 6.6).

Note that φ can be defined in the same way on marked dcc-polyominos. The coding difference is only on the marked step of path η, which is a north step and is coded by $\bar{x}$.

Using properties (7) and (8), we have the following lemma:

LEMMA 8. *If* A *is directed column-convex animal with bond perimeter* $2n+2$, *then* $w = \varphi(A)$ *is a word of* L *such that* $|w| = 2n$.

REMARK 9. Note that the notion of marked dcc-polyomino does not change anything to lemma 8 because the mark does not change the bond perimeter (if $\mathcal{A} = (A, \kappa_A)$ then $p(\mathcal{A}) = p(A)$).

Clearly, for dcc-polyominoes the decomposition in the above cases is unambiguous and we get directly from the definition of φ, that if A is a dcc-polyomino, then $\varphi(A)$ is a word of language L defined by equations (9) and (10).

Lemma 8 and the following ones can be easily proved by induction of n.

LEMMA 10. *If* A *is a directed column-convex animal with a directed site perimeter* s, *then* $w = \varphi(A)$ *is such that* $|w|_y + |w|_{\bar{y}} = s$.

LEMMA 11. *if* A *is a directed column-convex animal with* k *columns, then* $w = \varphi(A)$ *is such that* $|w|_y = k$

4 – THE REVERSE BIJECTION φ^{-1} BETWEEN DIRECTED COLUMN-CONVEX POLYOMINOES AND WORDS OF L.

In order to prove that φ is a bijection we give here an application φ' which is the reverse bijection φ^{-1} of φ. Let X be the alphabet $\{x, \bar{x}\}$, Y the alphabet $\{x, \bar{x}, y, \bar{y}\}$, and τ the morphism from Y^* into X^* defined by its action on the letters of Y:

$$\tau(x) = \tau(y) = x \text{ and } \tau(\bar{x}) = \tau(\bar{y}) = \bar{x}. \tag{11}$$

Let w be a word of L, then $\tau(w)$ is a Dyck word. We define the *decomposition* of a word w of L as follows:

w has the decomposition $w = aw_1bw_2$ with $w_1,w_2 \in Y^*$,
$a \in \{x,y\}$, $b \in \{\bar{x},\bar{y}\}$ if $\tau(w)=x\tau(w_1)\,\bar{x}\tau(w_2)$ with $\tau(w_1)$ and $\tau(w_2)$
Dyck words. (12)

For the construction of φ', we consider all the possible decomposition of a word of L using equation (9) and give the recursive construction which associates a dcc-polyomino. We use the implicit construction of a marked dcc-polyomino. In this construction, we denote the number of columns of a dcc-polyomino A by k_A.

Let w be a word of L, and denote the dcc-polyomino $\varphi'(w)$ by A.

case 1

$\varphi'(y\,\bar{y})$ is a dcc-polyomino A, such that $C^A = (1)$ and $G^A = \varepsilon$,

case 2

If the decomposition of w is $w = xw_1\bar{y}$, then w_1 is in L; if A′ is the dcc-polyomino $\varphi'(w_1)$, A is given by

for every $i \in [1, k_{A'}]$ $c_i^A = c_i^{A'} + 1$,

for every $i \in [1, k_{A'} - 1]$$g_i^A = g_i^{A'}$.

case 3

If the decomposition of w is $w = y\,\bar{x}\,w_1$, then w_1 is in L, if A′ is the dcc-polyomino $\varphi'(w_1)$, A is given by $c_1^A = 1, g_1^A = 1$,

for every $i \in [1, k_{A'}]$ $c_{i+1}^A = c_i^{A'}$,

for every $i \in [1, k_{A'} - 1]$ $g_{i+1}^A = g_i^{A'}$.

case 4

If the decomposition of w is $w = xw_1\bar{x}w_2$, then w_1 (resp. w_2) is in L; if A′ (resp. A″) is the dcc-polyomino $\varphi'(w_1)$ (resp. $\varphi'(w_2)$), the dcc-polyomino A is given by

for every $i \in [1, k_{A'}]$ $c_i^A = c_i^{A'} + 1$,

for every $i \in [1, k_{A'} - 1]$ $g_i^A = g_i^{A'}$.

$g_{k_{A'}}^A = c_{k_{A'}}^{A'}$,

for every $i \in [1, k_{A''}]$ $c_{i+k_{A'}}^A = c_i^{A''}$,

for every $i \in [1, k_{A''} - 1]$ $g_{i+k_{A'}}^A = g_i^{A''}$.

case 5

If the decomposition of w is $w = xw_1\bar{x}y\bar{x}w_2$, then w_1 (resp. w_2) is in M (resp. L); if $(A',\kappa_{A'})$ (resp. A″) is the dcc-polyomino $\varphi'(w_2)$). A is given by

for every $i \in [1, k_{A'} - 1]$ $c_i^A = c_i^{A'} + 1$.

for every $i \in [1, k_{A'} - 1]$ $g_i^A = g_i^{A'}$,

$c_{k_{A'}+1}^{A'} = 1$, $g_{k_{A'}}^{A'} = \kappa_{A'} - 1$, $g_{k_{A'}+1}^A = 0$,

for every $i \in [1, k_{A''}]$ $c_{i+k_{A'}+1}^A = c_i^{A''}$,

for every $i \in [1, k_{A''} - 1]$ $g_{i+k_{A'}+1}^A = g_i^{A''}$,

case 6

If the decomposition of w is $w = xw_1\bar{x}y\bar{y}$, then w_1 is in M, if $(A',\kappa_{A'})$ is the dcc-polyomino $\varphi'(w_1)$, A is given by

for every $i \in [1, k_{A'}]$ $c_i^A = c_i^{A'}+1$,
for every $i \in [1, k_{A'} - 1]$ $g_i^A = g_i^{A'}$,
$c_{k_{A'}+1}^A = 1$, $g_{k_{A'}}^A = \kappa_{A'} - 1$.

The definition of φ' for the words of M can be deduced from the definition for the words of L. Thus, the reader will easily write it. We have

LEMMA 12. *If* w *is a word of* L *such that* $|w| = 2n$, *then* $\varphi'(w)$ *is a directed column-convex animal with perimeter* $2n+2$.

REMARK 13. As in section 3, the notion of mark does not change the perimeter of a dcc-polyomino (the construction of a dcc-polyomino from a word of L is the same as that from a word of M). Thus, there is an analogous lemma for the words of M. The proof of Lemma 12 is similar to that of Lemma 8 in section 3. Thus, using lemmas 8 and 12, we have proved that φ' is the reverse bijection of φ, and the following

THEOREM 14. *There exists a bijection* φ *between directed column-convex polyominoes with perimeter* $2n+2$ *and the words of length* $2n$ *of language* L.

We also have the analogues of the other lemmas of section 3.

LEMMA 15. *If* w *is a word of* L *so that* $|w|_y+|w|_{\bar{y}} = s$, *then* $\varphi'(w)$ *is a directed column-convex animal with site perimeter* s.

LEMMA 16. *If* w *is a word of* L *so that* $|w|_y = k$, *then* $\varphi'(w)$ *is a directed column-convex animal with* k *columns.*

See in Figure 7 an example of bijection φ. In other sections we use equations (9) and (10) in order to give exact enumerations for dcc-polyominoes.

5 – FUNCTIONAL EQUATIONS FOR DIRECTED COLUMN-CONVEX POLYOMINOES

We have proved in sections 3 and 4 that the number $l_{n,s,k}$ of dcc-polyominoes, having a bond perimeter $2n+2$, a site perimeter s and k columns, is equal to the number of words w of L so that $|w| = 2n$, $|w|_y + |w|_{\bar{y}} = s$ and $|w|_y = k$. We deduce now from this result some functional equations which are satisfied by some generating functions of dcc-polyominoes.

We introduce the following generating functions in commutative variables x,y,z:

$$l(x, y, z) = \alpha(\theta(\mathbf{L})) \text{ and } m(x, y, z) = \alpha(\theta(\mathbf{M})),$$

where θ is the morphism from Y^* into $\{x, y, z\}^*$ defined by its actions on the letter of Y

$$\theta(x) = \theta(\bar{x}) = x,\ \theta(y) = y \text{ and } \theta(\bar{y}) = z.$$

Then, using equalities (9) and (10) and, for short, writing l (resp. m) for l (x, y, z) (resp. m(x, y, z)), we get

$$\begin{cases} l = yz + xyl + xzl + x^2l^2 + x^3ylm + x^2yzm, \\ m = xy + xzm + x^2l + xym + x^2lm + x^3ym^2 + x^3ym. \end{cases} \tag{13}$$

We computed the resultant of these two equations eliminating variable m, and we found equation $E(x, y, z, 1) = 0$ with

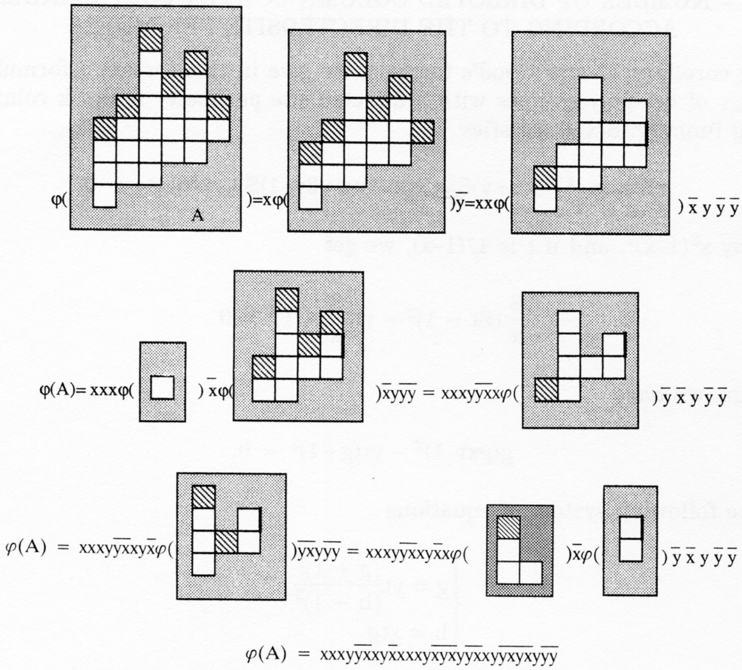


Figure 7. An example of bijection φ.

$$E(x,y,z,l) \;=\; x^4zl^3 \;+\; l^2(2x^3z^2 \;+\; x^3yz \;-\; 2x^2z \;-\; x^4y) \;+ \\ +\; l(x^2z^3 \;+\; 2x^2yz^2 \;-\; 2xz^2 \;-\; x^3yz \;-\; xyz \;+\; z) \;+\; xyz^3 \;-\; yz^2 \;=\; 0 \qquad (14)$$

Thus, using Lemmas 8, 10 and 11 we get

COROLLARY 17. *The generating function for the number* $s_{n,k}$ *of directed column-convex polyominoes with a directed site perimeter* n *and* k *columns*

$$S(x,\, y) \;=\; \sum_{\substack{n\ge 0\\ k\ge 0}} s_{n,k}\, x^{n-k} y^k$$

satisfies the algebraic equation $E(1,y,x,S) = 0$.

COROLLARY 18. *The generating function for the number* $p_{n,k}$ *of directed column-convex polyominoes with bond perimeter* $2n+2$ *and* k *columns*

$$P(x,\, y) \;=\; \sum_{\substack{n\ge 0\\ k\ge 0}} p_{n,k}\, x^{2n}\, y^k$$

satisfies the algebraic equation $E(x,yx,x,P) = 0$.

Using these two results, we give in the next sections an exact formula for the numbers $s_{n,k}$ and $p_{n,k}$.

6 – NUMBER OF DIRECTED COLUMN-CONVEX POLYOMINOES ACCORDING TO THE DIRECTEDSITE PERIMETER

Using corollary 17 and Good's formula, we give in this section a formula for the number $s_{n,k}$ of dcc-polyominoes with a directed site perimeter n and k columns. The generating function S(x,y) satisfies

$$xS^3+(x-1)\ (2x+y)S^2+x(x-1)(x+2y-1)S+yx^2(x-1)\ =\ 0.$$

Dividing by $x^2(1-x)^2$, and if t is $1/(1-x)$, we get

$$\frac{S}{x}(St-1)^2 - yt(\frac{S}{x}+1)^2 = 0$$

Let S be gx, we have

$$g(gxt-1)^2 - yt(g+1)^2 = 0.$$

We get the following system of equations

$$\begin{cases} g = yt\dfrac{(g+1)^2}{(h-1)^2}, \\ h = xtg \end{cases}$$

This system has a form which allows us to use Good's formula with u = yt, v = xt,

$$\varphi(g,h) = \frac{(g+1)^2}{(h-1)^2} \text{ and } \psi(g,h) = g.$$

Thus, we have

$$(g,\ u^m v^p)\ =\ \frac{1}{mp}\sum_{i=1}^{p}\sum_{j=0}^{m-1} k(\varphi^m,\ g^j h^i)(\psi^p,\ g^{m-1-j}h^{p-i}).$$

and consequently

$$(g,\ u^m v^p)\ =\ \frac{1}{m}\binom{2m}{m-1-p}\binom{2m+p-1}{p}.$$

Afterwards. we substitute the values of u and v in g and then the value of g in S, we get

$$S(x,\ y)\ =\ \sum_{k\geq 1}\ \sum_{i\geq 0}\ \sum_{r\geq 0}^{k-1} \frac{1}{k}\binom{2k}{k-r-1}\binom{2k+r-1}{r}\binom{k+r+i-1}{i} y^k x^{r+i+1}.$$

At length, we get the following

THEOREM 19. *The number of directed column-convex polyominoes with a directed site perimeter* n *and* k *columns is*

$$s_{n,k} = \frac{1}{k}\sum_{r\geq 0}^{M_s}\binom{2k}{k-r-1}\binom{2k+r-1}{r}\binom{n-2}{k+r-1}$$

with M_s *the minimum of* $k-1$ *and* $n-k-1$.

REMARK 20. If $n = k+1$, then $s_{n,k}$ is the Catalan number C_k and we find again the number of stair polyominoes which was stated in section 2.

Using the previous theorem, we get immediately

COROLLARY 21. *The number of directed column-convex polyominoes having a directed site perimeter* n *is*

$$s_n = \sum_{k=1}^{n-1}\frac{1}{k}\sum_{r=0}^{M_s}\binom{2k}{k-r-1}\binom{2k+r-1}{r}\binom{n-2}{k+r-1}$$

with M_s *the minimum of* $k-1$ *and* $n-k-1$.

REMARK 22. Using equation (14) with $x=1$, $y=z=x$, we obtain the following equation in which $S = \sum_{n\geq 0} s_n x^n$ is the solution:

$$S^3+(3x-3)S^2+(3x^2-4x+1)S+x^2(x-1) = 0.$$

It is possible to get an asymptotic result for s_n from this functional equation by using analysis techniques and then we get

$$s_n \approx \left(\frac{32}{5}\right)^n n^{-3/2}.$$

We give the table for $s_{n,k}$ and s_n in Figure 8.

7 – NUMBER OF DIRECTED COLUMN-CONVEX POLYOMINOES ACCORDING TO THE BOND PERIMETER

Using the same results as in the previous section, we give here a formula for the number $p_{n,k}$ of dcc-polyominoes with bond perimeter $2n+2$ and k columns. We have the following equation for $P(x,y)$

$$x^4P^3-2x^2(1-x^2)P^2+(1-x^2)(1-x^2-x^2y)P-yx^2(1-x^2) = 0. \tag{20}$$

Dividing by $(1-x^2)^2$, and if s is $x^2/(1-x^2)$, we get

$$s^2P^3-2sP^2+(1-ys)P-sy = 0.$$

If g is sP, we have

$$P(sP-1)^2-sy(P+1) = 0.$$

We get the following system of equations

$s_{n,k}$	1	2	3	4	5	6	7	8	s_n
1	1								1
2	1	2							3
3	1	6	5						12
4	1	12	27	14					54
5	1	20	85	112	42				260
6	1	30	205	492	450	132			1310
7	1	42	420	1582	2565	1782	429		6821
8	1	56	770	4172	10415	12562	7007	1430	36413

Figure 8. Table for the numbers $s_{n,k}$ and s_n.

$$\begin{cases} P = ys\,\dfrac{(P+1}{(g-1)^2}, \\ g = s\,P \end{cases}$$

This system has a form that allows us to use Good's formula, with $t = ys$, $u = s$,

$$\varphi(P,g) = \frac{(P+1)}{(g-1)^2} \text{ and } \psi(P,g) = P.$$

Thus, we have

$$(P,\ t^m u^r) = \frac{1}{mr} \sum_{i=1}^{r} \sum_{j=0}^{m-1} i(\varphi^m, P^j g^{i)}\ (\psi^r, P^{m-1-j} g^{r-i}),$$

and consequently

$$(P,\ t^m u^r) = \frac{1}{m} \binom{m}{r+1} \binom{2m+r-1}{r}.$$

Afterwards, we substitute the values of t and u

$$P(x,\ y) = \sum_{m\geq 1} \sum_{i\geq 0} \sum_{r=0}^{m-1} \frac{1}{m} \binom{m}{r+1} \binom{2m+r-1}{r} \binom{m+r+i-1}{i} y^m x^{2(m+r+i)}.$$

At length, we get the following

THEOREM 23. *The number of directed column-convex polyominoes with bond perimeter* $2n+2$ *and* k *columns is*

$$p_{n,k} = \frac{1}{k} \sum_{r=0}^{M_p} \binom{k}{r+1} \binom{2k+r-1}{r} \binom{n-1}{k+r-1},$$

with M_p *the minimum of* $k-1$ *and* $n-k$.

Summing over k, we get

COROLLARY 24. *The number of directed column-convex polyominoes having a bond perimeter* $2n+2$ *is*

$$p_n = \sum_{k=1}^{n} \frac{1}{k} \sum_{r=0}^{M_p} \binom{k}{r+1} \binom{2k+r-1}{r} \binom{n-1}{k+r-1},$$

with M_p *the minimum of* $k-1$ *and* $n-k$.

In Figure 9 we give the table for the values of p_n and $p_{n,k}$.

REMARK 25. Using equation (20) with $y=1$, it is easy to find an asymptotic value for p_n and we get

$$p_n \approx \left(\frac{3 + 2\sqrt[3]{100 + 5\sqrt{10}}}{6}\right)^n n^{-3/2}.$$

$p_{n,k}$	1	2	3	4	5	6	7	pn
1	1							1
2	1	1						2
3	1	4	1					6
4	1	9	9	1				20
5	1	16	37	16	1			71
6	1	25	105	106	25	1		263
7	1	36	240	446	245	36	1	1105

Figure 9. Table for the numbers $p_{n,k}$ and p_n.

8 – NUMBER OF DIRECTED COLUMN-CONVEX POLYOMINOES WITH A GIVEN AREA

In this section, we give an exact formula for the number r_n of dcc-polyominoes having an area n. Let X be the alphabet $\{a,x\}$.

DEFINITION 26. R *is the language of the words* w *of* X^* *satisfying*

(i) w *is in* $(xx+a)^*xx$,

(ii) $|w|$ *is even.*

For each dcc-polyomino A having k columns, we define the word $w = \rho(A)$ in X^* using the following construction:

– if A has one column, $C^A = (c_1^A)$ and then $\rho(A) = x^{2c_1^A}$,

– if A has k columns, then $\rho(A) = w_1w_2\cdots w_k$ with

for every $i \in [1, k-1]$, $w_i = x^{2g_i^A}ax^{2(c_i^A-g_i^A-1)}a$

and $w_k = x^{2c_k^A}$.

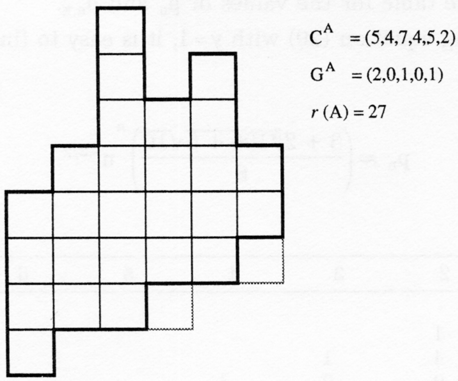
Clearly, w is a word of R. The number of columns of the dcc-polyomino A is

$$\frac{|\rho(A)|_a}{2} + 1,$$

and the area of A is

$$\frac{|\rho(A)|}{2}.$$

An example of this coding is given in Figure 10.



ρ (A) = xxxxaxxxxaaxxxxxxaxxaxxxxxxxxxxaaxxxxxxaxxaxxxxxxaxxxx

Figure 10. An example of bijection ρ.

LEMMA 27. *Map ρ is a bijection between the words of* R *of length* 2n *and the dcc-polyominoes having an area* n.

In order to prove it, we construct the reciprocal bijection ρ' of ρ. If w is a word of R, we construct two sequences of integers associated to w, that is $\rho'(w) = (\Xi(w), \Gamma(w)$.

1 – If $|w|_a = 0$, then $\Xi(w) = (|w|_x/2)$ and $\Gamma(w)$ is empty.

2 – If $|w|_a \neq 0$, in this case $w = w_1aw_2a\cdots aw_{2k+1}$ with for every $i \in [1, 2k+1]$, $w_i \in \{xx\}^*$ and $w_{2k+1} \neq \varepsilon$.. Then for every $i \in [1, k]$ we define

$$\Xi(w)_i = \frac{|w_{2i-1}| + |w_{2i}|}{2} + 1,$$

$$\Gamma(w)_i = \frac{|w_{2i-1}|}{2},$$

$$\text{and } \Xi(w)_{k+1} = \frac{|w_{2k+1}|}{2}.$$

Clearly, if we suppose that $C^A = \Xi(w)$ and $G^A = \Gamma(w)$, the two sequences C^A and G^A satisfy condition (5) and thus define a dcc-polyomino A with an area

$$r(A) = \sum_{i=1}^{k+1} \Xi(w)_i = \frac{|w|}{2}.$$

Using the definition of language R and lemma 27, it is easy to prove the

PROPOSITION 28. *The number of dcc-polyominoes having an area* n *and* k *columns is*

$$r_{n,k} = \binom{n+k-2}{n-k}.$$

Let r(t) be the generating function

$$r(t) = \sum_{n\geq 1} r_n t^n,$$

where r_n is the number of dcc-polyominoes with area n.

Language R is given by two following equations

$$\boldsymbol{R} = xx\,\boldsymbol{R} + a\,\boldsymbol{R_1} + xx,$$

$$\boldsymbol{R_1} = xx\,\boldsymbol{R_1} + a\,\boldsymbol{R}.$$

Then, using the morphism $\theta(x) = \theta(a) = t^{1/2}$ and lemma 28, we have $\theta(\alpha(\boldsymbol{R}) = r(t)$ and r(t) is given by

$$r(t) = \frac{t(1-t)}{1-3t+t^2}.$$

This function is the even part of the generating function of the Fibonacci numbers and we have

PROPOSITION 29. *The number of dcc-polyominoes having an area* n *is the Fibonacci number* $F_{2(n-1)}$.

Note that the number of stack polyominoes having a bound perimeter 2p+4 is also the Fibonacci number F_{2p} (see Ref. 5). Surprisingly, the parameter perimeter for stack polyominoes is transformed into the area parameter for the dcc-polyominoes.

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SAŽETAK

Prebrojavanje konveksnih životinja zadanog perimetra i površine, usmjerenih po stupcima

Maylis Delest i Serge Dulucq

Dane su točne formule za broj konveksnih životinja zadanog perimetra i površine, usmjerenih po stupcima. Dokaz je zasnovan na bijekcijama i kombinatornim rezultatima preuzetim iz algebarske teorije jezika.