

PUTOVANJE OD ST. PETERSBURGA DO MOSKVE ILI MODEL GRAVITACIJSKOG VLAKA U MATHCAD-U

JOURNEY FROM ST. PETERSBURG TO MOSCOW OR MODEL OF GRAVITY TRAIN IN MATHCAD

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Pregledni rad

Sažetak: Ovaj rad na pomalo humorističan način prikazuje fizikalni model polu fantastične ideje podzemnog tunela koji bi spajao dva grada, a kroz koji bi se putovalo tzv. gravitacijskim vlakom. Ideja postoji od vremena Roberta HOOKA, koji ju je prvi izložio Sir Isaac Newtonu. Ovaj je problem uz određena pojednostavljena opisan sustavom običnih diferencijalnih jednadžbi, a rješenje sustava jednadžbi dobiveno je numerički, korištenjem Mathcad aplikacije i naredbe Odesolve.

Ključne riječi: Mathcad, gravitacijski vlak, obične diferencijalne jednadžbe, Odesolve

Subject review

Abstract: This paper somewhat humorously shows a physical model of the semi-fantastic idea of an underground tunnel that would connect the two cities, which would be traveled through by the so called gravity train. The idea has existed since Robert Hook first presented it to Sir Isaac Newton. This problem is described by the system of ordinary differential equations. The solution of equation system is obtained numerically using Mathcad applications (with units) and function Odesolve.

Key words: Mathcad, gravity train, ordinary differential equations, Odesolve

1. INTRODUCTION

According to the story [1,2], prior to the construction of the railway "St. Petersburg-Moscow", Russian Tsar Nicholas I while holding a ruler with one hand, he drew a straight line with his pencil between the two capitals of the Russian Empire. In the area of the Valdai Hills the pencil hit a finger of the emperor, and at this point the path made a little detour, which will be mentioned later in this article. And another anecdote, not historic but from recent days, says that one man handed over a report on business trip to the accounting, in which the price of the train ticket from Moscow to St. Petersburg was slightly higher than the price of a return ticket. When the accountant questioned why such a difference, the accountable person advised to look at the Globe: from Moscow to St. Petersburg the train climbs up to the North Pole, and on the way back it rolls downhill to the equator.

To further reduce the path, Russian Tsar Nicholas I should not have drawn a straight line on the map but should have drilled a straight hole in the globe directly connecting St. Petersburg to Moscow!

Joking aside, the semi-fantastic project of the friction-free so-called gravity train driven by magnetic levitation in a straight underground tunnel, from which air is exhausted has long been discussed (Fig. 1). The first half of the way the train will accelerate downhill without a

traction locomotive and half way up it will go by inertia, slowing down without brakes to the destination where it will stop. This is one of the clearest examples of the transition of potential energy into kinetic energy and vice versa.

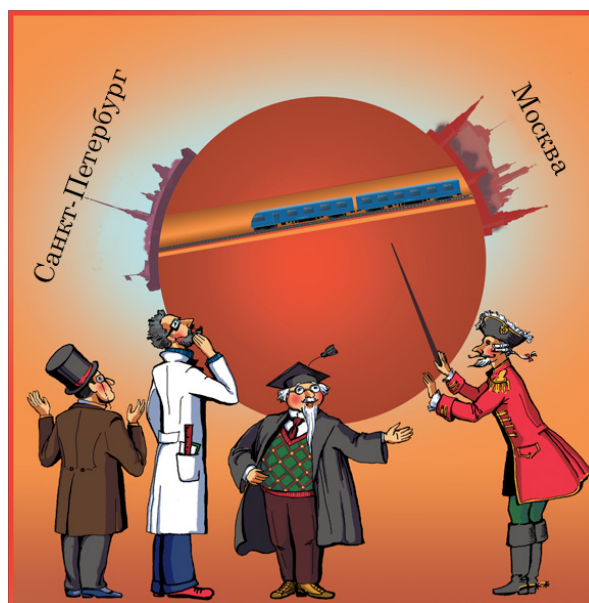


Figure 1. Scheme of the gravity train from St. Petersburg to Moscow [3]

2. GRAVITY TRAIN

On the Internet [3], ready-made formulas can be found by which one can estimate how long this train will be on the road and the maximum speed it will reach in the middle of the tunnel. We are not going to consider these formulas, but analyze the forces acting on the gravity train, compose and solve the equation of balance of power - get the function of position of train in the tunnel depending on time. Rejection of ready-made formulas allows us to further complicate our mathematical model of the gravity train bringing it closer to reality through the friction forces.

In Fig. 2, the simplest calculation model of the train is outlined: a straight tunnel with length L , in which the train is rolling, was drilled on planet Earth (a perfect sphere with radius R). The origin of Cartesian coordinate system, which will be used for reference, is placed at the center of Earth.

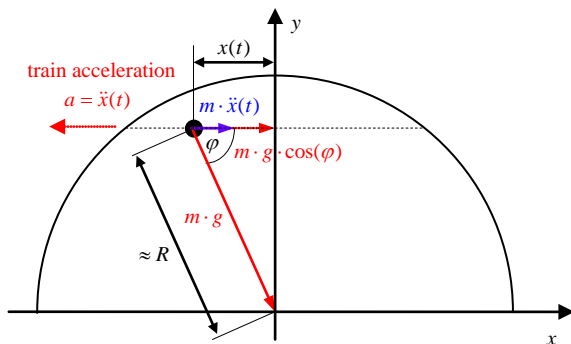


Figure 2. Free body diagram of the gravity train

We set dynamic equilibrium equation:

$$\sum F_x = 0; m \cdot x''(t) + m \cdot g \cdot \cos(\varphi) = 0 \quad (1)$$

Where $\cos(\varphi) = \frac{x(t)}{R}$ thus:

$$m \cdot x''(t) + m \cdot g \cdot \frac{x(t)}{R} = 0 \quad (2)$$

This shows that the force acting on our gravity train (the physical material point) along the x axis is an accelerating force (first half of the way) or retarding force (second half of the way), equal to the weight of the body ($m \cdot g$), multiplied by the ratio of the value of the x coordinate and the radius of the Earth $\frac{x}{R}$.

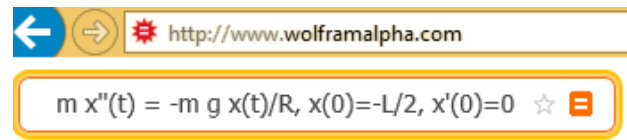
Position of this material point will depend on the time - it will be a function $x(t)$. If we take the first derivative of this function $x'(t)$, we obtain the speed of the train, and if we take the second derivative $x''(t)$ - it is the acceleration.

$$m \cdot x''(t) = -m \cdot g \cdot \frac{x(t)}{R} \quad (3)$$

Finally differential equation of movement [3] of our gravity train will be stemming from Newton's second law: the sum of forces acting on a body is equal to the product of its mass and its acceleration¹.

3. SOLUTION OF DIFFERENTIAL EQUATION

Figure 3 shows the solution of this differential equation using the Wolfram Alpha Site [4]. Not only it permits the equation itself, but also the start position of the train $x(0) = -\frac{L}{2}$ and zero initial velocity $x'(0) = 0$ are stated and separated by commas.



Differential equation solution:

$$x(t) = -\frac{1}{2} L \cos\left(\frac{\sqrt{g t}}{\sqrt{R}}\right)$$

Figure 3. Solution of the problem of gravity train by Wolfram Alpha [4]

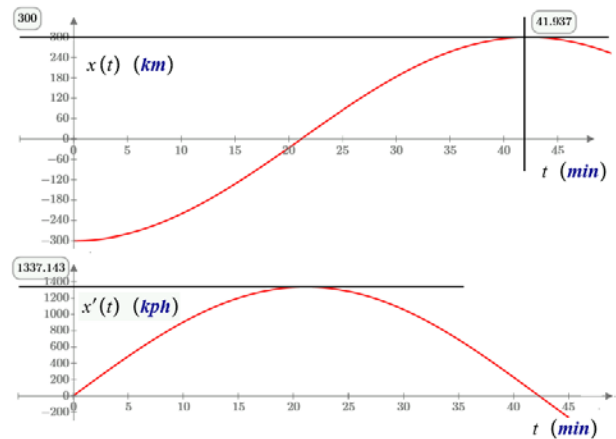


Figure 4. Position and speed of gravity train

In Figure 4 the function $x(t)$, generated by Wolfram Alpha Site [4] shown in Fig. 3, and its first derivative $x'(t)$ (speed) are displayed graphically. Our fantastic train will be on the road a little more than 42 minutes and to reach the speed of 1337 km/h in the middle of the tunnel. Then we could say that the speed of the train has exceeded the speed of sound, but in the tunnel there is no air, so there will be no sound.

Figures 3 and 4 show that the train in the tunnel will perform like a pendulum and oscillate from one city to another (from St. Petersburg to Moscow and back), with a period that does not depend on the distance between the cities. This distance will affect the train's average speed

¹ Teacher of God's law complains to the physics teacher: "Today I asked your pupil what is God's power. So he told me it's the work of God's Mass at God's Acceleration". Physics teacher: "I told him, and in the subject put a deuce. After all, God's work in God's Mass Acceleration should give squared Divinity, but not in the first degree."

and it's maximum in the middle of the path. These calculations in Mathcad are shown in Fig. 5.

$$\begin{aligned}
 R &:= 6400 \text{ km} & L &:= 600 \text{ km} \\
 x_0 &:= -\frac{L}{2} & y_0 &:= \sqrt{R^2 - x_0^2} = 6392.965 \text{ km} & h &:= R - y_0 = 7.035 \text{ km} \\
 x(t) &:= -\frac{1}{2} L \cdot \cos\left(\sqrt{\frac{g}{R}} \cdot t\right) & t &:= \frac{\pi}{\sqrt{\frac{g}{R}}} = 42.299 \text{ min} \\
 v(t) &:= \frac{d}{dt} x(t) \rightarrow \frac{L \cdot \sin\left(t \cdot \sqrt{\frac{g}{R}}\right) \cdot \sqrt{\frac{g}{R}}}{2} & v_{max} &:= \frac{L}{2} \cdot \sqrt{\frac{g}{R}} = 1336.9 \text{ kph}
 \end{aligned}$$

Figure 5. Calculation of gravity train in Mathcad

Figure 5 shows the complete calculation – after the data input (R and L), the initial train coordinates x_0 and y_0 are calculated as the maximum depth of the tunnel h , which for the route Moscow-Petersburg will be a little more than seven kilometers. This is a very important parameter. The fact is in our model the gravitational acceleration g is taken as a constant 9.807 m/s^2 , as built in Mathcad constant. In deeper tunnels we will need to take into account the change in the values of g depending on the depth of the tunnel. What affects this value? This is a separate issue. If, for example, we dig a tunnel through the center of the Earth (and such fantastic projects are also discussed), the value of gravitational acceleration must be zero in the middle of the shaft of the gravity lift.

It is not known, which is more difficult to do - to dig a long straight underground tunnel or provide a tunnel with high vacuum and insure that the train moves along the ideal suspension without friction. The equation shown in Fig. 5 can be supplemented by the power of the air resistance force and rolling friction of the wheels on the rails, thereby bringing our gravity train to actual operating conditions. Force of air resistance is usually assumed to be proportional to air density ρ , multiplying the cross-sectional area of the train S and the square of its velocity $x'(t)$. The strength of the rolling friction of the wheels on the rails with great simplifications (see below) shall be in proportion to the weight of the train component that is parallel to the axis y (pressure force of the train on the rails). This expanded differential equation can no longer be solved analytically (exactly) [5,6] and obtain a formula for the function $x(t)$, as shown in Fig.3. This complicated equation can be solved only numerically (approximately), and we get a table of values of $x(t)$ for different values of t . Fig. 6 shows how this is done in Mathcad Prime [6, 7, 8].

It is practical to mention that the Mathcad Prime built-in **Odesolve** function can work with units [9]. It is very useful for solving physical problems - whereas Mathcad 15 is not!

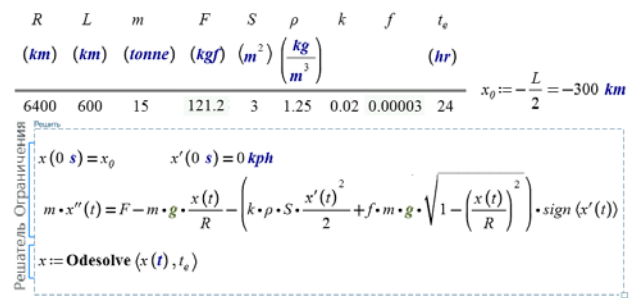


Figure 6. Simulation of motion of the gravity train in Mathcad Prime

In the problem shown in Fig. 6, addition to the values of R and L is introduced: the train mass m , traction force F , cross sectional area of the train S , air density ρ , friction coefficients between the train and air k and the rail f , and a time value t_e , which are included in the tabular function $x(t)$. The default tab is over 1000 points, but it can be changed in calculation. Two friction forces multiply Mathcad built-in step function **sign**, which returns zero if the argument is less than or equal to zero and one otherwise. This is done in order for the friction force of the train on the air always has to act against the movement of the train and the friction force of the wheels on the rails is equal to zero at zero speed of train. At high speeds, the train will be decelerated mainly due to the force of headwind and at low speeds - at the expense of the rolling friction of the wheels. This can be seen from the landing of the aircraft: first, it is hampered due to the brake pads and/or brake parachute, and then wheel brake begins to work. From the equation in Fig. 6 we can remove the square root of the tunnel for the Moscow Petersburg ratio $\frac{x(t)}{R}$ is very small. But for long tunnels, this ratio is large enough and cannot be ignored. In deeper tunnels we also need to take into account the change in air density. In our calculation it is accepted as a constant - the average density of air at sea level².

A built-in Mathcad function **Odesolve** numerically determines (solves) our ordinary (o) differential (d) equation (e). Prior to the train equation, initial position $x(0s) = x_0$, and its velocity $x'(0s) = 0 \text{ kph}$ (kilometer per hour) are given. From this starting point to the point t_e function **Odesolve** will calculate the value of the function being created $x(t)$, displayed on the graph of Fig. 7.

Taking into account the friction forces and correctly selected locomotive traction force F (121.2 kilogram of force), the train in the tunnel safely makes it to the end point and rolls back (if it is not delayed, wedged, for

² Speaking of the sea level, once upon a time a bridge across the Rhine was being built. On one side of the river it was done by the Germans, and on the other by the Swiss. When the two halves of the bridge began to connect in the middle of the river, it was found that the difference in their heights was almost half a meter. The reason for the error was that the German building standard zero height is the average level of the North Sea, whereas in Switzerland the Mediterranean. This difference is due to the rotation of the Earth, which, incidentally, should also be taken into account when calculating the gravitational movement of our train.

example, by the brake shoe under its wheel), repeating damped pendulum movement (Fig. 7) with a final stop in two days, but not in the middle of the tunnel, but somewhere under the Valdai Hills - where the train at a slight incline will be held by the traction force of the locomotive. This point is easy to calculate. We suggest to the reader, who can also further complicate our model, to take into account the changes in air density, as well as the value of the gravitational acceleration in the tunnel.

Coefficient k in formula of friction force between the train and the wind is not a constant but depends on the airflow - at low speeds laminar, transitional; at high speeds turbulent. Also, keep in mind that factor f is different if the motion of the train is in the open space from that in the tunnel. Rolling coefficient of friction f depends on the speed of the train - at point at which it began movement it may have one (high) value, and with the growth of velocity the other, lower, resulting in some jerky motion. All this information can be found in other resources [10] and added to our model.

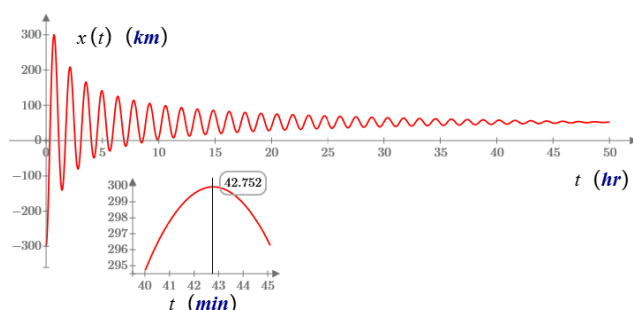


Figure 7. Calculation of train's gravitational movement with damping

We have shown both theoretically and with our numerical simulations that the movement of the gravity train does not depend on the length of the tunnel and lasts 42 minutes, which can be considered a kind of constant. This characterizes our planet and the associated gravitational constant and the density of Earth. For other lengths of the tunnel, as noted earlier, only the speed of the train will change - the average and maximum. For the tunnel Moscow-Petersburg it is easy to estimate the average speed of $600 \text{ km} / 42 \text{ minutes} = 857 \text{ km/h}$.

Finally, the finger of Nicholas I on the line "St. Petersburg-Moscow" is not just a historical anecdote. This train real detour is large and deep ravine, across which the embankment was paved and the bridge was built only in 2000 during the reconstruction of roads [2]. One additional point: It is easy to calculate that if the 600 km long railway between Moscow and St. Petersburg, lay not on the Earth's surface (the surface of a perfect sphere with a radius of 6400 km), but in the tunnel in a straight line (see. Fig. 1 and 2), the distance between the two cities would be reduced by only 220 m.

5. CONCLUSION

Speaking of the real, not fiction tunnels, do you know, in what kind of path they are built, if there are no other restrictions? The usual perception - they match to a

straight line or a circular arc. But it is not so. In the direct tunnel water will collect and must be continually pumped. Figure 7 can be interpreted as a graph that represents the flow of puddles of water poured at the entrance of a straight tunnel, where rod locomotive traction is replaced by wind in the tunnel. To arc such tunnel, echoing the Earth's circumference, it is slightly hard to dig and it is not rational for another reason. Mountain tunnels are usually built in such a way: of two points located on opposite slopes, two tunnel boring machines, laser controlled, begin to lay the tunnel in a straight line. The very same route of each half of the tunnel being built will rise slightly above the entry line. Boring machines shall meet in the center of the tunnel³ which is slightly higher than starting points. This sloped tunnel not only will not accumulate water, but if necessary, it will enable stalled vehicles to roll out due to their weight. Light at the end of the tunnel can be seen, only when you reach the tunnels mid... Underwater tunnels as, for example, mentioned by us in a tunnel under the English Channel footnote, are built, of course on more complex trajectories.

You can see the animation of the gravitational motion of the train, made in the Mathcad Online: St. Petersburg-Moscow [11], with friction and NYC-San Francisco [12], without friction.

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³ One last joke: once upon a time two Scottish brothers came with shovels before the Commission for the construction of a tunnel under the English Channel and told that they could dig a tunnel; one will dig from England, and the other will start from continental Europe. Members of the Commission smiled and asked: "What if you miss?!". "It's okay," said the brothers, "you get two tunnels for the price of one!".

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