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Conference Paper

The Magic of the Number Five*

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»Numero pondere et mensura Deus omnia condidit.«
Isaac Newton (1722)[1]

»But for most of us, philistine enough to accept the historically contingent nature of evolution, there is nothing specially deep about the number five. Pianists should ponder the challenge that our motor cortexes would have been set had Bach or Scarlatti sported eight deeply and ineffably named fingers per hand«.
Jonathan Cooke (1990)[2]

The appearance of the number five and fivefold symmetry in Nature and their uses in several fields of human activity are presented with a special emphasis on chemistry.

INTRODUCTION

The concept of number is one of the oldest and most useful concepts in the history of the human race.[3,4] (The reader should note that in this essay under the term number I will consider the natural numbers or non-negative integers). People have been aware of the number concept from the very beginnings of humankind. However, there was (and there still is) a difference in the use and appearance of numbers in different contexts, some numbers being used more often than others.[5] There is a variety of reasons for the frequent utilization of certain numbers in cultural contexts such as: religious (three in the Catholic faith, five in Islam); dramatic (three and nine by William Shakespeare (1564–1616) in Machbet); alchemistic (four by Robert Fludd (1574–1637), a notable alchemist and opponent of Johannes Kepler (1571–1630), who wrote a beautiful essay on the manifold qualities of the number four[6]); symbolic (different numbers depending on the political, cultural or religious circumstances, as, for example, the number 13 a fundamental holy number in the religion of the Aztecs); count-

*This article is dedicated to the contemporary Croatian poet dr. Boro Pavlović ing systems (seven days in a week, 12 months in a year); sports (nine is a significant number in baseball); *etc.* On the other hand, there is no difference in the frequency of appearance of numbers in computational work: There all numbers appear somewhat randomly.

In my case, some numbers are definitely more interesting and attractive than others. For example, I like the numbers 13, 5, 6, 2, 1 and 3 in that order. Other people may have similar sets of numbers that they like. Talking to several friends has confirmed this supposition; each of them has her or his special set of numbers.

All kinds of symbolic meanings are given to the number 13.[7] However, my relation to the number 13 is much simpler: 13 is the root word of the family name Trinajstić (13 = trinajst in Chakavian dialect of the Croatian language). I may even suffer from triskaidekaphilia, an unreasonable liking for the number 13. However, I am sure that there are quite a few people who suffer from triskaidekaphobia, an irrational fear of the number 13. Thus, many hotels do not have rooms or storeys with the number 13.

The number five is the second on the list of my favourite numbers. Whilst my attachment to the number 13 is personal, the interest in the number five is stimulated by observing the appearance (sometimes inexplicable) of this number in Nature and its usage in various fields of human interest. My attraction to the number five is also emotional because the Croatian coat-of-arms has a checkered 5 × 5 square pattern consisting of alternating (horizontally and vertically) red and white squares.[8] Other people may have other reasons for being interested in the number five.[9] However, some other people may not be so impressed by the number five. For example, the number nine is the magic number for Iyer.[10] He stated: »Every number has its character, its own distinctive coloring: 5, for instance, is the gray accountant, the user-friendly solid citizen, the John Major, if you like of integers,...But what of 9? It is, we all know, an odd number (very odd), and an early square. It is 6 on its head, a circle and a line, the highest digit and the last, with something of the darkness that attaches to last things. Yet it has strange magic in it.«

Similarly, Devi[11] said about the number five: »Perhaps the most important thing about 5 is that it is half of 10,...« Croatian poet Želimir Mađer, on the other hand, equates five with sadness (five = sadness).[12]

Human awareness of the number five is natural. All the time that we are awake, most of us see five fingers on each hand. Five fingers on a hand is the result of an anatomical incident,[2] which appears to be quite satisfactory since in the course of evolution it was not altered. In support of the above is the fact that some mammalian families have a different number of »fingers«.[13] However, there are people with more than five fingers on a hand or five toes on a foot. This abnormality, known as polydactylism (from Greek word daktylos for finger), is very rare. I have never met a person with more than five fingers on each hand or five toes on each foot, but in the literature such people do appear occasionally. For example, Dr. Hannibal Lecter, a mad psychiatrist in Thomas Harris' bestselling novel »The Silence of the Lambs«, is a polydactyl (since he has six fingers on his left hand).

The decimal (or decadic) system that we use today is founded on the practice of counting by tens. This counting system is the result of counting by using fingers. The quinary system was probably also used by early people. (Actually, it is still today found in a South American Arawakan language named Saraveca in which counting has solely base five.)[14] It may be that the use of the decimal system was much simpler than the use of the quinary system for practical purposes and, consequently, the former system has developed exclusively. In addition, the development of language to cover abstractions such as numbers was rather slow. Probably, the difficulties in producing numerical verbal expressions were the smallest in the case of the decimal system and this perchance also favoured the base 10 system. Possibly, for this reason, modern languages are built almost uniformly around the decimal system. A mathematician may argue that more interesting counting systems than quinary or decimal systems would emerge if Hominids had four (quartal or octal systems) or six (sextal or duodecimal systems) fingers on each hand.

At this point I wish to tell an instructive funny story entitled »Why With Five Fingers.«[15]:

»Somebody saw Nassruddin Khoja eating greedily good food with all five fingers.

'Why do you, Khoja, eat with five fingers?' This man asked Nasruddin Khoja wondering.

'Because I do not have six fingers', replied Nasruddin Khoja without interrupting his partake of food.«

In this essay, I will present some uses and occurrences of number five and fivefold symmetric objects in a variety of areas. I will not attempt to explain why the number five is used instead of some other small non-negative integer (see the story above). I will just state the facts. However, one may speculate that the occurrence of the number five in Nature is somehow related to evolution, whilst in the fine arts and literature it may be a requisite ingredient in the source of aesthetic pleasure. No introduction to the concept of fivefold symmetry will be given, since there are a number of texts available on the subject.[16,17]

The essay is structured as follows. In the next section, I will list the appearances of the number five in various contexts. The third section will contain a presentation of several fivefold symmetric objects. In the fourth section, I will give some examples of the use of fivefold symmetry in the literature. In the fifth section, I will discuss the use of the number five and fivefold symmetry in chemistry. I will end this essay with concluding remarks.

THE NUMBER FIVE IN VARIOUS CONTEXTS

The number five is the third odd integer and represents the sum of the first even (2) and the second odd numbers (3). It is also a prime number.

The number five is the fifth member of the Fibonacci series (so called by Francois Eduard Anatole Lucas (1842–1891) in 1877) introduced by Leonardo Pisano (1170–post 1240), usually known as Fibonacci,* to model the breeding of rabbits.[19] The rabbit problem was introduced in his book on mathematics and calculation entitled *Liber*

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* The use of the sobriquet *Fibonacci* (contraction of filius Bonacci, son of Bonaccio) for Leonardo Pisano probably originated with the mathematical historian Guillaume Libri in 1838. However, there is no evidence that Leonardo referred to himself in this way or was ever called so by his contemporaries.[18] Actually, Leonardo referred to himself as Leonardo Pisano Bigollo. No one has so far been able to explain the meaning of *Bigollo*. Furthermore, his father's name was Guilielmo. Bonaccio was most probably one of Leonardo's not far removed ancestors.

abaci (1202). However, it appears that the Fibonacci series may have been known to and used by ancient mosaicists.[20]

The Fibonacci series looks like 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233,$\cdots$ and can be generated by

$$F_n = F_n + F_{n-2} \tag{1}$$

where F_n is the n-th Fibonacci number. Note that $F_1 = F_2 = 1$, and by convention $F_0 = 0$.

The Fibonacci series has been the subject of many studies[21] and is closely related to the golden section (*sectio aurea*), that is, the division of a line segment AB by a point P, such that: AB : AP = AP : PB.[22] AB/AP or AP/PB is called the golden ratio (or golden mean). In a more general sense, the golden ratio is a proportion in which the ratio of the whole to the larger part is the same as the ratio of the larger part to the smaller. Greek architects and sculptors made use of it. The proportions of Phidias' (*ca.* 490–430 B.C.) Athena Parthenos and the Parthenon in Athens well illustrate this point.[23] The golden ratio was also known to early Croatian builders and was applied by them when they were building, in the 9th and 10th century, the first Christian churches in the Slavic world[24,25] (Some of them, such as the Church of the Holy Cross in Nin or the Church of the Holy Trinity in Zadar are still well-preserved and in good shape, even after the heavy shelling in the 1990–1993 war of Serbian and Montenegrine fascists against the Republic of Croatia). Likewise, modern architects use the golden ratio in their works.[26] The golden section also plays an important role in the measurement theory.[27]

The value of the golden ratio is $(1 + \sqrt{5})/2 \cong 1.618034$. The ratios of pairs of successive Fibonacci numbers F_n/F_{n-1} approach this value by increasing n. For example, $F_5/F_4 \cong 1.666667$, $F_{14}/F_{13} \cong 1.618026$ and $F_2/F_{24} \cong 1.618034$.

In many patterns that can be expressed in terms of numbers, a Fibonacci series is hidden. For example, such a number pattern is Pascal's triangle. Although it is called so because Blaise Pascal (1623–1662) made extensive use of this triangle, it is of very ancient standing, being probably more than a thousand years old. One of its hidden properties is that it contains the Fibonacci series. However, there is no record available to show that Pascal had noticed this property of the triangle.

In animal (the rabbit problem), insect and plant worlds, patterns showing regularity related to the Fibonacci series have been found.[28,29] For example, the genealogical tree of a bee is such a pattern with a hidden Fibonacci series.[22]

There is also a mathematical journal, named »Fibonacci Quarterly«, in which research concerning Fibonacci numbers is published.

The number five is the third Catalan number. The Catalan series was discovered by Leonhard Euler (1707–1783) for counting triangulations. (A triangulation of an n-gon is a division of the inside into triangles. For example, a pentagon has five different triangulations.) However, it is named after the Belgian mathematician Eugene Charles Catalan (1814–1894) who rediscovered it in connection with a different problem.[30] The Catalan series looks like 1, 2, 5, 14, 42, 132, 429, 1430, 4862,$\cdots$ and can be generated by

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{2}$$

where C_n is the n-th Catalan number.

The number five is also, of course, a pentagonal number.[22] Pentagonal numbers belong to the series whose name is derived from an association with patterns shaped like regular pentagons (see Figure 1). They are part of a special class of numbers called polygonal numbers.

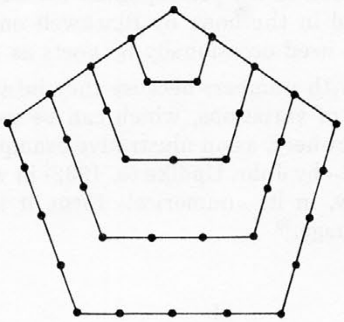


Figure 1. Diagram of the first five pentagonal numbers.

The pentagonal number series is: 1, 5, 12, 22, 35, 51, 70, 92, 117, 145,$\cdots$ and the n-th pentagonal number, P_n is given by

$$P_n = \frac{n}{2}(3n - 1) \tag{3}$$

Pierre de Fermat (1601–1665) found that there is a strikingly simple relationship between the polygonal number series and the natural number series. This may be expressed by the following simple statement: Every number is either pentagonal, or the sum of two, three, four of five pentagonal numbers.

There are also some special occurrences of 5 in mathematics: It is only quintic and higher polynomials that cannot be solved by radicals. Dimension 5 is often some sort of critical dimension (even as regards phase transitions in physics or chemistry), where *e.g.* it is the first dimension d where the only regular polytopes are hypertetrahedron, the hyperoctahedron and the hypercube – all $d \geq 5$ having exactly these regular polytopes. All the alternating groups A_n are simple for $n \geq 5$. Indeed, it appears that A_5 is the smallest nontrivial simple group. Also, the graph K_5 plays a crucial role in Kuratowski's theorem for planar graphs.[31] (K_5 is a complete graph with five vertices, each of degree four, and 10 edges. It has the shape of a pentagon containing a star, see Figure 2 later in the text.

The number five is present in the structure of the Bible.[32] For example, the Bible begins with the Pentateuch (from Greek $\pi\epsilon\nu\tau\epsilon$ = five and $\tau\epsilon\upsilon\chi o\sigma$ = tool, vessel, book),[33] a collective name for the five books of Moses, called Genesis, Exodus, Leviticus, Numbers and Deuteronomy. The number five is also an important number in the Islamic faith and in the Maya and Aztec religions. In Islam, the number five is a favourable number. It represents the number of calls to prayers each day, the number of ways of praying, the number of elements of hadj, *etc.* Mayas considered the number five to be a »perfect« number. It was a numerical symbol of the maize deity.[34] In Aztec mythology, this deity was the master of dance and music, thus symbolizing love, spring, dawn, plays, *etc.*[35]

Artists have been inspired by the number five. There is an interesting painting by Charles Demuth from 1928 entitled »I Saw the Figure 5 in Gold«. It may be found in the Alfred Stieglitz Collection of the Metropolitan Museum of Art in New York. This painting is also reproduced in the book by Blackwell on geometry in architecture.[26] The number five has been used occasionally by poets as a symbol or metaphor.[36–38]

Poets do like to play with numbers because they intuitively realize that a number is an element in the field of variations, which can be used in many ways by an imaginative person.[39] I present here, as an illustrative example, a part of a rather lengthy poem »On Top of the Point« by John Updike (b. 1932) in which all the first 10 integers are used and, consequently, in its »numerical« form, it is not confined to the boundaries of the English language:[40]

	1		
	2		
4	3	5	
6	8	7	
9		10	»Hey«

In Roman times five witnesses had to be present at the closing of a deal in order for the venture to be considered legal. This fact and the possibility of corrupted witness(es) was used by Costain for the plot of his historical novel on the fate of the cup of the last Supper entitled »The Silver Chalice.«[41]

Let me end this section by pointing out that the number five is also present in sports. For example, the pentathlon is an athletic contest involving participation of each competitor in five different events. It has recently been abolished from regular competition in favour of the decathlon. The modern pentathlon is a very demanding composite contest which consists of five very different categories of competition: a 300 m freestyle swim, a 4000 m crosscountry run, a 5000 m 30-jump equestrian steeplechase, épée fencing and target shooting at 25 meters. Basketball is an example of an indoor game between two teams of five players each. It is considered by many as the most perfect game to play or to watch. The beauty and the grace of a game of basketball is rooted in the athletic ability of each player and in the coordinated togetherness of all five of them in the action. Some of their actions are so well coordinated that we are reminded of Henry Matisse's famous painting »Dance« (presently at the Hermitage in St. Petersburg). The game of basketball is full of symmetries, broken symmetries and fivefold symmetric structures that are often so rapidly created (and destroyed) that they can be analyzed only by a slow motion camera.

FIVEFOLD SYMMETRIC OBJECTS

The most elementary fivefold symmetric objects are the regular pentagon and the five-pointed star. The regular pentagon and the five-pointed star, viewed as objects in the 2-dimensional Euclidean plane, have 10 symmetry operations (five rotations and five reflections) and, thus, they belong to the group 5m of order 10. Both these objects are related: each pentagon contains a star and each star contains a pentagon, *i.e.* the central area of the star has the shape of the pentagon which again may contain the star. This is illustrated in Figure 2.

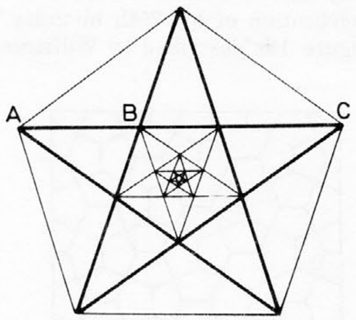


Figure 2. The regular pentagon and the five-pointed star. Labels A, B and C are to illustrate the golden ratio (AC : BC = BC : AB). A progression of star shapes within pentagons is also shown.

The construction of the five-pointed star is subjected to the golden ratio and, consequently so is the construction of the related pentagon.

The regular pentagon and the five-pointed star are pretty structures and are objects of manifold symbolism related to the number five. The five-pointed star in Greek times was used as a symbol and badge of the Pythagorean brotherhood. By this sign they recognized a fellow member. (The motto of the Pythagoreans was »All is number«.) This star was also used by alchemists and magicians as one of their most potent symbols. The five-pointed star is found in early sacred edifices, such as the baptistery of the Cathedral in Split and in the rosette of the Cathedral in Amiens. The five-pointed star is the symbol of a human being: five points – head, two arms and two legs – and a pentagonal center-torso. Leonardo de Vinci (1452–1519) has nicely illustrated this in his well-known drawing.[42]

The five-pointed star was the symbol of the Third International, because it symbolized the union of workers from all five continents. The (red, yellow, blue, white, *etc.*) five-pointed star is one of the symbols of many countries. In the case of the former socialist countries, this was a red star which symbolized the power and union of the working class of the world to which they belong. Fifty white stars representing fifty states of the Union appear on the flag of the United States of America. The state of Texas displays the five-pointed star prominently in its flag and state seal, and indeed Texas is often termed »The Lone Star State« (where I have spent altogether four happy years of my life in Austin and Galveston). A number of five-pointed stars is also a symbol of good quality (see, for example, the ranking of restaurants by Michelin, a tourist guide) and, by coincidence, of the ranking of generals in the US Army, the highest rank being a five-star general. It appears that a five-pointed star is amongst the most often used symbols nowadays.[43]

The pentagon is in many ways an unusual polygon.[44] For example, a plane surface cannot be completely covered with regular pentagons, because fivefold symmetry is incompatible with packing requirements.[45] Of the regular polygons, only equilateral triangle, square and regular hexagon can be used alone for floor tiling.[46] However, an interesting and unusual pentagonal tile pattern may be seen in the street tiling in Cairo and occasionally in the mosaics of Moorish buildings.[26] Pentagons in this pattern are equilateral but not equiangular. An example of such a pattern is given in Figure 3. This pattern was also mentioned by Lothar Collatz (b. 1910) in a lecture on geometric ornaments given during the celebration of his 75th birthday.[47] One way of generating the pentagonal pattern in Figure 3 is described by Williams.[48]

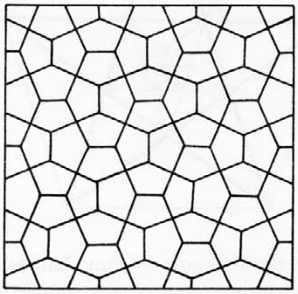


Figure 3. A pentagonal tile pattern with pieces that are equilateral but not equiangular.

Kepler was also interested in the pentagonal-symmetry of a plane.[49] The first stages of 5-fold symmetric plane tiling by Kepler are shown in a recent issue of *Mathematical Chemistry Newsletter.*[50] Also, tiling involving pentagonally symmetric tiles often occurs in »Penrose« tilings which are now believed of relevance in quasicrystals – one characteristic of these species is that their diffraction patterns exhibit a unique five-fold symmetry.[51,52]

I have found a just-opened jewelry shop (Schmuck-Galerie) in downtown Düsseldorf (Friedrichstr. 6, 4000 Düsseldorf 1) called »Pentagramm« (owners: Angelika Koch and Peter Tierling). There is a prominently displayed pentagon at the entrance. »Art & So« is inscribed in the pentagon. I talked to the owners and asked them why they had chosen this particular and unusual name for their shop. Their answer was they had selected this name simply because they liked it. Interestingly enough, I did not find any piece of jewelry in the shape of a pentagon displayed in the shop window.

An intriguing structure is Petersen's graph,[53] which in one of its geometric representation (with apparent symmetry D_{5h}) combines the pentagon and the five-pointed star (see Figure 4).

Petersen's graph has been introduced in mathematics to provide a counterexample to the Tait conjecture concerned with resolving the four colour problem.[54, 55] It has been found important in discussions of various mathematical problems[53] and it appears to be one of the important graphs in chemistry where it depicts possible routes for the isomerization of trigonal bipyrimidal complexes with five different ligands.[56] The rearrangement graph for the trigonal bipyramid was first discussed by Balaban *et al.*[57] It is also the incidence graph for the 10 lines and 10 planes resulting from 5 general points in 5-dimensional space, as noted by Cayley in 1878. Petersen's graph may have other geometric representations with different apparent symmetries. Some of these structures are shown in Figure 5.

Obviously, Petersen's graph has all the symmetry properties that can be associated with its geometrical representations, which one may regard as special subgroups of the actual symmetry group of the graph. It is found that the order of the symmetry group for Petersen's graph is 120 and that the group is fully symmetric on five objects S_5.[58,59] Since Petersen's graph is a structure which has found application in mathematics and chemistry, it was selected for the logo of the Journal of Mathematical Chemistry (published since 1987 by Baltzer of Basel), a journal devoted to novel applications of mathematical methods in the chemical realm. Similarly, the Journal of Graph Theory (published since 1977 by Wiley of New York) has also Petersen's graph as its logo.

The line graph of Petersen's graph is a bilaterally symmetric structure (see Figure 6).

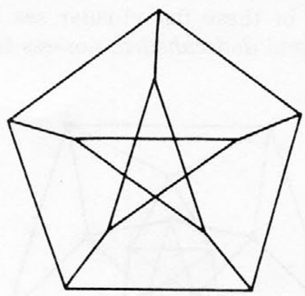


Figure 4. Petersen's graph in geometrical representation with D_{5h} symmetry.

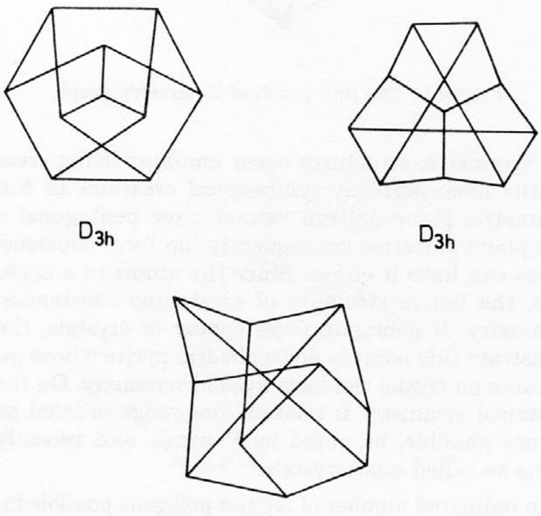


Figure 5. Geometrical representation of Petersen's graph with different symmetries.

However, the centre of this graph contains a five-pointed star and its boundary has the shape of the pentagon. Both Petersen's graph and the corresponding line graph are distance-regular graphs.[60]

Pentagonal (fivefold) structures also occur in Nature.[48,61,62] The fivefold symmetry is very frequent amongst flowers.[63] Families such as *Caryophyllaceae* or *Rosaceae* and many others possess fivefold symmetry. Individual examples are *Herniaria glabra L., Cydonia Oblonga, Gentiana pumila, etc.* Some lower animals also exhibit fivefold symmetry. For example the sea star (*Astropecten Aurantiacus*), the starfish (*Ophiotrix capillaris*) and the sand dollar (*Echinarachnius parma*) have pentagonal structures which are well suited to the sedendary habits of these sea creatures and to facing threats to their safety from all sides. The Bank of the West in Galveston has quite appropriately chosen the sand dollar for its logo. Galveston (Texas) is an island in the shallow Mexican Gulf which abounds in these flat circular sea urchins. Skeletons of some *Radiolaria*, such as *Cirrorrhegma dodecahedra*, possess fivefold symmetry.[64]

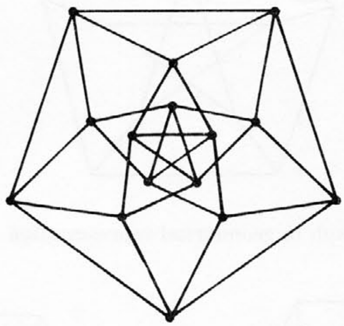


Figure 6. The line graph of Petersen's graph.

Whilst fivefold symmetric structures occur amongst living creatures, they cannot be found amongst the most perfectly symmetrical creations of Nature, crystals. If a translationally symmetric plane pattern cannot have pentagonal symmetry, neither can a stack of such plane patterns; consequently, no form constructed of translationally symmetric layers can have it either. Since the atoms in a crystal are arranged in such pattern layers, the lattice structure of crystalline substances cannot possess a fivefold axis of symmetry. If pentagon faces appear in crystals, they are irregular. A good example to illustrate this point is dodecahedral pyrite whose pentagonal faces are irregular indeed, because no crystal can have fivefold symmetry. On the other hand, if the condition of translational symmetry is relaxed, long-range ordered patterns based upon fivefold symmetry are possible, as noted by Penrose, and recently believed to have been observed as the so-called quasicrystals.[51,52,65,66]

While there is an unlimited number of regular polygons possible in 2D space, only five regular convex polyhedra can be constructed in 3D space. These are the tetrahedron, the cube (or hexahedron), the octahedron, the icosahedron and the dodecahedron. The surface of a regular polyhedron is bounded by congruent regular polygons and each vertex is symmetrically equivalent to each other. The simplest polygons that can form the surface are the equilateral triangle, the square and the pentagon. It is also clear that an angle of a polyhedron cannot be formed by fewer than three faces and that an angle may be formed by joining three, four and five equilateral triangles. With six such triangles, the angle flattens into a plane. The same will happen if four squares are united. Similarly, three regular pentagons at an angle are the maximum. But hexagons and higher polygons are all ruled out. This argument for the limitation to five regular polyhedra is based on Euler's formula:[16,17,22,67] $V + F - E = 2$, where V is the number of vertices. F the number of faces and E the number of edges.

The regular polyhedra have very high symmetries and one important common characteristic, *i.e.*, none of their rotational axes is unique, but each axis is associated with several axes equivalent to itself. They can be classified into three symmetry classes. Here, I am interested in the icosahedron and the dodecahedron because they are the only regular polyhedra which possess fivefold axes of rotation. They, of course, belong to the same symmetry class. The fivefold, threefold and twofold rotation axes intersect the midpoints of faces, the vertices and the edges of the dodecahedron, respectively. The corresponding axes intersect the vertices and midpoints of faces and edges of the icosahedron. Consequently, the icosahedron and the dodecahedron exhibit a dual relationship with regard to their faces and vertex figures.

These five regular polyhedra are also referred to as the Platonic solids[16,17,22] since they are prominent objects in Plato's philosophy. However, Theaetetos (c. 380), a mathematician, a pupil of Socrates (c. 470–399) and Theodoros of Cyren, a friend of Plato, discovered the octahedron and the icosahedron. Theaetetos was also the first to write on the five regular solids.[68] Plato (c. 427–347) in Timaeus[69] associated the regular tetrahedron, the cube, the octahedron and the icosahedron with four elements: fire, earth, air and water, respectively. In the dodecahedron (see Figure 7), which is aesthetically the most appealing convex polyhedron, Plato saw the outer shape of the Universe.

Leonardo da Vinci (1452–1519) was also attracted to the beautiful form of the dodecahedron[70] as well as many other people.[16,17,22,43] There are a number of Escher's prints with fivefold symmetric objects.[72] A few of his prints such as »Order and Chaos«, »Gravity« and »Reptiles« contain, for example, the dodecahedron or dodecahedral star.[71]

The icosahedron is also an attractive structure. Superimposition of several figures in some modern dances has a spatial structure of the icosahedron. Thus, in the theory of dance, the icosahedron is called the dance crystal.[72] It is interesting to note that Crick and Watson[73] suggested that the symmetry group of a small virus which appears to be a polyhedron must be that of one of the five regular polyhedra. Their suggestion initiated investigations of the shapes of viruses. The findings were interesting: Most

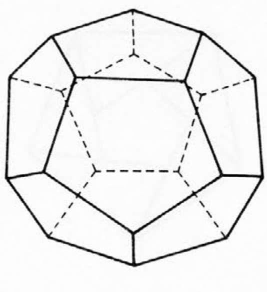


Figure 7. Dodecahedron (V = 20, F = 12, E = 30).

small polyhedral viruses have the symmetry of the icosahedron.[74] The icosahedron appears quite appropriately as the logo of »Polyhedron«, an international journal for inorganic and organometallic chemistry, published by Pergamon Press. Several hydroboranes are based upon icosahedral structures.

As it can be seen from Figure 7, twelve pentagons make a dodecahedron. German renaissance painter and graphic artist Albrecht Dürrer (1471–1528) also tried to pack pentagons. He was perhaps the first to construct a fold-up cardboard dodecahedron:[75] 12 pentagons fold into a dodecahedron. Five pentagons arranged around a sixth in a flower-like pattern can be folded to make half of the dodecahedron. This pattern is shown in Figure 8.

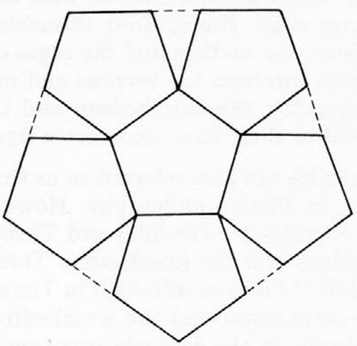


Figure 8. The pattern for one-half of the dodecahedran. Six regular pentagons and five isosceles triangles make a larger pentagon.

It is pleasing to note that a similar pattern is exhibited by the apple blossom, the dogwood blossom, and blossoms of some other plants.

Playing with the pattern in Figure 8 may lead to some non-periodic regular arrangements as described by Mackay.[76] Six pentagons in Figure 8 combined make a larger pentagon and the resulting five isosceles triangles when joined together with still another smaller regular pentagon produce a pentagon which is identical to six other pentagons (see Figure 9).

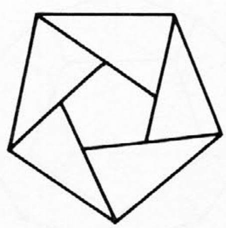


Figure 9. Five isosceles triangles and a smaller pentagon make a pentagon identical to each of the six pentagons in Figure 8.

If this construction is repeated on an ever-increasing scale, then the hierarchical packing of pentagons builds up like a pentagonal snowflake.

There are also other families of polyhedra with smaller degrees of regularity.[77,78] One such family are the semiregular or Archimedean polyhedra which are similar to the Platonic solids in that all their faces are regular and their vertices are congruent. The difference is that their faces are not polygons of the same kind. There are 13 semiregular polyhedra.[79] These are the truncated tetrahedron, truncated cube, truncated octahedron, cuboctahedron, truncated cuboctahedron, rhombicuboctahedron, snub cube, truncated dodecahedron, icosidodecahedron, truncated icosahedron, truncated icosidodecahedron, rhombicosidodecahedron and snub dodecahedron. All 13 semiregular polyhedra possess twofold and threefold rotation axes. Structures from the truncated cube to snub cube possess fourfold rotation axes, whilst the last six semiregular polyhedra have fivefold axes of rotation. It is interesting to note that Stanko Bilinski, a renowned Croatian mathematician, discovered and reported in his dissertation in 1943, and later published (1948),[80] the 14th Archimedean polyhedron which differs from the rhombicuboctahedron only in that the upper part, which consists of 5 squares and 4 equilateral triangles, is rotated for $\pi/4$ radians. The same geometric figure was later (in 1957) rediscovered by Russian mathematician Aškinuze.[81] However, there are different definitions of semiregular polyhedra, and if one (along with Grünbaum[77] and others[82]) requires all their vertices to be point-group equivalent, then this 14th semiregular polyhedron is excluded from the family of Archimedean polyhedra.

It is pleasing to note at this point that two of the most popular ballgames possess either pentagonal symmetry (basketball) or are played with a fivefold symmetric object (soccerball).

Buildings other than geodesic domes but with fivefold symmetry are very rare.[83] This is a pity because the effect of the pentagonal shape, especially in a tall building or sky-scraper, could be quite attractive. The best known pentagonal building in the world is the Pentagon, headquarters of the U.S. Department of Defense near Washington, D.C. This building is a hollow low pentagonal prism, built so during the Second War in order to save steel. The hollow part of the Pentagon has also the shape of the pentagonal prism. However, the rooms in the Pentagon are not pentagonal. Some ten years ago, I saw the Pentagon from a distance, but I have never been in a pentagonal room. It seems strange that the architects are biased against pentagonal rooms, because this shape has many appealing features. The corners are more open than those in rectangular rooms, the walls are not parallel, there is a strong focus on the axis in the center, it may diminish claustrophobic feelings, *etc.*

It is interesting enough to report that the European postal system is not prejudiced against pentagonal letters. A letter in the form of a pentagon was mailed to my address at the Rugjer Bošković Institute in Zagreb by Professor Sven J. Cyvin from Trondheim in Norway. I got it without any problem. In addition, my family name Trinajstić was short-handed to 13-ić. A photo of this letter is shown in Figure 10.

In a science-fiction classic about life in a 2D world entitled »Flatland« by a noted Shakespearean scholar Edwin A. Abbott (1838–1926), whose hobby was the study of higher mathematics, almost all houses are pentagonal.[84] The following lines explain why pentagonal houses are preferred in Flatland:

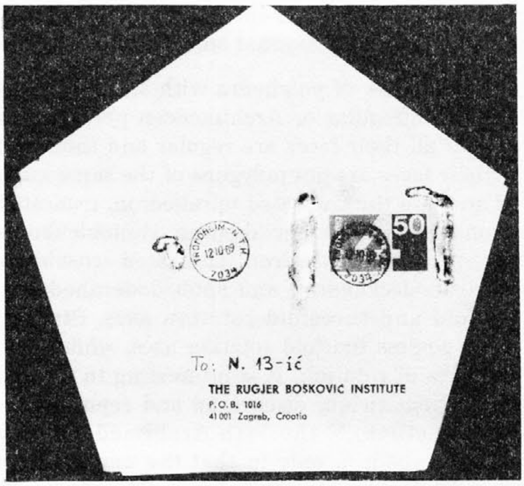
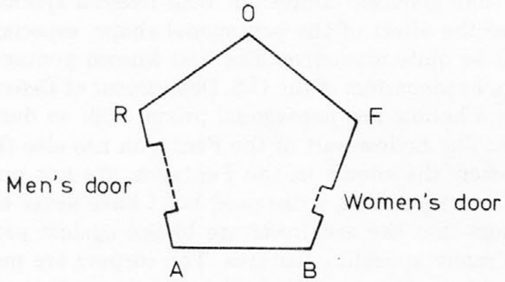


Figure 10. A photo of a pentagon-shaped letter which arrived at my Zagreb address without any difficulty from Trondheim.

»The most common form for the construction of a house is five-sided or pentagonal, as in the annexed figure. The two Northern side RO, OF, constitute the roof, and for the most part have no doors; on the East is a small door for the Women; on the West a much larger one for the Men; the South is usually doorless.



Square and triangular houses are not allowed, and for this reason. The angles of a Square (and still more those of an equilateral Triangle), being much more pointed than those of a pentagon, and the lines of inanimate objects (such as houses) being dimmer that the lines of Men and Women, it follows that there is no little danger lest the points of a square or triangular house residence might do serious injury to an inconsiderate or perhaps absent-minded traveler suddenly therefore, running against them: and as early as the eleventh century of our area, triangular houses were universally forbidden by Law, the only exceptions being fortification, powder-magazines, barracks, and other state buildings, which it is not desirable that the general public should approach without circumspection.

At this period, square houses were still everywhere permitted, though discouraged by a special tax. But, about three centuries afterwards, the Law decided that in all towns containing a population above ten thousand, the angle of a Pentagon was the smallest house-angle that could be allowed consistently with the public safety. The good sense of the community has seconded the efforts of the Legislature; and now, even in the country, the pentagonal construction has superseded every other. It is only now and then in some very remote and backward agricultural district that an antiquarian may still discover a square house.« Unfortunately, even in Flatland rooms are not pentagonal, though in some houses the Hall and the Cellar have the shape of an irregular pentagon.

Not only houses, but some people are also pentagonal in Flatland: »Our Professional Men and Gentlemen are Squares (···) and Five-Sided Figures or Pentagons.« The sons have one more side than the fathers, but his rule does not apply to all classes: »It is a Law of Nature with us that a male child shall have one more side than his father, so that each generation shall rise (as a rule) one step in the scale of development and nobility. Thus, the son of a Square is a Pentagon; the son of a Pentagon, a Hexagon; and so on.

But this rule applies not always to the Tradesmen, and still less often to the Soldiers, and to the Workmen; ...«

My interest in the book by Abbott was aroused after I read the article based on the Nobel prize lecture delivered on December 12, 1975 in Stockholm by Vladimir Prelog on chirality in chemistry.[85] In this article, Prelog described chiral objects in Flatland. The simplest chiral object in Flatland is scalene. However, in the book there are all kinds of amusing objects that can easily stimulate the imagination of a scientist. Actually, this book, although written in the early 1880's, is still a first-rate fictional introduction to the concepts of relativity and multidimensions of space. Furthermore, besides being fascinating reading, »Flatland« helps the reader to understand certain aspects of modern science better than most texts.

FIVEFOLD SYMMETRY IN THE LITERATURE

Symmetry in general and fivefold symmetry in particular provide artists and writers with a rich source of themes, images and metaphors.[16,17,25,42,61,62,70,86–90] Here, I will present some examples of the use of fivefold symmetry in the literature.

To detect symmetry in the literature is not an easy task, though in some case the writer may use, in a direct way, a symmetric and/or asymmetric object as a descriptive means or a metaphor. The abstractness of language makes literary works difficult to analyze in terms of symmetry operations. Symmetry operations such as rotations, reflections or translations do not appear discernible in a simple way in the literature; they are, if they exist, usually hidden. Lately, however, efforts have been made[86] to associate the structure of a literary work (*e.g.*, poem, novel, short story, play) with a diagram or a geometric object with some characteristic symmetry or antisymmetry. A diagram may be used, for example, to visualize the characters in a novel or a play.[91] A whole literary work may be associated with some geometric (spatial) form or it may be associated with several spatial forms, tightly or loosely related.[86,92,93] Very often in poetry, we find poems in graphical shapes which are isomorphic with diagrams that possess recognizable symmetric or asymmetric characteristics. A good example to illustrate this point is a funnel-like bilateral symmetric shape of a poem by the German neoromantic poet Christian Morgenstern (1871–1914), appropriately entitled »Die Trichter«.[94]

To associate the structure of a literary work with a 3D object is not simple because we deal with non-spatial objects to which we ascribe spatial features. Such an analysis and interpretation are usually highly subjective and a given non-spatial configuration may be associated with different symmetric or asymmetric figures, depending on the imagination of the person carrying out such a study.[86]

I have already stated in section two that numbers are often used by poets. Here, I present a poem entitled »Quinta essentia«[95] by an important modern Croatian poet Boro Pavlović (b. 1922) in which he plays »in quantized harmony«, as a poet and Nobel-prize winning scientist Roald Hoffmann (b. 1937) would say,[96] with the first five numbers in five languages (English, Latin, Greek, Italian and Croatian):

QUINTA ESSENTIA

one	unus	eis	un(o)	je(da)n
one	una	mia	una	je(d)na
one	unum	en	un	je(d)no
two	duo	dyo	due	dva
two	duae	dyein	due	dvije
three	tres	treis	tre	tri
four	quattour	tessares	quattro	čet(i)ri
five	quinque	penta	cinque	pet

PANTA
&
TUTTA
QUANTA

The formal structure of this poem may be associated with an inverted pentagonal pyramid whose base is a regular pentagon (see Figure 11):

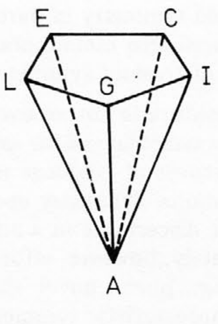


Figure 11. The geometry of the poem »Quinta essentia« by Boro Pavlović. The labels have the following meaning: A = Arabic numerals, C = Croatian language, E = English language, I = Italian language, G = Greek language and L = Latin language.

Each vertex of a pentagon represents one language and each edge the translation process from one language into the other. The bottom of the pyramid is labelled by Arabic numerals. Rotation by $2\pi/5$ radians always brings a different code for a given number at the bottom of the pyramid, which serves as a decoding device.

Since the meaning is always the same, all the vertices are of the same kind, but since the coding process (language) is different, the vertices are labelled by different letters. Therefore, the hidden structure of this poem possesses pentagonal symmetry.

I have already pointed out that the possible origin of the number five is the fact that there are five fingers on each hand. Hands have also been used by writers in different contexts. For example, the Croatian writer Ranko Marinković (b. 1913) has produced a wonderful short story entitled »Hands«[97] in which the hands possess a life of their own. The author aims to show in this story that the hands are something more than simply the executive organs of the spirit. Each hand differs: the right hand is more aggressive and stronger than the left hand. However, when they work, the work is efficiently performed only if they act together. (We should remember that two hands are chiral objects, identical in shape, but not superimposable because one is right and one is left. They represent a prototypical chiral pair.[85]) In this story, Marinković has shown that each hand individually is unique (an asymmetric figure, but by acting together, they form a symmetric figure).

It is amusing to note that there is a horror story related to the above by William Fryer Harvey entitled »The Beast With Five Fingers,«[98] in which one severed hand possesses all the (even evil) characteristics of a human being. It walks, talks, writes, hates, *etc.* the same as hands in the above mentioned story. The element of horror in the story is produced by the persistent persecution of the main character (Eustace Borlsover) by the severed right hand of his uncle (Adrian Borlsover) because he has offended it. But there is nothing unfathomable about the hand as Eustace said to Saunders (the secretary of the late Adrian) who admitted that he was scared stiff by the hand's continuous harassment: »You've no need to be. There's nothing supernatural about the hand, Saunders, I mean, it seems to be governed by the laws of time and space. It's not the sort of thing that vanishes into thin air or slides through oaken doors...« Here, I point to a recent movie »The Addams Family« that pokes fun at horror films, and in which a hand called »the Thing« is an irreplaceable servant to the whole family.

There is also a fair number of other stories in which hands have a life and a purpose of their own. For example, Guy de Maupassant (1850–1893) wrote two stories in which a hand is central: »The Hand« and »The Withered Hand«. Sir Arthur Conan Doyle (1859–1930) one: »The Brown Hand« and Theodore Dreiser (1871–1945) also one: »The Hand«. To these, it is appropriate to add William Jacob's (1863–1943) classic spine-chiller »The Monkey's Paw«. Here, I wish to mention a sinister tale by Thomas Burke (1886–1945) entitled: »The Hands of Mr. Ottermole«,[99] perhaps one of the finest short stories in the detective story genre. In this story, at the climatic end, the main character (the sergeant Mr. Ottermole) expressed the following idea, which might be considered the idea common to all the three stories about hands mentioned here: »...Can't ideas live in nerve and muscle as well as in brain? Couldn't it be that parts of our bodies aren't really us, and couldn't ideas come into those parts all of a sudden, like ideas come into – into' – $\cdots$ – 'into *my hands*'?«

A poem about the hand entitled: »The hand that signed the paper« by Dylan Thomas (1914–1953) is terrifying. A few lines of this poem are given below:

»The hand that signed the paper felled a
city;
Five sovereign fingers taxed the breath,
Doubled the globe of death and halved a
country;
These five kings did a king to death.«

Much nobler are the following lines from the prose poem »The Hands« by Srećko Karaman (1909–1964):[100] »Hands are the strongest in the prayer. And much more in the gratitude.«

An example of the direct use of a fivefold symmetric object in the literature, *e.g.*, the pentagon, may be found, for instance, in a spectacular novel »Poland« by James A. Michener (b. 1907).[101]

One chapter (Chapter 63) in the part entitled »Gevurah« of the monumental novel »Foucalt's Pendulum« by Umberto Eco[102] contains discourse about the first ten numbers and the human body. Each of these numbers enumerates some part of the body. For example, in the words of the author: »...two arms and two legs make four, and four is a beautiful number when you consider that animals have four legs and little children go on all fours, as the Sphinx knew. We hardly have to discuss five, the fingers of the hand, and then with both hands you get the other sacred number, ten. There have to be ten commandments because, if there were twelve, when priest counts one, two, three, holding up his fingers, and comes to the last two, he'd have to borrow a hand from the sacristan.« However, our body does not know the number three. But, the matching of a male and female may lead to the number three through their marital union when they produce a child.

Probably the most prominent pentagram in the literature is the one used by Johann Wolfgang Goethe (1749–1832) in »Faust«[103] (the first part of this drama appeared in 1808 and the second part in 1832 after the death of its author). The pentagram was used by sorcerers and magicians as a powerful device of magic.[7] Similarly, the pentagram in »Faust« is used as a mighty device by dr. Faust to exorcise Mephistopheles, the devil:

FAUST. Das Pentagramma macht dir Pein?
Ei sage mir, du Sohn der Hölle,
Wenn das dich bannt, wie kommst du denn herein?
Wie ward ein solcher Geist betrogen?
MEPHISTOPHELES. Beschaut es recht! Es ist nicht gut gezogen;
Der eine Winkel, der nach aussen zu,
Ist wie du siehst, ein wenig offen.

THE NUMBER FIVE AND FIVEFOLD SYMMETRY IN CHEMISTRY

Amongst other applications, the Fibonacci numbers and Catalan numbers have found use in chemistry[104,105] for counting valence structures of polycyclic conjugated systems. Fibonacci numbers correspond to the numbers of Kekulé valence structure[106,107] for the zig-zag chains of benzene rings called fibonaccenes[108] or fibonacenes*[109] (polyphenanthrenes) and helicenes, whilst Catalan numbers represent the total number of

* Balaban[109] proposed the term »fibonacenes« (with one c) instead of etymologically more suitable »fibonaccenes« for simplicity and for similarity with the established term »polyacenes«

valence structures (Kekulé structures, Dewar structures and higher excited valence structures) for benzenoid hydrocarbons.[110]

There are only two benzenoid hydrocarbons with five Kekulé valence structures: phenanthrene and tetracene, and only benzene has the total of five neutral valence bond structures (two Kekulé structures and three Dewar structures). Fibonacci numbers were also identified as the numbers of conjugated circuits of fibonaccenes and helicenes[111] and, thus, they are related to a simple VB resonance-theoretic model named the conjugated-circuit model.[112,113] A recent review article describes many other uses of Fibonacci numbers in chemistry.[114]

The number five additionally appears in a number of areas of chemistry. For example, there are 3 isomeric alkanes, 5 isomeric alkenes, 8 isomeric alkyl radicals, 4 isomeric primary alcohols, 3 isomeric secondary alcohols, 1 tertiary alcohol, 9 isomeric esters, 14 isomeric glycols, *etc.* with five carbon atoms. Similarly, there are 22 polyhex hydrocarbons[115] with five hexagons. The number five also appears in the count of d-orbitals affording the colors to most beautiful and valuable gem stones.

Pascal's triangle has found many uses in chemistry and physics. Let me mention here only two examples of its recent utilization. The Kekulé structures of benzenoid hydrocarbons may be counted using the truncated Pascal's triangle[116] and a new class of geometric fractals may be derived from Pascal's triangle, of which the Sierpinski gasket[117,118] is a special case.[119]

In chemistry, a hydrocarbon, $C_{20}H_{20}$, named dodecahedrane, is known whose carbon skeleton has the structure of the dodecahedron, the most beautiful of Platonic solids.[120,121] Many borides and ice structures involve dodecahedra, as attractively illustrated in the Pauling and Hayward book on chemical architecture.[122]

In recent years, there has been considerable interest in Archimedean polyhedra in general, and in the truncated icosahedron in particular.[123] The reason is the experimental observation of a remarkably stable cluster consisting of 60 carbon atoms, formed by laser vaporization of graphite in a high-pressure supersonic laser.[124] It has been proposed that this cluster takes the »unusually beautiful« form of the truncated icosahedron (see Figure 12) with carbon atoms at each of the vertices and σ-bonds along each edge. Since the late autumn 1990, the buckminsterfullerene and the family

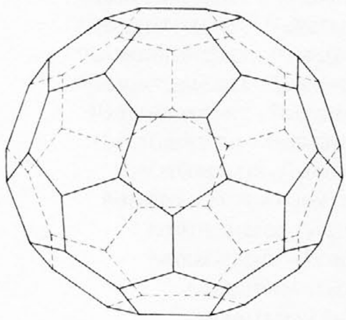


Figure 12. The truncated icosahedron that serves as a model for the structure of the stable C_{60} cluster.

of related C_n ($n \geq 26$; n = even) clusters called fullerenes have been readily available by the Krätschmer-Huffman method.[125] After their preparation of C_{60}, an explosion of research articles on the fullerenes has occurred.[126] Presently, the research on buckminsterfullerene and fullerenes is the hottest research topic in chemistry.[127] Fullerenes are the third allotropic form of carbon, or rather a family of allotropes, after graphite and diamond.

The truncated icosahedron has 60 vertices, 90 edges and 32 faces, 12 of which are pentagonal and 20 hexagonal. This geometrical structure is of considerable interest in the design of geodesic polyhedra, *i.e.*, polyhedra constructed on the surface of a sphere by a network of intersecting geodesic lines (*i.e.*, line segments on great circles). Richard Buckminster Fuller (1895–1983) developed the art of construction of geodesic (dome-like) polyhedra[128] for the C_{60} and the name buckminsterfullerene was suggested by Kroto *et al.*[124,129] for the C_{60} cluster with the geometry of the truncated icosahedron.

TABLE I

The characteristic polynomial of buckminsterfullerene

$$
\begin{aligned}
|A + xI| = {} & x^{60} - 90x^{58} + 3825x^{56} \\
& + 24x^{55} - 102160x^{54} \\
& - 1920x^{53} + 1925160x^{52} \\
& + 72240x^{51} - 27244512x^{50} \\
& - 1700640x^{49} + 300906380x^{48} \\
& + 28113600x^{47} - 2661033600x^{46} \\
& - 347208896x^{45} + 19180834020x^{44} \\
& + 3327625680x^{43} - 114118295000x^{42} \\
& - 25376437920x^{41} + 565407465144x^{40} \\
& + 156652575440x^{39} - 2346799508400x^{38} \\
& - 792175427520x^{37} + 8189116955350x^{36} \\
& + 3308173115904x^{35} - 24056403184260x^{34} \\
& - 11466942645600x^{33} + 59443188508110x^{32} \\
& + 33076275953760x^{31} - 123163094844616x^{30} \\
& - 79417625268960x^{29} + 212712221820840x^{28} \\
& + 158412719276240x^{27} - 303315997028160x^{26} \\
& - 261359090670624x^{25} + 351861389316780x^{24} \\
& + 354145195147200x^{23} - 324375523213200x^{22} \\
& - 390055074762240x^{21} + 228227031040884x^{20} \\
& + 344185906596720x^{19} - 122654402736360x^{18} \\
& - 238553091055200x^{17} + 29617003666920x^{16} \\
& + 126428882536240x^{15} + 4679380503120x^{14} \\
& - 49433493646080x^{13} - 8131429397135x^{12} \\
& + 13627897407360x^{11} + 3576552321006x^{10} \\
& - 2527365617120x^{9} - 831616531095x^{8} \\
& + 310065067080x^{7} + 108565938200x^{6} \\
& - 26034025632x^{5} - 7440712560x^{4} \\
& + 1566501120x^{3} + 186416640x^{2} \\
& - 54743040x + 2985984 \\
= {} & (x+3)(x+1)^9 (x-2)^4 (x^2+x-3)^5 (x^2-3x+1)^3 (x^2-x-1)^5 (x^2-x-4)^4 (x^4+3x^3-2x^2-7x+1)^3
\end{aligned}
$$

Other names which could be used for the C_{60} cluster are footballene (in Europe) and soccerballene (in the USA), because of its structural similarity with the surface structure of the modern soccerball.

The C_{60} cluster is a very stable conjugated structure. It possesses 12.500 Kekulé structures.[130,131] Its reconance energy (computed, for example, by the conjugated-circuit model with Herndon's parameters)[58,113] is comparable in the normalized form (RE/N; N=60; RE/60=0.120 eV) to that of benzene (RE/6=0.140 eV).[130,132] Buckminsterfullerene is also a hard[133] and inert[134] molecule. I also give here its characteristic and acyclic (matching) polynomials.[131,135,136] These polynomials are given in Table I and Table II. Their zeros can be used in the computation of topological resonance energ (TRE) of C_{60}. TRE of buckminsterfullerene in the normalized form (TRE/N; N=60) is equal to 0.0274β. Note, that TRE/N for benzene (N=6) is 0.046β. Similarly, the zeros of the characteristic polynomials can be used to generate a Hückel-based resonance energy ($RE/60=0.553\beta$) of C_{60}.

When the characteristic and acyclic polynomials are given in the forms such as in Tables I and II, their pictures have, as stated by Brendsdal *et al.*,[136] »an intrinsic beauty, not quite unlike a modern poem«. It reminds me of some poems by the Croatian poet Dražen Mazur (b. 1951).

The last coefficient a_N^{ac} of the acyclic polynomial is equal to the Kekulé-structure count K of C_{60}: a_N^{ac} = K = 12.500. Perhaps, the evident »fiveness« of the Kekulé-structure count, written differently: $K = 2^5 5^5$ for buckminsterfullerene is also some kind of manifestation of the fivefold symmetries of the C_{60} cage.[131]

The number of spanning trees in buckminsterfullerene[137] is 375 291 866 372 898 816 000, or, factorized into powers of prime numbers: $2^{25} \times 3^4 \times 5^3 \times 11^5 \times 19^3$. The authors of this computation also noted with amusement that, in the above numbers, the sum of the bases, (2+3+5+11+19), is equal to the sum of the powers, (25+4+3+5+3), both being 40.

TABLE II

The acyclic polynomial of buckminsterfullerene

$$x^{60} - 90x^{58}$$
$$+ 3825x^{56} - 102120x^{54}$$
$$+ 1922040x^{52} - 27130596x^{50}$$
$$+ 298317860x^{48} - 2619980460x^{46}$$
$$+ 18697786680x^{44} - 109742831260x^{42}$$
$$+ 534162544380x^{40} - 2168137517940x^{38}$$
$$+ 7362904561730x^{36} - 20949286202160x^{34}$$
$$+ 49924889888850x^{32} - 99463457244844x^{30}$$
$$+ 165074851632300x^{28} - 227043126274260x^{26}$$
$$+ 256967614454320x^{24} - 237135867688980x^{22}$$
$$+ 176345540119296x^{20} - 104113567937140x^{13}$$
$$+ 47883826976580x^{16} - 16742486291340x^{14}$$
$$+ 4310718227685x^{12} - 783047312406x^{10}$$
$$+ 94541532165x^{8} - 6946574300x^{6}$$
$$+ 269272620x^{4} - 4202760x^{2}$$
$$+ 12500$$

The enumeration of all C_{60} cages associated to trivalent polyhedra with five- and six-sided faces produced 1812 cages (not distinguishing mirror images).[138–140] The earlier computation of Liu *et al.*[141] was erroneous. They obtained 1790 isomeric C_{60} cages. A valuable result of these enumerations is that the most stable of isomeric C_{60} cages is the buckminsterfullerene structure.

In 1965, a hydrocarbon, $C_{10}H_{10}$, was proposed[142] whose carbon skeleton resembled the Pentagon and which was named the pentaprismane (see Figure 13). In this figure, the flat drawing emphasized the prism containing two regular pentagons in parallel planes conjoined by five-membered rings. Incidentally, the name: »pentaprismane« is much prettier than the IUPAC name »hexacyclo[5.3.0.0.^{2,6}0.^{3,10}0.^{4,9}0^{5,8}] decane«.[143]

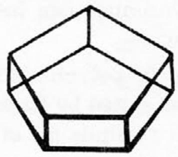


Figure 13. The model for the carbon skeleton of pentatprismane as a regular pentagonal prism (of D_{5h} symmetry).

Empirical force-field calculations[144,145] predicted pentaprismane to be loaded with strain (135–145 kcal/mol). Apparently, pentagonal systems are not easily constructed. Many efforts at preparing this ring system have failed. However, in 1981, pentaprismane was synthesized and isolated[146] in quite high yields (40–45%). Pentaprismane belongs to the D_{5h} symmetry group, similarly as, for example, ruthenocene $(C_5H_5)_2Ru$. Ruthenocene is a sandwich-like compound[147] in which ruthenium is sandwiched between two cyclopentadienyl ring systems in the *syn* position. Because of its symmetry (D_{5h}), the structure of ruthenocene may also be modeled by a regular pentagonal prism (see Figure 14) which is, in this case, much taller than that representing pentaprismane.

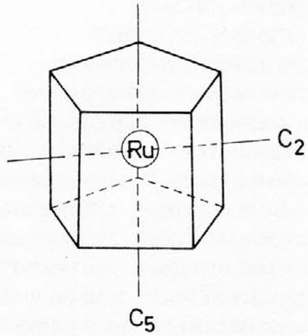


Figure 14. The structure of ruthenocene represented by a regular pentagonal prism. The twofold and fivefold axes of rotation are indicated.

Ferrocene,[148,149] exhibiting somewhat similar sandwiched molecular structure, in which iron is sandwiched between two cyclopentadienyl ring systems in the *anti* position (see Figure 15), is also a pentagonally symmetric compound that played a central role in Fischer and Wilkinson's Nobel Prize in chemistry (1973).

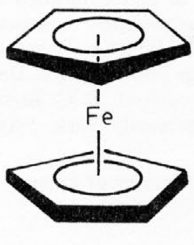


Figure 15. The structure of ferrocene. Iron is sandwiched between two cyclopentadienyl ligands in the *anti* position.

Cyclopentane, which is a parent molecule of cyclolpentadienyl moiety, exhibits a broken 5-fold symmetry which, as Kilpartrick and Pitzer[150] proposed, is elegantly restored through pseudorotation.

CONCLUDING REMARKS

In this essay, I have collected the appearances and uses of the number five in different areas. In many cases, it is understandable why the number five appeared, but in some cases we do not comprehend why five rather than four or six. In some sports, the number five probably emerged as an optimum number after much trial and error. The game of basketball may be (as it was early in its YMCA days) played by twelve or four players (as it is nowadays done in parks and streets), but, the optimum number of players is established to be five on each side in the regular professional or amateur game.

I also noted some occurrences and utilizations of the fivefold symmetric objects, such as the pentagon, five-pointed star, isocahedron, dodecahedron, truncated isocahedron. They are used for many reasons: symbolic, esthetic, mathematical, practical, *etc.* Some plants and animal possess fivefold symmetry. The reasoning behind that must be related to evolution, since it seems that evolution follows the symmetry rules. For example, the representatives of a given animal species in order to survive must have an optimum size and shape, which is ordinarily very symmetric. The symmetry requirements for the (pentagonal) spatial organization apparently appear indirectly in the genetic code of all living matter, *e.g.*[151]

Several examples of the use of the number five and fivefold symmetry in the literature are also shown. These latter examples are not as abundant as those with other types of symmetry.[86] To conclude, I may say that the use of the number five and fivefold symmetry in the art and literature is related, in some way, to the search for unusual beauty inherent and induced by this particular type of symmetry. However, we should remember that the aesthetic effects resulting from symmetry of an object or a literary work lie in the mental process of perception, and this process is highly individualistic.

The number five and fivefold symmetry are abundantly present in chemistry. Only a few examples are listed in this essay. However, even in this limited number of examples, it is easily seen that five-fold symmetric objects in chemistry are the molecules with unusual, but attractive, structures which posses out-of-ordinary properties.

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REFERENCES

1. M. Rozsondai and B. Roszondai, *Comput. Math. Appl.* **17** (1989) 837–885.
2. J. Coke, *Nature* **347** (1990) 14–15.
3. L. Conant, *The Number Concept, Its Origin and Development*, Macmillan, New York, 1923.
4. D. Smeltzer, *Man and Number*, Emerson, New York, 1958.
5. *e.g.* R. A. Raimi, *Sci. Am.* **221** (1969) 109–120.
6. C. G. Jung and W. Pauli, *Naturerklärung und Psyche* (a translation from German into Croatian), Globus, Zagreb, 1989. Appendix II contains Robert Fludd's essay on the number four (in Latin and a translation into Croatian).
7. J. Chevalier and A. Cheerbrant, *Dictionary of Symbols* (a translation from French into Croatian), Matica hrvatska, 1983, pp. 713–714.
8. M. Grakalić, *Croatian Coat of Arms* (in Croatian), Matica hrvatska, 1990.
9. *e.g.* A. Lakhtakia, *Symmetry* **1** (1990) 93.
10. P. Iyer, The Many Lives and Tricks of 9, *Times* (September 16, 1991), 78.
11. S. Devi, *Figuring* (The Joy of Numbers), Harper & Row, New York 1977. pp. 22–23.
12. Ž. Mađer, *Excuso*, Privlačica, Privlaka, 1986.
13. A. E. Brehm, *Der Farbige Brehm* (translation from German into Croatian), Prosvjeta, Zagreb, 1982.
14. D. Wells, *The Penguin Dictionary of Curious and Interesting Numbers*, Penguin, London, 1987.
15. *Nassrudin-Khoja Anecdotes* (translated from Croatian into English by NT), Svjetlost, Sarajevo, 1987, p. 181.
16. H. Weyl, *Symmetry*, University Press, Princeton, NJ, 1952.
17. I. Hargittai and M. Hargittai, *Symmetry through the Eyes of a Chemist,* VCH, Weinheim, 1986.
18. Leonardo Pisano Fibonacci, *The Book of Squares,* Academic Press, Boston, 1987, p. XV.
19. D. J. Struik, Ed., A. *Source Book in Mathematics: 1200—1800,* Harvard University Press, Cambridge, MA, 1969, pp. 2–4.
20. *e.g.*, R. E. M. Moore, *Science* **161** (1968) 1358–1359.
21. N. N. Vorob'ev, *Fibonacci Numbers,* Blaisdell, New York, 1961.
22. H. E. Huntley, *The Divine Proportion*, Dover, New York, 1970.
23. S. Batušić, *Art in Pictures* (in Croatian), Matica hrvatska, Zagreb, 1967.
24. Z. Črnja, *Cultural History of Croatia* (in Croatian, but there is also an English translation available), Epoha, Zagreb, 1965.
25. M. Pejaković and N. Gattin, *Early Croatian Sacral Architecture* (in Croatian), Matica hrvatska, Zagreb, 1988.
26. W. Blackwell, *Geometry in Architecture,* Wiley, New York, 1984.
27. A. P. Stakhov, *Comput. Math. Appl.* **17** (1989) 613–638.

28. T. A. Cook, *The Curves of Life*, Dover, New York, 1979.
29. R. Dixon, *Comput. Math. Appl.* **17** (1989) 535–538.
30. D. I. Cohen, *Basic Techniques in Combinatorial Theory,* Wiley, New York, 1978, pp. 121–132.
31. K. Kuratowski, *Fund. Math.* **15** (1930) 271–283.
32. *The Holy Bible Containing the Old and New Testaments,* King James Version, 1973 Edition.
33. *Webster's Ninth New Collegiate Dictionary,* Merriam-Webster, Springfield, MA, 1985, p. 870.
34. R. Cavendish and T. O. Ling, *Mythology* (a translation from English in Croatian), Mladost, Zagreb, 1982.
35. I. Nicholson, *Mexican and Central American Mythology* (a translation from English into Croatian), Keršovani, Opatija, 1984.
36. E. E. Cummings, *Collected Poems,* Harcourt, Brace & Co., New York, 1938.
37. M. Stojević, Ed., *Chakavian Poetry in the 20th Century* (in Croatian), Izdavački centar, Rijeka, 1987.
38. N. Trinajstić, in: *Fivefold Symmetry,* I Hargittai, Ed., World Scientific, Singapore, 1992. pp. 395–405.
39. S. Buchanan, *Poetry and Mathematics*, The University of Chicago Press, Chicago, 1975.
40. J. Updike, *Seventy Poems* (a translation from English into Croatian), Prosvjeta, Zagreb, 1981, p. 132.
41. T. B. Costain, *The Silver Chalice,* Doubleday, Garden City, NY, 1952.
42. G. Pischel, *Storia Universale dell'Arte* (a translation from Italian into Croatian), Mladost, Zagreb, 1966. Vol. 2, p. 159.
43. K. Miyazaki, *An Adventure in Multidimensional Space,* Wiley, New York, 1987.
44. *e.g.*, J. J. Seidel, *Annals New York Acad. Sci.* **319** (1979) 497–507; A. L. Loeb, *Comput. Math. Appl.* **17** (1989) 33–48.
45. B. Grünbaum and G. C. Shephard, *Tilings and Patterns,* Freeman, San Francisco, 1987.
46. *e.g.* D. Chavey, *Comput Math. Appl.* **17** (1989) 147–155.
47. L. Collatz, *Geometrische Ornamente,* Universität, Augsburg, 1985.
48. R. Williams, *The Geometrical Foundation of Natural Structure,* Dover, New York, 1979, p. 204.
49. J. Kepler, *Harmonice Mundi,* Opera Omnia, Vol. 5, Frankfurt, 1864.
50. *International Society for Mathematical Chemistry Newsletter,* Issue 9, May 31, 1992, p. 3.
51. P. J. Steinhardt, *Sci. Am.* **74** (1986) 586–597.
52. G. G. Hall, *Molecular Solid State Physics,* Springer, Berlin, 1991.
53. F. Harary, *Graph Theory,* 2nd printing, Addison – Wesley, Reading, MA, 1971, pp. 89–90.
54. J. Petersen, *L'Intermédiaire des Mathématiciens* **5** (1898) 225–227. A translation from French into English may be found in Ref. 55, pp. 197–199.
55. N. L. Biggs, E. K. Lloyd, and R. J. Wilson, *Graph Theory,* 1736–1936, Clarendon, Oxford, 1977, Chapter 10.
56. J. Brocas, M. Gielen, and R. Willem, *The Permutational Approach to Dynamic Stereochemistry,* McGraw-Hill, New York, 1983, p. 647.
57. A. T. Balaban, D. Farcasiu and R. Banica, *Rev. Roum. Chim.* **11** (1966) 1205–1227.
58. M. Randić, *Chem. Phys. Lett.* **42** (1976) 283–287.
59. M. Randić, *Croat. Chem. Acta* **49** (1977) 643–655.
60. A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance Regular Graphs,* Springer, Berlin, 1989.
61. W. I. Michailenko and A. W. Kaschtshcenko, *Natur-Geometrie-Architektur,* VEB, Berlin, 1986.
62. W. Hahn, *Symmetrie als Entwicklungsprinzip in Natur und Kunst,* Langewiesche, Königstein, 1989, see also W. Gilde, Gespiegelte Welt, VEB, Berlin, 1979.
63. F. Weberling, in: *Symmetrie,* H. Stark, Ed., Aulis, Kòln, 1985, pp. 141–186; see also M. Hargittai, in: *Fivefold Symmetry,* I. Hargittai, Ed., World Scientific, 1992, pp. 529–541.
64. D'Arcy W. Thompson, *On Growth and Form,* 2nd edition, reprinted, University Press, Cambridge, 1942, Vol. II, p. 726.
65. D. Levine and P. J. Steinhardt, *Phys. Rev.* **B34** (1984) 2477–2480; M. Rochetti, *Phil. Mag.* **B56** (1987) 237–249.

66. I. Hargittai, Ed., *Quasicrystals, Networks and Molecules of Fivefold Symmetry,* VCH, New York, 1990.
67. S. Bilinski, *Nastavni vjestnik* **51** (1943) 281–285.
68. G. Sarton, *Introduction to the History of Science* Reprinted, Carnegie Institution, Washington, 1968. Vol. 1, p. 116.
69. Plato, *Timaeus and Critias,* Penguin, Harmondsworth, Middlesex, 1986, p. 77.
70. L. Pacioli, *De Divina Proportione,* Venice, 1509, reprinted in 1956.
71. V. Devidé, *Entertaining Mathematics* (in Croatian), Školska knjiga, Zagreb, 1988. pp. 99–102.; R. A. Dunlap, in: *Fivefold Symmetry,* I. Hargittai Ed., World Scientific, 1992, pp. 489–504.
72. A. Maletić, *Motion and Dance* (in Croatian), Kulturno-prosvjetni sabor Hrvatske, Zagreb, 1983, pp. 151–153.
73. J. Watson and F. Crick, *Nature* **177** (1956) 473–475.
74. D. L. D. Caspar and A. Klug, *Cold Spring Harbor Symp. Quantum Biol.* **27** (1961) 1–13.
75. A. L. Mackkay, *Symmetry* **1** (1990) 3–17; see also D. W. Crowe, in: *Fivefold Symmetry*, I. Hargittai Ed., World Scientific, Singapore, 1992., pp. 465–487.
76. A. L. Mackay, *Pysics Bull.* (November, 1976) 495–497.
77. B. Grünbaum, *Convex Polytopes*, Wiley, New York, 1967.
78. H. S. M. Coxeter, *Regular Polytopes,* 3rd edition, Dover, New York, 1973.
79. R. Pearce and S. Pearce, *Polyhedra Primer,* Nostrand-Rheinhold, New York, 1978.
80. S. Bilinski, *Rad JAZU* **271** (1948) 1–111.
81. V. G. Aškinuze, *Mat. Prosvešč.* **1** (1957) 107–118.
82. N. W. Johnson, *Can J. Math.* **18** (1966) 169–200.
83. D. Pedoe, *Geometry and Visual Arts,* Dover, New York, 1976.
84. E. A. Abbott, *Flatland,* Dover, New York, 1952, 6th revised edition.
85. V. Prelog, *Nobel Lecture* (Stockholm, December 12, 1975). This lecture was published in *Science* **193** (1976) 17–24, and its translation from English into Croatian in *Croat. Chem. Acta* **48** (1976) 194–208.
86. B. Pavlović and N. Trinajstić, *Comput. Math. Appl.* **12B** (1986) 197–227.
87. V. Molnar and F. Molnar, *ibid.* **12B** (1986) 291–301.
88. L. A. Cummings, *ibid* **12B** (1986) 981–997.
89. L. de Freitas, *ibid.* **17** (1989) 887–889.
90. L. A. Cummings, in: *Fivefold Symmetry,* I. Hargittai, Ed., World Scientific, Singapore, 1992. pp. 407–422.
91. S. Lasić, *The Structure of Krleža's »Banners«* (in Croatian), Liber, Zagreb, 1974.
92. C. H. Whitman, *The Geometric Structure of the Illiad,* Norton, New York, 1965.
93. M. Rose, *Shakespearean Design,* Belknap, Cambridge, MA, 1972.
94. C. Morgenstern, in: *Galgenlieder Gingganz und Horatius Travestitus,* Zbinden, Basel, 1972. p. 34.
95. B. Pavlović, in: *Encyclopaedia Poetica* (in Croatian), Exhibition, Zagreb, 1958.
96. R. Hoffmann, *The Metamict State,* University of Central Florida Press, Orlando, FL, 1987. p. 101.
97. R. Marinković, *Ruke* (in Croatian, but there is a translation into German available: Hände; in: Erzählungen, Steingruben, Stuttgart, 1961) 4th ed., Mladost, Zagreb, 1981, pp. 134–145.
98. W. F. Harvey, in: *The Ghouls,* P. Haining, Ed., Futura, London, 1976, Book two, pp. 31–55.
99. T. Burke, in: *Great Tales of Detection*, D. L. Sayers, Ed., Everyman's Library, London, 1976, pp. 154–170.
100. Š. Š. Ćorić, *45 Croatian Emigrants Writers* (in Croatian), Croatian Writer's Association and Croatian P. E. N. Center, Zagreb, 1991, p. 113.
101. J. A. Michener, *Poland,* Corgi, London, 1984, p. 685.
102. U. Eco, *Foucalt's Pendulum*, Secker & Warburg, London, 1989, pp. 360–365.
103. J. W. Goethe, *Faust*, Goldmann, München, 1928, p. 45.
104. H. Hosoya, *Fibonacci Quart.* **11** (1973) 255–266.
105. H. Hosoya, *ibid.* **14** (1976) 173–179.
106. S. J. Cyvin and I. Gutman, *Kekulé Structures in Benzenoid Hydrocarbons,* Springer, Berlin, 1988.

107. *e.g.* M. Randić, S. Nikolić, and N. Trinajstić, *Croat. Chem. Acta* **61** (1988) 821.
108. P. G. Anderson, in: *Fibonacci Numbers and Their Applications*, A. N. Phillippou, G. E. Bergum, and A. F. Horadam, Eds., Reidel, Dordrecht, 1986.
109. A. T. Balaban, *Math. Chem.* (Mülheim/Ruhr) **24** (1979) 29–38.
110. N. Ohkami and H. Hosoya, *Bull Chem. Soc. Japan* **52** (1979) 1624–1633.
111. *e.g.* M. Randić, B. M. Gimarc, S. Nikolić, and N. Trinajstić, *Gazz. Chim. Ital.* **119** (1989) 1–11.
112. M. Randić, *Chem. Phys. Lett.* **38** (1976) 68–70.
113. N. Trinajstić, *Chemical Graph Theory,* CRC Press, Boca Raton, FL, 1992, 2nd revised edition, Chapter 9.
114. S. El–Basil and D. J. Klein, *J. Math. Chem.* **3** (1989) 1–23.
115. N. Trinajstić, *ibid.* **5** (1990) 171–175.
116. D. J. Klein and N. Trinajstić, *J. Mol. Struct. (Theochem)* **206** (1990) 135–142.
117. J. Gleick, *Chaos*, Viking, New York, 1988, 6th printing, p. 100.
118. W. A. Seitz, D. J. Klein, and G. E. Hite, *Discrete Appl. Math.* **19** (1988) 339–348.
119. N. S. Holter, A. Lakhtakia, V. K. Varandan, V. V. Varandan, and R. Messier, *J. Phys.* **A19** (1986) 1753–1759.
120. L. A. Paquette, R. J. Ternansky, and D. W. Balogh, *J. Amer. Chem. Soc.* **104** (1982) 4502–4503.
121. R. J. Ternansky, D. W. Balogh, and L. A. Paquette, *ibid.* **104** (1982) 4503–4504
122. L. Pauling and R. Hayward, *The Architecture of Molecules,* Freeman, San Francisco, 1964.
123. See a special issue of *Acc. Chem. Res.* (Volume 25, November 3, March 1992, pp. 97–175) on buckminsterfullerenes.
124. H. W. Kroto, J. R. Heath, S. C. O'Brien, R. F. Curl, and R. E. Smalley, *Nature* **318** (1985) 162–163.
125. W. Krätschmer, L. D. Lamb, K. Fostiropoulos, and D. R. Huffman, *ibid.* **347** (1990) 354–358.
126. R. E. Smalley, *Acc. Chem. Res.* **25** (1992) 98–105.
127. See, for example, *Science Watch* (Vol. 2, No. 1, January 1992).
128. R. N. Marks, *The Dymaxion World of Buckminter Fuller,* Southern Illinois University Press, Carbondale, 1960.
129. H. W. Kroto, *Angew. Chem. Int. Edit. Engl.* **31** (1992) 111–129.
130. D. J. Klein, T. G. Schmalz, G. E. Hite, and W. A. Hite, *J. Amer. Chem. Soc.* **108** (1986) 1301–1302.
131. V. Elser, E. Brendsdal, S. J. Cyvin, J. Brunvoll, B. N. Cyvin, and D. J. Klein, in: *Quasicrystals, Networks and Molecules with Fivefold Symmetry*, I. Hargittai, Ed., VCH, New York, 1990, pp. 265–295.
132. M. Randić, S. Nikolić, and N. Trinajstić, *Croat. Chem. Acta* **60** (1987) 595–604.
133. D. Plavšić, S. Nikolić, and N. Trinajstić, *J. Mol. Struct. Theochem,* **277** (1992) 213–237.
134. D. Amić and N. Trinajstić, *J. Chem. Soc. Perkin Trans. II* (1990) 1595–1598.
135. K. Balasubramanian and X. Liu, *J. Comput. Chem.* **9** (1988) 406–415.
136. E. Brendsdal, S. J. Cyvin, B. N. Cyvin, J. Brunvoll, D. J. Klein, and W. A. Seitz, in: *Quasicrystals, Networks and Molecules with Fivefold Symmetry*, I. Hargittai, Ed., VCH, New York, 1990, pp. 257–263.
137. T. J. N. Brown, R. B. Mallion, P. Pollak, B. R. M. de Castro, and J. A. N. F. Gomes, *J. Comput. Chem.* **9** (1991) 1118–1124.
138. D. Babić, unpublished result.
139. D. E. Manulopoulos, *Chem. Phys. Lett.* **192** (1992) 330.
140. X. Liu, T. G. Schmalz, and D. J. Klein, *ibid.* **192** (1992) 331.
141. X. Liu, D. J. Klein, W. A. Seitz, and T. G. Schmalz, *J. Comput. Chem.* **12** (1991) 1265–1269.
142. H. P. Schultz, *J. Org. Chem.* **30** (1965) 1361–1364.
143. A. Nickson and E. F. Silversmith, *Organic Chemistry: The Name Game,* Pergamon Press, New York, 1987.
144. N. L. Allinger, M. T. Tribble, M. A. Miller, and D. H. Wertz, *J. Amer. Chem. Soc.* **95** (1971) 1637–1648.
145. E. M. Engler, D. J. Androse, and P. v. R. Schleyer, *ibid.* **95** (1973) 8005–8025.
146. P. E. Eaton, Y. S. Or and S. J. Branca, *ibid.* **103** (1981) 2134–2136.
147. J. P. Dunitz and L. E. Orgel, *Nature* **171** (1953) 121–122.

148. T. J. Kealy and P. L. Pauson, *ibid.* **168** (1951) 1039–1040.
149. S. A. Miller, J. A. Tebboth, and J. F. Tremaine, *J. Chem. Soc.* (1952) 632–635.
150. J. E. Kilpatrick and K. S. Pitzer, *J. Chem. Phys.* **17** (1949) 1064–1075.
151. R. W. Hendrix, J. W. Roberts, F. W. Stahl, and R. A. Weisberger, Eds., Lambda II, Cold Spring Harbor Laboratory, Cold Spring Harbor, 1983.

SAŽETAK

Čarobnost broja pet

Nenad Trinajstić

Prikazana je pojava broja pet i petorostruke simetrije u prirodi, te njihova upotreba u nekoliko područja ljudskog djelovanja, s osobitim naglaskom na kemiji.