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A New Coding for Column-Convex Directed Animals

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The present article contains two new results: a closed form expression for the number of column-convex directed (ccd–) animals having a given bond perimeter, directed site perimeter and number of columns, as well as a certain logarithmic function, a part of which is the ccd-animals two perimeters & columns generating function. Finally, an attempt has been made to formulate, in a more immediate way, the original proof of Delest and Dulucq[1] concerning the number of ccd-animals with a given area.

INTRODUCTION

A column-convex directed (ccd–) animal is defined as a plane region bounded by two internally disjoint self-avoiding plane lattice paths having a common origin and a common terminus. The upper path can make three kinds of steps: (1,0), (0,1) and (0,–1), while the step-set of the lower path is restricted to (1,0) and (0,1). Further, the upper path is required to terminate in a (1,0)-step. See Figure 1 for an example.

Two ccd-animals are considered to be equal if and only if there exists a translation superposing one of them to the other.

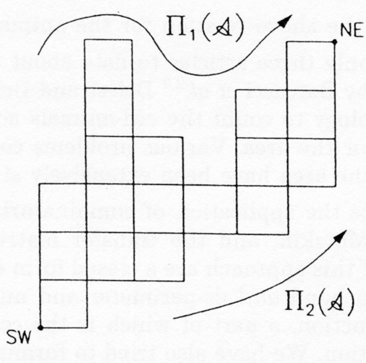

Figure 1. A ccd-animal. Columns: 6, bond perimeter: 30, ds-perimeter: 13, directed corners: 2.

We commonly denote the upper and the lower path delimiting a ccd-animal $\mathcal{A}$ by $\Pi_1(\mathcal{A})$ and $\Pi_2(\mathcal{A})$, respectively. The common origin of the two paths will be called the southwest pole of $\mathcal{A}$, while the common terminus will be called the northeast pole.

The unit squares with vertices at integer points of the plane lattice are called cells. Any non-empty intersection between a ccd-animal $\mathcal{A}$ and an infinite strip of the form $\langle i, i+1\rangle\times\mathbf{R}$ ($i \in \mathbf{Z}$) is called a column of $\mathcal{A}$ (the rows are defined similarly). Observe that from our choice of step-sets for $\Pi_1(\mathcal{A})$ and $\Pi_2(\mathcal{A})$ it follows that the columns of $\mathcal{A}$ are actually rectangles of unit width. Of course, rectangles are convex sets and that is why our animals are called column-convex. Further, these column-convex animals are also called directed because for every cell of a ccd-animal $\mathcal{A}$, except for the one which contains the SW-pole, either one of the cell's left and lower neighbours also belongs to $\mathcal{A}$.

The minimal and the maximal ordinate of the i-th column of a ccd-animal $\mathcal{A}$ will be denoted by $y_1(\mathcal{A})$ and $Y_1(\mathcal{A})$, respectively.

Obviously, the area of a ccd-animal $\mathcal{A}$ is equal to the number of cells contained in $\mathcal{A}$. The bond perimeter of $\mathcal{A}$ is what is usually called the perimeter of $\mathcal{A}$, *i.e.* the length of the boundary of $\mathcal{A}$. However, in this article we are also interested in another kind of perimeter, which is called the directed site (ds-) perimeter. The ds-perimeter of a ccd-animal $\mathcal{A}$ is defined as the number of those cells outside $\mathcal{A}$ whose left or lower neighbor cell lies inside $\mathcal{A}$.

If a cell c does not belong to $\mathcal{A}$, but the left and the lower neighbors of c both belong to $\mathcal{A}$, we shall say that c is a directed corner of $\mathcal{A}$.

Now, let $\mathcal{A}$ be a ccd-animal with k columns, bond perimeter $2k+2v$ and d directed corners. The boundary of $\mathcal{A}$ consists of k tops of columns, k bottoms of columns and 2v vertical edges. The v vertical edges are left-hand and v are right-hand (*e.g.*, by a right-hand vertical edge we mean an edge of the boundary which is the right-hand border of some cell that belongs to $\mathcal{A}$). Imagine going around the animal in, say, clockwise direction. Every right-hand vertical edge gives rise to a new cell which contributes to the ds-perimeter. The same is true of every top of a column, unless it is the bottom of some directed corner of $\mathcal{A}$. Hence, we conclude that

$$\text{the ds-perimeter of } \mathcal{A} = v+k-d \tag{1}$$

The reader may check the above relation for the animal shown in Figure 1.

As far as I know, the only three articles to date about the ccd-animals are those by Delest and Dulucq[1] and by Barcucci *et al.*[2,3] Delest and Dulucq[1] use Schützenberger's algebraic language methodology to count the ccd-animals according to the bond perimeter or the ds-perimeter or the area. Various problems connected with ccd-animals enumeration according to the area have been extensively studied by Barcucci *et al.*[2,3]

Our new coding permits the application of combinatorial tools, such as the cycle lemma of Dvoretzky and Motzkin, and the transfer matrix method in ccd-animals enumeration. The results of this approach are a closed form expression for the number of ccd-animals with a given bond *and* ds-perimeter and number of columns, as well as a certain logarithmic function, a part of which is the ccd-animals two perimeters & columns generating function. We have also tried to formulate, in a more immediate way, the original proof of Delest and Dulucq[1] concerning the number of ccd-animals with a given area.

The Cycle Lemma

The following apparently simple lemma, which was originally found in 1947 by Dvoretzky and Motzkin,[4] proves to be very useful in many enumeration arguments.

Lemma 1 (the cycle lemma). Let $z_1, z_2, \cdots z_n$ be integers less or equal to one such that $z_1 + z_2 + \cdots + z_n = p$ ($p \in \mathbf{N}$). Then, the sequence $z_1 z_2 \cdots z_n$ has exactly p cyclic permutations

$$z_i\ z_{i+1}\ \cdots\ z_n\ z_1\ \cdots\ z_{i-1}$$

whose partial sums are all positive.

In the case $p=1$, the cycle lemma may be restated in the following way:

*Lemma 2.** Let $S = z_1 z_2 \cdots z_n$ be a sequence of arbitrary integers such that $z_1 + z_2 + \cdots + z_n = 1$. Then there is exactly one cyclic permutation of S whose partial sums are all positive.

Example 3. The only two cyclic permutations of 1,1,–3,1,1,1,1,–2,1 that have positive partial sums are 1,1,1,1,–2,1,1,1,–3 and 1,1,1,–2,1,1,1,–3,1.

The only one cyclic permutation of 1,5,–6,4,–2,3,0,–4 whose partial sums are all positive is 4,–2,3,0,–4,1,5,–6.

The New Coding for Column-Convex Directed Animals

Let $\mathcal{A}$ be a ccd-animal with k columns. We put

$$f(\mathcal{A}) = a_1 b_1 a_2 b_2 \cdots a_k b_k\ , \tag{2}$$

where $a_1, \cdots, a_k$ and $b_1 \cdots, b_k$ are the numbers defined by

$$\begin{aligned} a_1 &= y_1(\mathcal{A}) - y_1(\mathcal{A})\ , \\ a_1 &= y_1(\mathcal{A}) - y_{i-1}(\mathcal{A}) && (i=2, \cdots, k)\ , \\ b_1 &= y_1(\mathcal{A}) - y_{i+1}(\mathcal{A}) && (i=1, \cdots k-1)\ , \\ b_k &= y_k(\mathcal{A}) - y_k(\mathcal{A}) + 1\ . \end{aligned} \tag{3}$$

Example 4. The animal in Figure 1 is encoded by the sequence 3,–1,3,0,–2–1,–1,0,2,–1,1,–2 .

Theorem 5. Let $\mathcal{A}$ be a ccd-animal having k columns, bond perimeter $2k+2v$ and directed site perimeter s and left $c = f(\mathcal{A})$. We assert that:

a) c is an integer sequence of length 2k.

b) The sum of all positive (resp. negative) terms of c is v (resp. $-v+1$).

c) The positive terms can occupy only odd positions, while the negative terms are present in some or none of the even positions and in exactly $k+v-s$ odd positions.

d) The partial sums of c are all positive and the total sum equals one.

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* Graham, Knuth and Patashnik[5] call Lemma 2 Raney's cycle lemma and give reference to Raney's paper,[6] where this lemma was employed in a combinatorial proof of the one-variable Lagrange inversion formula.

Conversely, if a sequence c satisfies conditions a), b), c) and d), then it encodes one and only one cdd-animal with k columns, bond perimeter 2k+2v and directed site perimeter s.

*Proof.***

a) is evident.

b) Assume, for convenience, that the SW-pole of $\mathcal{A}$ is at the origin (0,0). Now notice that an a_i is positive if there exist left-hand vertical edges with abscissa i–1; and when there are x ($x \geq 1$) such edges, then $a_i = x$. Hence, the sum of all the positive a_i's (which is at the same time the sum of all the positive terms in $c = f(\mathcal{A})$, since the b_i's are all nonpositive) equals the number of left-hand vertical edges of $\mathcal{A}$, that is v.

The negative a_i's ($i \in \mathbf{k}$) count the right-hand vertical edges with abscissa i–1 along the upper boundary $\Pi_1(\mathcal{A})$. The negative b_i's ($i \in \mathbf{k-1}$) count the right-hand vertical edges with abscissa i along the lower boundary $\Pi_2(\mathcal{A})$. However, if there are z right-hand vertical edges with abscissa k, b_k will be $-z+1$. That is why the sum of all the negative terms in c is $-v+1$.

c) By (1), an animal with k columns, v right-hand vertical edges and ds-perimeter s must have exactly k+v–s directed corners. Clearly, if $\mathcal{A}$ has k+v–s directed corners, then $\Pi_1(\mathcal{A})$ has k+v–s downwards steps, so that there are precisely k+v–s negative a_i's.

d) First, observe that

$$\begin{aligned} \sum_{i=1}^{j} a_i + \sum_{i=1}^{j-1} b_i &= Y_j(\mathcal{A}) - y_j(\mathcal{A}) \qquad (j \in \mathbf{k}) \text{ and} \\ \sum_{i=1}^{j} a_i + \sum_{i=1}^{j} b_i &= Y_j(\mathcal{A}) - y_{j+1}(\mathcal{A}) \qquad (j \in \mathbf{k-1}) \, . \end{aligned} \tag{4}$$

Now, the positiveness of the partial sums of c follows from the fact that $\mathcal{A}$ has a connected interior. The sum of all terms of c equals 1 because of b). Part d) is thus proved.

The proof that f is one-to-one is easy and we leave it to the reader.

Thus, the task of counting ccd-animals with k columns, bond perimeter 2k+2v and ds-perimeter s is now reduced to the less troublesome one of counting the sequences that have the above properties a), b), c) and d).

Theorem 6. The number of ccd-animals having bond perimeter 2k+2v, directed site perimeter s and k columns is

$$a_{k,\,v,\,s} = \frac{1}{k}\binom{k}{s-v}\binom{k+v-2}{s-k-1}\binom{s-1}{v}. \tag{5}$$

Proof. We shall say that a sequence which possesses properties a), b) and c) of Theorem 5 is a type(α) sequence. In order to define a type(α) sequence, we first have to choose the k+v–s odd positions that will be occupied by negative numbers. After that, we fix the negative and the nonnegative terms. The number of ways to do all this is

** In this proof, the terms of $f(\mathcal{A})$ will again be denoted in the way it was done in (2) and (3), *i.e.* by a_1, b_1, a_2, b_2, $\cdots$

$$n(\alpha) = \binom{k}{s-v}\binom{k+v-2}{s-k-1}\binom{s-1}{v}. \tag{6}$$

Let $c = a_1b_1 \cdots a_kb_k$ be a type(α) sequence. The cycle lemma assures that c has exactly one cyclic shift (say c_*) whose partial sums are all positive. c_* cannot be of the form

$$b_1a_{i+1} \cdots b_ka_1b_1 \cdots a_i$$

since, by property c), b_i's are all nonpositive. Hence, c_* is one of the following sequences:

$$\begin{array}{l} a_1b_1a_2b_2 \cdots a_kb_k \\ a_2b_2 \cdots a_kb_ka_1b_1 \\ \cdot \\ \cdot \\ a_kb_ka_1b_1 \cdots a_{k-1}b_{k-1}\,. \end{array} \tag{7}$$

But it is easy to see that the sequences listed in Eq. (7) are all of type(α). Thus, the type(α) sequences can be assembled in groups of cardinality k in such a way that each group contains exactly one sequence that possesses property d). The number of type(α) sequences having property d), *i.e.* the number of sequences having properties a), b) c) and d), is therefore given by

$$\frac{n(\alpha)}{k} = \frac{1}{k}\binom{k}{s-v}\binom{k+v-2}{s-k-1}\binom{s-1}{v}. \tag{8}$$

Theorem 5 ans Eq. (8) immediately imply the present theorem.

Remark. The cycle lemma can also be successfully applied in various enumerations of diagonally convex directed animals (Svrtan and Feretić[7]).

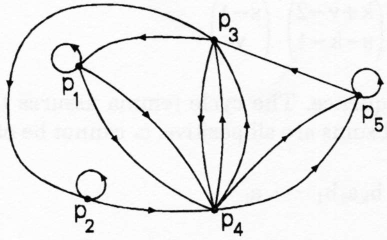
The Transfer Matrix Method

Let D be a directed graph (or diagraph) with vertices $p_1, \cdots, p_k$. The adjacency matrix of D is the k by k matrix $A = [a_{ij}]$, where a_{ij} is the number of arcs of D running from p_i to p_j (see Figure 2). A directed walk is a sequence of arcs $s=(a_1, \cdots, a_l)$ such that $(\forall\ i = 1, \cdots, l-1)$ the arc a_{i+1} starts at the same vertex where a_i finishes. In this paper, the directed walks of length l will be called l-walks.

Next, let $B(t) = (I-tA).^{-1}$ Observe that the elements of B(t) are rational functions of t. Matrix $B(t) = [b_{ij}(t)]$ has an amazing property: for $i, j \in (1, \cdots, k)$ and $l \in \mathbf{N}_0$, the coefficient of t^l in $b_{ij}(t)$ is the number of l-walks in D with the starting point (*origin*) p_i and ending point (*terminus*) p_j.

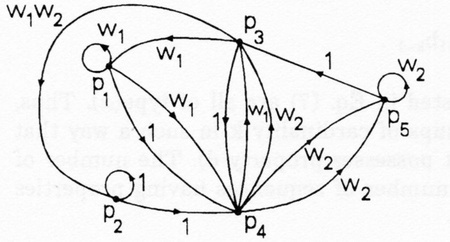
More information about directed walks in a diagraph can be obtained by assignation of *weights* to its arcs. Let D again be a diagraph with vertices $p_1, \cdots, p_k$ and let to every arc a of D be assigned w(a), a monomial in variables $w_1, \cdots w_n$. w(a) is called the weight of a. Let A_w be a matrix whose (i, j)-entry is the sum of weights associated to all arcs running from p_i to p_j (see Figure 3). A_w is called the adjacency weight matrix of D.

Then, let $B_w=(I-tA_w)^{-1}$. The matrix $B_w = [b^{\bullet}_{ij}]$ has the following property: the coefficient of $t^lw_1^{i_1} \cdots w_n^{i_n}$ in $b^{\bullet}_{ij} = b^{\bullet}_{ij}(t, w_1, \cdots, w_n)$ is the number of those l-walks $s=(a_1, \cdots, a_l)$ with origin p_i and terminus p_j which satisfy the condition



$$A = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{matrix}$$
$$\quad p_1 \; p_2 \; p_3 \; p_4 \; p_5$$

Figure 2. A directed graph and its adjacency matrix (the labels around A serve just to facilitate the reading).



$$A_1 = \begin{bmatrix} w_1 & 0 & 0 & w_1+1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ w_1 & w_1w_2 & 0 & 1 & 0 \\ 0 & 0 & w_1+w_2 & 0 & 2w_2 \\ 0 & 0 & 1 & 0 & w_2 \end{bmatrix} \begin{matrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{matrix}$$
$$\quad p_1 \; p_2 \; p_3 \; p_4 \; p_5$$

Figure 3. Weighted digraph and its adjacency weight matrix.

$$\prod_{i=1}^{l} w(a_i) = w_1^{j_1} \cdots w_n^{j_n}. \tag{9}$$

To illustrate the utility of this property of B_w in directed walks enumeration, we give the following example:

Example 7. For the diagraph in Figure 3, the coefficient of $t^{20}w_1^8w_2^5$ in $b_{23}^{\bullet}$ is the number of 20-walks $s=(a_1, \cdots, a_{20})$ from p_2 to p_3, such that

(A) there are exactly 8 i's for which
$a_1 \in \{p_1p_1$, upper arc p_1p_4, p_3p_1, p_3p_2, left arc $p_4p_3\}$
(B) there are exactly 5 i's for which
$a_1 \in \{p_3p_2$, right arc p_4p_3, upper and lower arc p_4p_5, $p_5p_5\}$.

Namely, properties (A) & (B) are equivalent to the condition

$$w(a_1)w(a_2) \cdots w(a_{20}) = w_1^8w_2^5.$$

The properties of B and B_w quoted in this section are well known in the graph theory. For the proof see, for instance, Svrtan and Veljan.[8]

The Two Perimeters & Columns »Over-generating« Function for ccd-Animals

First, we are going to put the type(α) sequences (defined in the proof of Theorem 6) in a one-to-one correspondence with some family of directed walks in the weighted diagraph E of Figure 4.

For $m \in \mathbf{N}_0$, let $\varphi(m)$ be a sequence (word) consisting of m letters x (notation: $\varphi(m) = x^m$). Next, let $\varphi(-m) = y^m$ and let $\psi(-m) = z^m$.

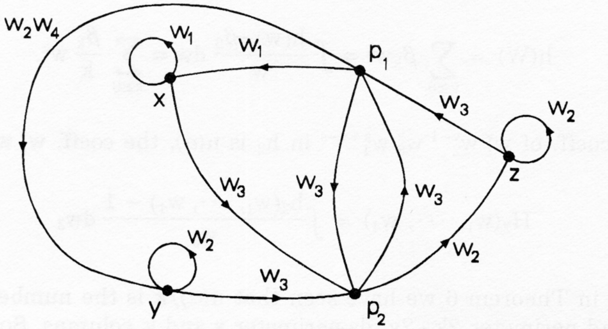


Figure 4. Weighted digraph E.

Now, to a type(α) sequence $c = a_1b_1 \cdots a_kb_k$, we associate the word

$$g_1(c) = p_1\,\varphi(a_1)\,p_2\,\psi(b_1)\,p_1\,\varphi(a_1) \cdots p_2\,\psi(b_{k-1})p_1\,\varphi(a_k)p_2\,\psi(b_k)\,p_1 \qquad (10)$$

$g_1(c)$ is the vertex sequence of a directed walk in diagraph E. That directed walk will be denoted by $g_2(c)$. Let us state without proof:

Lemma 8. g_2 is a bijection between the type(α) sequences and $2k+2v-1$–walks in E $s=(a_1, \cdots, a_{2k+2v-1})$ which have the following five properties:

(*i*) s starts and ends at p_1,

(*ii*) $a_i \in \{xx, p_1x\}$ for exactly v i's,

(*iii*) $a_i \in \{yy, p_1y, p_2z, zz\}$ for exactly v–1 i's,

(*iv*) $a_1 \in \{xp_2, yp_2, p_1p_2, p_2p_1\}$ for exactly 2k i's,

(*v*) $a_1 = p_1y$ for exactly k+v–2 i's.

In terms of weights the properties (*ii*) – (*v*) can be recast as

$$\prod_{i=1}^{2k+2v-1} w(a_i) = w_1^{v}\, w_2^{v-1}\, w_3^{2k}\, w_4^{k+v-s}. \qquad (11)$$

Let $h_1 = h_1(t, w_1, \cdots, w_4)$ be the (p_1p_1)-entry of what for diagraph E is B_w. In view of what was said in the preceding section, we argue that the paths which correspond to the type(α) sequences are enumerated by the coefficient of $t^{2k+2v-1}w_1^v w_2^{v-1} w_3^{2k} w_4^{k+v-s}$ in h_1. This coefficient is equal to the coeff. of $w_1^v w_2^{v-1} w_3^{k} w_4^{k+v-s}$ in

$$h_2(w_1, w_2, w_3, w_4) = h_1(1, w_1, w_2, w_3^{1/2}, w_4).$$

Lemma 9. The number of type(α) sequences n(α) is the coeff. of $w_1^v w_2^{v-1} w_3^k w_4^{k+v-s}$ in

$$h_2(w_1, \cdots, w_4) = \frac{1}{1 - [(1-w_1)^{-1}(1-w_2)^{-1} + w_2 w_4 (1-w_2)^{-2}]\cdot w_3} \quad . \tag{12}$$

Proof. An elementary calculation of the (p_1,p_1)-entry of B_w (*i.e.* of h_1), which the reader could carry out by himself.

For any power series h(w), we have

$$h(W) = \sum_{k\geq 0} \beta_k w^k \Rightarrow \int \frac{h(w)-\beta_0}{w}\, dw = \sum_{k\geq 0} \frac{\beta_k}{k} w^k. \tag{13}$$

Since the coeff. of $w_1^v\, w_2^{v-1}\, w_3^k\, w_4^{k+v-s}$ in h_2 is n(α), the coeff. $w_1^v\, w_2^{v-1}\, w_3^k\, w_4^{k+v-s}$ in

$$H_2(w_1, \cdots, w_4) = \int \frac{h_2(w_1, \cdots, w_4) - 1}{w_3}\, dw_3 \tag{14}$$

is n(α)/k. But, in Theorem 6 we have seen that n(α)/k is the number of ccd-animals having the bond perimeter 2k+2v, ds-perimeter s and k columns. So we have:

Theorem 10. The number of ccd-animals having the bond perimeter 2k+2v, directed site perimeter s and k columns is the coefficient of $w_1^v w_2^{v-1} w_3^k w_4^{k+v-s}$ in

$$H_2(w_1, \cdots, w_4) = -\ln \left\{1 - \left[\frac{1}{(1-w_1)(1-w_2)} + \frac{w_2 w_4}{(1-w_2)^2}\right]\cdot w_3\right\}. \tag{15}$$

Proof. It suffices to calculate the integral in Eq. (14).

Remark. Let $H(w_1,\cdots, w_4) = w_2 w_3 H_2(w_1, w_2w_3, w_1w_2w_3w_4, w_3^{-1})$. Another way of stating Theorem 10 is:

The number of ccd-animals having the bond perimeter 2p, directed site perimeter s and k columns is the coeff. of $w_1^p w_2^p w_3^s w_4^k$ in

$$\begin{aligned} H_2(w_1, \cdots, w_4) &= \\ &= -w_2 w_3 \ln \left\{1 - \left[\frac{1}{(1-w_1)(1-w_2w_3)} + \frac{w_2}{(1-w_2w_3)^2}\right]\cdot w_1 w_2 w_3 w_4\right\} \end{aligned} \tag{16}$$

Thus, we have obtained a four-variable power series, a part of which is the three-variable ccd-animals two perimeters & columns generating function.

Enumeration by the Area

Let $\mathcal{A}$ be a ccd-animal with an area n and k columns. Recall that $y_i(\mathcal{A})$ and $y_i(\mathcal{A})$ denote the minimal and the maximal ordinates of the ith column of $\mathcal{A}$. $\Pi_2(\mathcal{A})$ denotes the lower boundary of $\mathcal{A}$.

Observe that the numbers

$$c_i = y_i(\mathcal{A}) - y_{i+1}(\mathcal{A}) \quad (i=1, \cdots, k-1) \tag{17}$$

are all positive, because otherwise there would be no contact between the animal's i^{th} and $i+1^{st}$ column. Obviously, the length of the k^{th} column $c_k = y_k(\mathcal{A}) - y_k(\mathcal{A})$ is also positive.

Secondly, since $\Pi_2(\mathcal{A})$ makes only east and north steps, the numbers

$$d_i - y_{i+1}(\mathcal{A}) - y_i(\mathcal{A}) \qquad (i=1, \cdots, k-1) \tag{18}$$

are nonnegative. Further, we have

$$\sum_{i=1}^{k} c_i + \sum_{i=1}^{k-1} d_i = \sum_{i=1}^{k-1} (c_i + d_i) + c_k = \sum_{i=1}^{k} [Y_i(\mathcal{A}) - y_i(\mathcal{A})] = n. \tag{19}$$

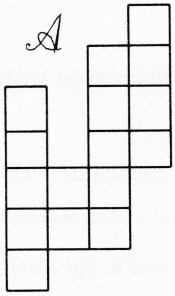


Figure 5.

On the other hand, when the sequence

$$e(\mathcal{A}) = c_1 \cdots c_k d_1 \cdots d_{k-1} \tag{20}$$

is given, it is possible to reconstruct the animal $\mathcal{A}$ by drawing it from the right to the left. Thus, we have

Theorem 11. e is a bijection between the ccd-animals with the area n and k columns and sequences s having the following three properties:

1) s is an integer sequence of length $2k-1$;

2) The first k terms of s are positive while the others are nonnegative;

3) The sum of all terms of s is equal to n.

Example 12. For the animal in Figure 5, we have $e(\mathcal{A}) = 4,2,3,4,1,0,2$.

Counting the sequences s which satisfy 1), 2) and 3), we get

Corollary 13. The number of ccd-animals having an area n and k columns is

$$b_{n,k} = \binom{n+k-2}{n-k}. \tag{21}$$

Thus, the number of all ccd-animals with an area n is

$$\sum_{k=1}^{n} \binom{n+k-2}{n-k} = \left| \begin{matrix} \text{new index} \\ i=n-k \end{matrix} \right| = \sum_{i\geq 0} \binom{2n-2-i}{i} . \tag{22}$$

But, the last sum in Eq. (22) can be calculated by means of the well-known identity***

$$\sum_{i\geq 0} \binom{p-1}{i} = F_p \qquad (p \in \mathbf{N}_0) , \tag{23}$$

where F_p's are the Fibonacci numbers:

$$F_0 = 1, F_1 = 1, \qquad F_{p+2} = F_{p+1} + F_p \qquad (\forall p \in \mathbf{N}_0). \tag{24}$$

Thus, we obtain

Theorem 14. The number of ccd-animals with the area n is the 2n-nd Fibonacci number F_{2n-2}

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SAŽETAK

Novi kod za vertikalno konveksne usmjerene životinje

Svjetlan Feretić

U ovom je radu dokazana formula zatvorenog tipa za broj vertikalno konveksnih usmjerenih (vku–) životinja sa zadanim opsegom, usmjerenim okruženjem i brojem stupaca. Zatim je dobivena jedna logaritamska funkcija od četiri varijable $w_1 \cdots, w_4$ kod koje se broj životinja s opsegom 2p, usmjerenim okruženjem s i k stupaca pojavljuje kao koeficijent u $w_1{}^p w_2{}^p w_3{}^s w_4{}^k$. Učinjen je i pokušaj da se razmatranja M. Delest i S. Dulucqa[1] o broju vku–životinja sa zadanom površinom formuliraju na jednostavniji način.

*** An interesting proof for (23) can be found in the book of Graham *et al.*[5] (p. 288).