

TRIBONACCI DIOPHANTINE QUADRUPLES

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ABSTRACT. In this paper, we show that there does not exist a quadruple of positive integers $a_1 < a_2 < a_3 < a_4$ such that $a_i a_j + 1$ ($i \neq j$) are all members of the Tribonacci sequence $(T_n)_{n \geq 0}$.

1. INTRODUCTION

A Diophantine m -tuple is a set a_1, \dots, a_m of positive rational numbers, or integers, such that $a_i a_j + 1$ is a square for all $1 \leq i < j \leq m$. The problem of finding Diophantine m -tuples has a long history. Diophantus found the rational quadruple $\{1/16, 33/16, 17/4, 105/16\}$ and only long after Fermat found the integer quadruple $\{1, 3, 8, 120\}$. In the integer case, it is known that there are infinitely many Diophantine quadruples of integers. Recently, there has been remarkable progress by Dujella ([5]), who showed that there is no Diophantine sextuple and that there can be at most finitely many Diophantine quintuples and all of them are, at least in theory, effectively computable. However, in the rational case, it is not even known if m must be universally bounded. Gibbs ([7]), found a few examples with $m = 6$.

Natural generalizations of this problem have been studied by replacing the squares by higher powers of fixed, or variable exponents in [1, 2, 8–10], or by members of nondegenerate binary recurrences $(u_n)_{n \geq 0}$, by Fuchs, Luca and Szalay in [6]. In this last setting, some research was done concerning Diophantine triples (a, b, c) of distinct positive integers, such that $ab + 1, ac + 1$ and $bc + 1$ are all three members of a particular recurrence $(u_n)_{n \geq 0}$. Particularly, Luca and Szalay showed that there are no Diophantine triples with values in the Fibonacci sequence (see [11]) and that the only Diophantine

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triple with values in the Lucas companion $(L_n)_{n \geq 0}$ of the Fibonacci sequence is $(a, b, c) = (1, 2, 3)$ (see [12]).

In this paper, we look at Diophantine quadruples with value in the Tribonacci sequence. The *Tribonacci sequence* $(T_n)_{n \geq 0}$ starts with $T_0 = 0$ and $T_1 = T_2 = 1$ and each term afterwards is the sum of the preceding three terms

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \quad \text{for all } n \geq 0.$$

We prove the following theorem.

MAIN THEOREM. *There do not exist positive integers $a_1 < a_2 < a_3 < a_4$ such that $a_i a_j + 1 = T_{n_{i,j}}$, with $1 \leq i < j \leq 4$, for some integers positive $n_{i,j}$.*

We conjecture that in fact there are only finitely many Tribonacci triples (a, b, c) , that is triples of positive integers $a < b < c$ such that $ab + 1$, $ac + 1$, $bc + 1$ are all three Tribonacci numbers, but we do not know how to attack this question. We leave this as an open question for the reader.

2. PROPERTIES

The characteristic polynomial of the Tribonacci sequence is

$$\Psi(x) = x^3 - x^2 - x - 1.$$

It has a real root

$$\alpha = \frac{1}{3} \left(1 + (19 - 3\sqrt{33})^{1/3} + (19 + 3\sqrt{33})^{1/3} \right)$$

and two complex conjugated roots

$$(2.1) \quad \beta = \alpha^{-1/2} e^{i\theta} \quad \text{and} \quad \gamma = \alpha^{-1/2} e^{-i\theta} \quad \text{with } \theta \in (\pi/2, \pi).$$

Using a result of Dresden ([4]), we have the following Binet-like formula for T_n :

$$(2.2) \quad T_n = c_\alpha \alpha^{n-1} + c_\beta \beta^{n-1} + c_\gamma \gamma^{n-1},$$

where $c_z = (z - 1)/(4z - 6)$. Dresden also showed that the contribution of the complex roots β and γ , which have absolute value less than 1, to the right-hand side of (2.2) is very small. More precisely, he proved that the inequality

$$(2.3) \quad |T_n - c_\alpha \alpha^{n-1}| < \frac{1}{2} \quad \text{holds for all } n \geq 0.$$

Another well-known property of the Tribonacci numbers which is useful to us is the following (see [3]):

$$(2.4) \quad \alpha^{n-2} \leq T_n \leq \alpha^{n-1} \quad \text{for all } n \geq 1.$$

3. FIRST OBSERVATIONS

Let $a_1 < a_2 < a_3 < a_4$ be a Tribonacci Diophantine quadruple. Then

$$(3.1) \quad a_1 a_2 + 1 = T_x, \quad a_2 a_3 + 1 = T_y, \quad a_3 a_4 + 1 = T_z, \quad a_1 a_4 + 1 = T_w,$$

for some positive integers x, y, z and w . We see easily that

$$(3.2) \quad 4 \leq x < \min\{y, w\} \leq \max\{y, w\} < z.$$

Combining the equalities in (3.1), we conclude that

$$(3.3) \quad (T_x - 1)(T_z - 1) = (T_y - 1)(T_w - 1).$$

From inequalities (2.4) we have $\alpha^{n-2.5} < T_n - 1 < \alpha^{n-1}$, for all $n \geq 4$. Thus,

$$\alpha^{x+z-5} < (T_x - 1)(T_z - 1) < \alpha^{x+z-2}$$

and

$$\alpha^{y+w-5} < (T_y - 1)(T_w - 1) < \alpha^{y+w-2}.$$

Hence, considering equality (3.3), we deduce that

$$(3.4) \quad |(x + z) - (y + w)| \leq 2.$$

In the rest of this paper, we work on the Diophantine equation (3.3) by distinguishing two cases:

$$x + z \neq y + w \text{ and } x + z = y + w.$$

 4. THE CASE $x + z \neq y + w$

By using formula (2.2) and inequality (2.3), we have that

$$(4.1) \quad T_n = c_\alpha \alpha^{n-1} + e(n), \text{ with } |e(n)| < 1/2.$$

Thus, by expanding equation (3.3) this can be rewritten as

$$\begin{aligned} c_\alpha^2 \alpha^{x+z-2} - c_\alpha^2 \alpha^{y+w-2} &= c_\alpha(1 - e(z))\alpha^{x-1} + c_\alpha(1 - e(x))\alpha^{z-1} \\ &\quad + c_\alpha(e(w) - 1)\alpha^{y-1} + c_\alpha(e(y) - 1)\alpha^{w-1} + e(x) \\ &\quad + e(z) - e(y) - e(w) - e(x)e(z) + e(y)e(w). \end{aligned}$$

Dividing both sides of the above equation by $c_\alpha^2 \alpha^{x+z-2}$ and taking absolute values, we get

$$\begin{aligned} \left| 1 - \alpha^{-(x+z-y-w)} \right| &< \frac{3}{2c_\alpha} \left(\frac{1}{\alpha^{z-1}} + \frac{1}{\alpha^{x-1}} + \frac{\alpha^{y-z}}{\alpha^{x-1}} + \frac{\alpha^{w-z}}{\alpha^{x-1}} \right) + \frac{2.5}{c_\alpha^2 \alpha^{x+z-2}} \\ (4.2) \quad &< \frac{1}{\alpha^{x-1}} \left(\frac{3}{c_\alpha} \left(1 + \frac{1}{\alpha} \right) + \frac{5}{2c_\alpha^2 \alpha^5} \right) < \frac{8}{\alpha^{x-1}}, \end{aligned}$$

where we have used (3.2) and that $|e(n) - 1| < 3/2$. On the other hand, by inequality (3.4) and since $x + z \neq y + w$, we obtain

$$(4.3) \quad \min_{|x+z-y-w| \leq 2} |1 - \alpha^{-(x+z-y-w)}| > 0.4563.$$

Thus, by (3.2), (4.2) and (4.3), we get $x = 4$ or 5 .

Now, from equation (3.3) we have the equality

$$(T_x - 1)T_z - T_\lambda T_\delta = T_x - T_\lambda - T_\delta,$$

where $\lambda = \min\{y, w\} \leq \delta = \max\{y, w\}$. Replacing T_z , T_λ , T_δ according to the equation (4.1) in the above equation, we conclude that

$$(T_x - 1)c_\alpha \alpha^{z-1} - c_\alpha^2 \alpha^{\lambda+\delta-2} = c_\alpha(e(\delta) - 1)\alpha^{\lambda-1} + c_\alpha(e(\lambda) - 1)\alpha^{\delta-1} - e(\lambda) \\ - e(\delta) + e(\lambda)e(\delta) - (T_x - 1)(e(z) - 1) + 1.$$

Dividing both sides of above equation by $(T_x - 1)c_\alpha \alpha^{z-1}$, and taking absolute values, we get

$$(4.4) \quad |1 - (T_x - 1)^{-1}c_\alpha \alpha^{\lambda+\delta-z-1}| < \frac{1/2}{\alpha^{z-\lambda}} + \frac{1/2}{\alpha^{z-\delta}} + \frac{9/(4c_\alpha \alpha^4)}{\alpha^{z-5}} < \frac{1.4}{\alpha^{z-\delta}},$$

where we used the fact that $z - 5 \geq z - \lambda \geq z - \delta$ (by (3.2)) and that $T_x - 1 \geq 3$. However, by inequality (3.4) and since $x = 4$ or 5 , we obtain that $|\lambda + \delta - z - 1| \leq 6$. We find that

$$(4.5) \quad \min_{\substack{x \in \{4,5\} \\ |\lambda+\delta-z-1| \leq 6}} |1 - (T_x - 1)^{-1}c_\alpha \alpha^{\lambda+\delta-z-1}| > 0.179.$$

Thus, combining (3.2), (4.4) and (4.5) we conclude that $z - \delta = 1, 2$ or 3 . Returning to inequality (3.4), we get that $5 \leq \lambda \leq 10$.

Going back one more time to equality (3.3), we rewrite it as

$$(T_x - 1)T_z - (T_\lambda - 1)T_\delta = T_x - T_\lambda.$$

Replacing T_z , T_δ according to (4.1), dividing by $(T_\lambda - 1)c_\alpha \alpha^{\delta-1}$ and taking absolute values, we get

$$(4.6) \quad |1 - (T_x - 1)(T_\lambda - 1)^{-1}\alpha^{z-\delta}| < \frac{5}{\alpha^{\delta-1}}.$$

By analyzing the minimum value of the left hand side in (4.6), we get

$$(4.7) \quad \min_{\substack{x \in \{4,5\} \\ 5 \leq \lambda \leq 10 \\ z-\delta \in \{1,2,3\}}} |1 - (T_x - 1)(T_\lambda - 1)^{-1}\alpha^{z-\delta}| > 0.08.$$

Hence, from inequalities (4.6) and (4.7) we conclude that $\delta \leq 7$ and, in particular, that $6 \leq z \leq 10$.

Let us record what we have proved so far.

LEMMA 4.1. *Let $4 \leq x < \min\{y, w\} \leq \max\{y, w\} < z$ be positive integers such that*

$$x + z \neq y + w \quad \text{and} \quad (T_x - 1)(T_z - 1) = (T_y - 1)(T_w - 1).$$

Then

$$4 \leq x \leq 5, \quad 5 \leq y, w \leq 7 \quad \text{and} \quad 6 \leq z \leq 10.$$

5. THE CASE $x + z = y + w$

Substituting β and γ given by (2.1) in equation (2.2), we conclude that

$$(5.1) \quad T_n - 1 = c_\alpha \alpha^{n-1} + \frac{\tilde{c}(n)}{\alpha^{\frac{1}{2}(n-1)}} - 1 = c_\alpha \alpha^{n-1} \left(1 + \frac{\tilde{c}(n)/c_\alpha}{\alpha^{\frac{3}{2}(n-1)}} - \frac{1/c_\alpha}{\alpha^{n-1}} \right),$$

where $\tilde{c}(n) := c_\beta e^{i\theta(n-1)} + c_\gamma e^{-i\theta(n-1)}$.

Combining the above equation with (3.3) and performing the respective calculations, we get an equality with dominant terms $1/\alpha^{\lambda-1}$ and $1/\alpha^{x-1}$ in each of the two sides. Grouping these terms in one side, we obtain

$$(5.2) \quad \begin{aligned} \frac{1/c_\alpha}{\alpha^{\lambda-1}} - \frac{1/c_\alpha}{\alpha^{x-1}} &= \frac{\tilde{c}(\delta)/c_\alpha}{\alpha^{\frac{3}{2}(\delta-1)}} + \frac{\tilde{c}(\lambda)/c_\alpha}{\alpha^{\frac{3}{2}(\lambda-1)}} - \frac{\tilde{c}(z)/c_\alpha}{\alpha^{\frac{3}{2}(z-1)}} - \frac{\tilde{c}(x)/c_\alpha}{\alpha^{\frac{3}{2}(x-1)}} + \frac{1/c_\alpha}{\alpha^{z-1}} \\ &- \frac{1/c_\alpha}{\alpha^{\delta-1}} - \frac{\tilde{c}(\lambda)/c_\alpha^2}{\alpha^{\frac{3}{2}(\lambda-1)+\delta-1}} - \frac{\tilde{c}(\delta)/c_\alpha^2}{\alpha^{\frac{3}{2}(\delta-1)+\lambda-1}} \\ &+ \frac{\tilde{c}(x)/c_\alpha^2}{\alpha^{\frac{3}{2}(x-1)+z-1}} + \frac{\tilde{c}(z)/c_\alpha^2}{\alpha^{\frac{3}{2}(z-1)+x-1}} + \frac{(\tilde{c}(\lambda)\tilde{c}(\delta) - \tilde{c}(x)\tilde{c}(z))/c_\alpha^2}{\alpha^{\frac{3}{2}(x+z-2)}}. \end{aligned}$$

Multiplying both sides of (5.2) by $c_\alpha \alpha^{x-1}$ and taking absolute values, we conclude that

$$(5.3) \quad \begin{aligned} |1 - \alpha^{-(\lambda-x)}| &< \frac{|\tilde{c}(\delta)|/\alpha^2}{\alpha^{\delta-x}} + \frac{|\tilde{c}(\lambda)|/\alpha}{\alpha^{\frac{1}{2}(x-1)}} + \frac{|\tilde{c}(z)|/\alpha^{5/2}}{\alpha^{z-x}} + \frac{|\tilde{c}(x)|}{\alpha^{\frac{1}{2}(x-1)}} \\ &+ \frac{1}{\alpha^{z-x}} + \frac{1}{\alpha^{\delta-x}} + \frac{|\tilde{c}(\lambda)|/(c_\alpha \alpha^6)}{\alpha^{\delta-x}} + \frac{|\tilde{c}(\delta)|/(c_\alpha \alpha^6)}{\alpha^{\delta-x}} \\ &+ \frac{|\tilde{c}(x)|/(c_\alpha \alpha^5)}{\alpha^{\frac{1}{2}(x-1)}} + \frac{|\tilde{c}(z)|/(c_\alpha \alpha^5)}{\alpha^{\frac{1}{2}(x-1)}} \\ &+ \frac{(|\tilde{c}(\lambda)||\tilde{c}(\lambda)| + |\tilde{c}(\lambda)||\tilde{c}(\lambda)|)/(c_\alpha \alpha^{15/2})}{\alpha^{\frac{1}{2}(x-1)}} \\ &< \frac{2.1}{\alpha^{\delta-x}} + \frac{0.3}{\alpha^{\frac{x}{2}}} \leq \frac{2.4}{\alpha^\ell}, \end{aligned}$$

where $\ell := \min\{x/2, \delta - x\}$. In the above inequality, we used the facts that $|\tilde{c}(n)| \leq |c_\beta|/2 < 0.1$, for all $n \geq 4$, as well as inequalities (3.2). Thus, noting also that $|1 - \alpha^{-(\lambda-x)}| \geq 1 - 1/\alpha > 0.45$, we obtain from (5.3) that $\alpha^\ell < 5.4$, which leads to $\ell < 2.8$.

If $\ell = x/2$, we then get that $x = 4$ or 5 . Returning to equation (3.3) and replacing T_λ , T_δ and T_z (from the first part of (5.1)), we get an equality with dominant terms $c_\alpha^2 \alpha^{\lambda+\delta-2}$ and $(T_x - 1)c_\alpha \alpha^{z-1}$. Separating them into one side and putting all the rest into the other side, we obtain

$$\begin{aligned} (T_x - 1)c_\alpha \alpha^{z-1} - c_\alpha^2 \alpha^{\lambda+\delta-2} &= \frac{\tilde{c}(\delta)c_\alpha \alpha^{\lambda-1}}{\alpha^{\frac{1}{2}(\delta-1)}} + \frac{\tilde{c}(\lambda)c_\alpha \alpha^{\delta-1}}{\alpha^{\frac{1}{2}(\lambda-1)}} + \frac{\tilde{c}(\lambda)\tilde{c}(\delta)}{\alpha^{\frac{1}{2}(\lambda+\delta-2)}} \\ &- \frac{\tilde{c}(z)(T_x - 1)}{\alpha^{\frac{1}{2}(z-1)}} - \frac{\tilde{c}(\delta)}{\alpha^{\frac{1}{2}(\delta-1)}} - \frac{\tilde{c}(\lambda)}{\alpha^{\frac{1}{2}(\lambda-1)}} - c_\alpha \alpha^{\lambda-1} - c_\alpha \alpha^{\delta-1} + T_x. \end{aligned}$$

As usual, we divide both sides of the above equality by $c_\alpha^2 \alpha^{\lambda+\delta-2}$ and take absolute values to get:

$$\begin{aligned}
|(T_x - 1)c_\alpha^{-1} \alpha^{z-(\lambda+\delta)+1} - 1| &< \frac{|\tilde{c}(\delta)|/c_\alpha}{\alpha^{\frac{3}{2}(\delta-1)}} + \frac{|\tilde{c}(\lambda)|/c_\alpha}{\alpha^{\frac{3}{2}(\lambda-1)}} + \frac{|\tilde{c}(\lambda)||\tilde{c}(\delta)|/c_\alpha^2}{\alpha^{\frac{3}{2}(\lambda+\delta-2)}} \\
&+ \frac{(|\tilde{c}(z)|/c_\alpha^2)(T_x - 1)}{\alpha^{\frac{1}{2}(z-1)+\lambda+\delta-2}} + \frac{|\tilde{c}(\delta)|/c_\alpha^2}{\alpha^{\frac{3}{2}(\delta-1)+\lambda-1}} \\
(5.4) \quad &+ \frac{|\tilde{c}(\lambda)|/c_\alpha^2}{\alpha^{\frac{3}{2}(\lambda-1)+\delta-1}} + \frac{1/c_\alpha}{\alpha^{\delta-1}} + \frac{1/c_\alpha}{\alpha^{\lambda-1}} + \frac{T_x/c_\alpha^2}{\alpha^{\lambda+\delta-2}} \\
&< \frac{1}{\alpha^{\lambda-1}} \left(\frac{|c_\beta|}{c_\alpha \alpha^2} + \frac{|c_\beta|^2}{4c_\alpha^2 \alpha^8} + \frac{3|c_\beta|}{c_\alpha^2 \alpha^{13/2}} + \frac{|c_\beta|}{c_\alpha^2 \alpha^6} \right. \\
&\left. + \frac{2}{c_\alpha} + \frac{7}{c_\alpha^2 \alpha^4} \right) < \frac{5}{\alpha^{\lambda-1}}.
\end{aligned}$$

In the above chain of inequalities, we used inequality (3.2) in each term of the right side and the fact that $T_x \leq T_5 = 7$. Further, since $z - (\lambda + \delta) = -x$ and $x = 4$, or 5 , we get that

$$\min_{x \in \{4,5\}} |(T_x - 1)c_\alpha^{-1} \alpha^{z-(\lambda+\delta)+1} - 1| > 0.15.$$

Therefore, it follows from (3.2) and (5.4) that $\lambda = 5$ or 6 . Once more, we return to equation (3.3). Replacing T_z , T_δ according to (5.1), dividing by $(T_\lambda - 1)c_\alpha \alpha^{\delta-1}$, and taking absolute values, we have

$$\begin{aligned}
(5.5) \quad |1 - (T_x - 1)(T_\lambda - 1)^{-1} \alpha^{z-\delta}| &< \frac{1}{\alpha^{\delta-1}} \left(\frac{|c_\beta|}{2c_\alpha \alpha^2} + \frac{|c_\beta|}{2c_\alpha \alpha^{5/2}} + \frac{2}{c_\alpha} \right) \\
&< \frac{3.4}{\alpha^{\delta-1}}.
\end{aligned}$$

Now, we estimate the minimum value of the left hand side in (5.5), where $z - \delta = \lambda - x$, $x = 4$ or 5 , and $\lambda = 5$ or 6 , to get

$$(5.6) \quad \min_{\substack{x \in \{4,5\} \\ \lambda \in \{5,6\}}} |1 - (T_x - 1)(T_\lambda - 1)^{-1} \alpha^{\lambda-x}| > 0.08.$$

Hence, from inequalities (5.5) and (5.6) we conclude that $\delta \leq 7$ which implies that $6 \leq z \leq 9$.

But, if $\ell = \delta - x$ then $\lambda \leq \delta \leq x + 2$. In particular, $x + z = \lambda + \delta \leq 2x + 4$, and so $z \leq x + 4$. To summarize, we proved the next result.

LEMMA 5.1. *Let $4 \leq x < \min\{y, w\} \leq \max\{y, w\} < z$ be positive integers such that $x + z = y + w$ and $(T_x - 1)(T_z - 1) = (T_y - 1)(T_w - 1)$. Then*

$$4 \leq x \leq 5, \quad 5 \leq y, w \leq 7 \quad \text{and} \quad 6 \leq z \leq 9,$$

or

$$x + 1 \leq y, w \leq x + 2 \quad \text{and} \quad x + 2 \leq z \leq x + 4.$$

6. THE PROOF OF THE MAIN THEOREM

6.1. *Case $x + z \neq y + w$.* Listing the values of T_x, T_y, T_z, T_w , with x, y, z, w in the range given by Lemma 4.1, we note that (3.3) has no solutions. So, there is no Tribonacci Diophantine quadruple that satisfies (3.1) in this case.

6.2. *Case $x + z = y + w$.* From the first conclusion of Lemma 5.1, we obtain by simple verification that the only solution of (3.3) is $(x, y, z, w) = (4, 5, 6, 5)$. Hence, from (3.1) we get

$$a_1 a_2 = 3, \quad a_2 a_3 = 6, \quad a_3 a_4 = 12, \quad a_1 a_4 = 6,$$

which does not lead to an integer solution (a_1, a_2, a_3, a_4) . So, there is no Tribonacci Diophantine quadruple that satisfies (3.1) in this case.

From the second conclusion of Lemma 5.1, we can rewrite the equation (3.3) as

$$(T_x - 1)(T_{x+i+j} - 1) = (T_{x+i} - 1)(T_{x+j} - 1),$$

where $1 \leq i \leq j \leq 2$. Using identity (5.1) in above equation, and by making the respective calculations and simplifications, we get

$$\begin{aligned} & \frac{1/c_\alpha}{\alpha^{x+j-1}} + \frac{1/c_\alpha}{\alpha^{x+i-1}} - \frac{1/c_\alpha}{\alpha^{x+i+j-1}} - \frac{1/c_\alpha}{\alpha^{x-1}} \\ &= \frac{\tilde{c}(x+i)\tilde{c}(x+j)/c_\alpha^2}{\alpha^{\frac{3}{2}(2x+i+j-2)}} - \frac{\tilde{c}(x)\tilde{c}(x+i+j)/c_\alpha^2}{\alpha^{\frac{3}{2}(2x+i+j-2)}} - \frac{\tilde{c}(x+i+j)/c_\alpha}{\alpha^{\frac{3}{2}(x+i+j-1)}} \\ & - \frac{\tilde{c}(x)/c_\alpha}{\alpha^{\frac{3}{2}(x-1)}} + \frac{\tilde{c}(x)/c_\alpha^2}{\alpha^{\frac{3}{2}(x-1)+x+i+j-1}} + \frac{\tilde{c}(x+i+j)/c_\alpha^2}{\alpha^{\frac{3}{2}(x+i+j-1)+x-1}} + \frac{\tilde{c}(x+j)/c_\alpha}{\alpha^{\frac{3}{2}(x+j-1)}} \\ & + \frac{\tilde{c}(x+i)/c_\alpha}{\alpha^{\frac{3}{2}(x+i-1)}} - \frac{\tilde{c}(x+i)/c_\alpha^2}{\alpha^{\frac{3}{2}(x+i-1)+x+j-1}} - \frac{\tilde{c}(x+j)/c_\alpha^2}{\alpha^{\frac{3}{2}(x+j-1)+x+i-1}}. \end{aligned}$$

Multiplying both sides of the last equation by $c_\alpha \alpha^x$, and taking absolute values, we conclude that

$$\begin{aligned} & \left| \frac{1}{\alpha^{j-1}} + \frac{1}{\alpha^{i-1}} - \frac{1}{\alpha^{i+j-1}} - \alpha \right| \\ & < \frac{|c_\beta|^2/2c_\alpha}{\alpha^{\frac{1}{2}(4x+3i+3j-6)}} + \frac{|c_\beta|}{\alpha^{\frac{1}{2}(x+3i+3j-3)}} + \frac{|c_\beta|}{\alpha^{\frac{1}{2}(x-3)}} + \frac{|c_\beta|/c_\alpha}{\alpha^{\frac{3}{2}(x-1)+i+j-1}} \\ & + \frac{|c_\beta|/c_\alpha}{\alpha^{\frac{3}{2}(x+i+j-1)-1}} + \frac{|c_\beta|}{\alpha^{\frac{1}{2}(x+3j-3)}} + \frac{|c_\beta|}{\alpha^{\frac{1}{2}(x+3i-3)}} + \frac{|c_\beta|/c_\alpha}{\alpha^{\frac{3}{2}(x+i-1)+j-1}} \\ & + \frac{|c_\beta|/c_\alpha}{\alpha^{\frac{3}{2}(x+j-1)+i-1}}. \end{aligned}$$

It is easy to check that for $x \geq 6$ and $1 \leq i \leq j \leq 2$, the above inequality is impossible. Thus, $x = 4$ or 5 , and from Lemma 5.1 we obtain values for y, w and z which were already considered, and which do not lead to positive integer solutions to equation (3.1). This completes the proof of the main theorem.

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