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BOUNDS FOR DIOPHANTINE QUINTUPLES

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Abstract. A set of m positive integers $\{a_1, \ldots, a_m\}$ is called a Diophantine m-tuple if the product of any two elements in the set increased by one is a perfect square. The conjecture according to which there does not exist a Diophantine quintuple is still open. In this paper, we show that if $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$, then $b > 3\,a$; moreover, $b > \max\{21\,a,\ 2\,a^{3/2}\}$ in case $c > a + b + 2\sqrt{ab+1}$.

1. Introduction

A set of m positive integers $\{a_1, \ldots, a_m\}$ with the property that the product of any two elements in the set increased by one is a perfect square is called *a Diophantine m-tuple*.

The first example $\{1, 3, 8, 120\}$ of a Diophantine quadruple was found by Fermat. In general, for a given Diophantine triple $\{a, b, c\}$, the set $\{a, b, c, d_+\}$ is always a Diophantine quadruple, where

$$d_+ = a + b + c + 2abc + 2\sqrt{(ab+1)(ac+1)(bc+1)}$$

(see [1]). A set of the type $\{a, b, c, d_+\}$ is called a *regular* Diophantine quadruple. Fermat's set $\{1, 3, 8, 120\}$ is regular, and, according to [17], in [18] and independently in [1] it is conjectured that any Diophantine quadruple is regular, implying the folklore conjecture that there does not exist a Diophantine quintuple. In [8] Dujella obtained results very close to settling this conjecture by proving that there does not exist a Diophantine sextuple and that there exist only finitely many Diophantine quintuples. Later bounds on the number of Diophantine quintuples are provided in [9,11,16]. Historical and recent

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developments of the study of Diophantine m-tuples are found on Dujella's webpage: http://web.math.pmf.unizg.hr/∼duje/dtuples.html.

The existence of Diophantine sets boils down to solving systems of generalized Pell equations. Recently, the first author ([6, Lemma 2.4]; see also Lemma 3.4 below) improved the lower bound for hypothetical solutions to the system relevant in the study of Diophantine quintuples (note that a similar assertion is obtained in [23]). Using this lemma, he investigated the properties that Diophantine quintuples should have, and in particular updated the known upper bounds for the fourth element d ([6, Theorem 2.1]) and for the number of Diophantine quintuples ([6, Theorem 1.3]).

The aim of this paper is to show the non-existence of Diophantine quintuples such that the two smallest elements are rather close to each other. More precisely, we prove the following.

Theorem 1.1. *There exists no Diophantine quintuple* $\{a, b, c, d, e\}$ *with* $a < b < c < d < e$ *and* $b \leq 3\,a$.

Theorem 1.2. *There exists no Diophantine quintuple* $\{a, b, c, d, e\}$ *with* $a < b < c < d < e$, $c > a + b + 2\sqrt{ab+1}$ *and* $b \leq \max\{21\,a,\ 2\,a^{3/2}\}$.

In addition to the above mentioned result [6, Lemma 2.4], a key role in our deliberations plays an amelioration (Theorem 2.2 in Section 2) of Rickert's theorem (see [21]; cf. also [5, Theorem 3.2], [12, Theorem 2.5]). The improvements were obtained by adjusting an idea found in Bennett's paper [3] to our situation. The proofs of Theorems 1.1 and 1.2 are given in the last section.

2. A version of Rickert's theorem

For any irregular quadruple $\{a, b, c, d\}$ with $a < b < c < d$, a lower bound for the second element b has been obtained by using a version ([10, Lemma 5]) of the Baker-Davenport reduction method ([2, Lemma]) in the proof of [13, Theorem 1.2]. Computations performed in order to establish the above mentioned theorem (and described at length on the last page of [13]) can be summarized as follows.

Lemma 2.1. *Suppose that* $\{a, b, c, d\}$ *is a Diophantine quadruple with* $a < b < c < d_+ < d$.

- *If* $b < 2a$, *then* $b > 21000$.
- *If* $2a \leq b \leq 8a$, *then* $b > 130000$.
- *If* $b > 8a$, *then* $b > 2000$.

As already mentioned, the existence of Diophantine sets hinges on the solvability of a certain system of generalized Pell equations. It is also well known that solutions to such systems appear as common terms to several second-order linear recurrent sequences. In order to get a better upper bound for the index of appearance of a hypothetical solution in the relevant sequence, we slightly improve [12, Theorem 2.5], which is a version of Rickert's theorem ([21]).

Theorem 2.2. *Let* a, b *and* N *be integers with* $0 < a \le b-5$, $b > 2000$ *and* $N \ge 3.706a'b^2(b-a)^2$, *where* $a' = \max\{b-a, a\}$. *Assume that* N *is divisible by* ab. *Then the numbers* $\theta_1 = \sqrt{1+b/N}$ *and* $\theta_2 = \sqrt{1+a/N}$ *satisfy*

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > \left(\frac{1.413 \cdot 10^{28} a'bN}{a}\right)^{-1} q^{-\lambda}$$

for all integers p_1, p_2, q *with* $q > 0$, *where*

$$\lambda = 1 + \frac{\log(10a^{-1}a'bN)}{\log(2.699a^{-1}b^{-1}(b-a)^{-2}N^2)} < 2.$$

The proof of the original result, as well as of all subsequent versions of it, relies on a very general construction recalled below.

Lemma 2.3. ([5, Lemma 3.1]) *Let* $\theta_1, \ldots, \theta_m$ *be arbitrary real numbers and* $\theta_0 = 1$. *Assume that there exist positive real numbers* l, p, L *and* P *with* $L > 1$ *such that for each positive integer* k, *we can find integers* p_{ijk} $(0 \le i, j \le m)$ *with nonzero determinant,*

$$|p_{ijk}| \le pP^k \quad (0 \le i, j \le m)$$

and

$$\left|\sum_{j=0}^{m} p_{ijk}\theta_j\right| \le lL^{-k} \quad (0 \le i \le m).$$

Then

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \ldots, \left|\theta_m - \frac{p_m}{q}\right|\right\} > cq^{-\lambda}$$

holds for all integers $p_1, \ldots, p_m, q$ *with* $q > 0$, *where*

$$\lambda = 1 + \frac{\log P}{\log L} \quad \text{and} \quad c^{-1} = 2mpP\left(\max\{1, 2l\}\right)^{\lambda-1}.$$

Proof of Theorem 2.2. We apply Lemma 2.3 with $m = 2$ and θ_1, θ_2 as in Theorem 2.2. For $0 \le i, j \le 2$ and arbitrary integers a_i (which will be specialized to 0, a and b in due time), let $p_{ij}(x)$ be the polynomial defined by

$$p_{ij}(x) = \sum_{ij} \binom{k+\frac{1}{2}}{h_j} (1+a_jx)^{k-h_j} x^{h_j} \prod_{l \ne j} \binom{-k_{il}}{h_l} (a_j - a_l)^{-k_{il}-h_l},$$

where $k_{il} = k + \delta_{il}$ with δ_{il} the Kronecker delta, $\sum_{ij}$ denotes the sum over all non-negative integers h_0, h_1, h_2 satisfying $h_0 + h_1 + h_2 = k_{ij} - 1$, and $\prod_{l \ne j}$ denotes the product from $l = 0$ to $l = 2$ omitting $l = j$ (which is the expression (3.7) in [21] with $\nu = 1/2$). As seen in the proof of [12, Theorem 2.5], we have

$$f(x) := 2^{2k-1} \prod_{l \neq j} (a_j - a_l)^{k_{il}+h_l} p_{ij}(x) \in \mathbb{Z}[x].$$

Noting

$$\begin{pmatrix} -k_{il} \\ h_l \end{pmatrix} = (-1)^{h_l} \begin{pmatrix} k_{il} + h_l - 1 \\ h_l \end{pmatrix},$$

we see from [4, Lemma 4.1] (or [3, Lemma 3.2]) that $P_2(k)$ divides the greatest common divisor, denoted by $\Pi_2(k)$, of the coefficients of $f(x)$, where $P_2(k)$ is the product over all primes p satisfying $p > \sqrt{2k+1}$, $\gcd(p, 2k) = 1$ and $\{(k-1)/p\} > 3/4$ (as usual, $\{t\}$ denotes the fractional part of a real number t); in particular, we have $\Pi_2(k) \geq P_2(k)$. Denoting by $J_l(k)$ the open interval $((k-1)/l, 4(k-1)/(4l-1))$, we obtain

$$P_2(k) \geq \prod_{l=1}^{\left[\frac{k-1}{\sqrt{2k-1}}\right]} \prod_{\substack{p \in J_l(k) \\ \gcd(p,2k)=1}} p. \tag{2.1}$$

Now we appeal to Corollary 2* of Theorem 7* (the case $(c, d) = (15, 70877)$ in the table) and the Note added in proof from [22] in order to get a lower bound for the right-hand side of (2.1). If $k \geq 903683$, then

$$\begin{aligned}\sum_{l=1}^{13} \sum_{\substack{p \in J_l(k) \\ \gcd(p,2k)=1}} \log p &> \sum_{l=1}^{13} \left(\frac{4}{4l-1} - \frac{4}{15(4l-1)\log\frac{4(k-1)}{4l-1}} - \frac{1.000081}{l} \right)(k-1) \\ &\quad - \log k \\ &> 0.47064(k-1) - \log k > 0.4706k,\end{aligned}$$

which implies that $\Pi_2(k) > e^{0.4706k} > 1.6^k$. For each $1 \leq k \leq 903682$, put

$$g(k) = \sum_{l=1}^{\left[\frac{k-1}{\sqrt{2k+1}}\right]} \sum_{\substack{p \in J_l(k) \\ \gcd(p,2k)=1}} \log p - k \log 1.6.$$

Then, with the help of a computer we find that $g(k) > -31.342$, where the minimal value of $g(k)$ is attained in the case of $k = 607$. This estimate yields

$$\Pi_2(k) > \frac{1.6^k}{e^{31.342}} > \frac{1.6^k}{4.09 \cdot 10^{13}}, \tag{2.2}$$

which also holds for $k \geq 903683$.

If we take $a_0 = 0$, $a_1 = a$ and $a_2 = b$, the proof of [12, Theorem 2.5] shows that

$$p_{ijk} := 2^{-1}[4ab(b-a)^2N]^k \Pi_2(k)^{-1} p_{ij}(1/N) \in \mathbb{Z},$$

which together with (2.2) and the proof of [15, Theorem 21] implies that

$$|p_{ijk}| < pP^k, \qquad \left|\sum_{j=0}^{2} p_{ijk}\theta_j\right| < lL^{-k},$$

where

$$\begin{aligned}
p &= \frac{4.09 \cdot 10^{13}}{2}\left(1 + \frac{a'}{2N}\right)^{1/2} < 2.046 \cdot 10^{13},\\
P &= \frac{32\left(1 + \frac{3b-a}{2N}\right) ab(b-a)^2 N}{1.6\zeta} < \frac{10a'bN}{a}\\
&\left(\zeta = \begin{cases} a^2(2b-a) & \text{if } b-a \geq a,\\ (b-a)^2(a+b) & \text{if } b-a < a \end{cases}\right),\\
l &= \frac{4.09 \cdot 10^{13}}{2} \cdot \frac{27}{64}\left(1 - \frac{b}{N}\right)^{-1} < 8.628 \cdot 10^{12},\\
L &= \frac{1.6}{4ab(b-a)^2 N} \cdot \frac{27}{4}\left(1 - \frac{b}{N}\right)^2 N^3 > \frac{2.699N^2}{ab(b-a)^2}.
\end{aligned}$$

Moreover, the assumption $N \geq 3.706a'b^2(b-a)^2$ implies that

$$\lambda = 1 + \frac{\log(10a^{-1}a'bN)}{\log(2.699a^{-1}b^{-1}(b-a)^{-2}N^2)} < 2$$

and

$$\begin{aligned}
c^{-1} &< 4 \cdot 2.046 \cdot 10^{13} \cdot \frac{10a'bN}{a}\left(17.256 \cdot 10^{12}\right)^{\lambda - 1}\\
&< \frac{1.413 \cdot 10^{28} a'bN}{a}.
\end{aligned}$$

This completes the proof of Theorem 2.2.

$\square$

3. Proofs of the main results

Let $\{a, b, c\}$ be a Diophantine triple with $a < b < c$ and r, s, t the positive integers satisfying $ab + 1 = r^2$, $ac + 1 = s^2$, $bc + 1 = t^2$. Assume that $\{a, b, c, d\}$ is a Diophantine quadruple. Then, there exist positive integers x, y, z satisfying $ad + 1 = x^2$, $bd + 1 = y^2$, $cd + 1 = z^2$. Eliminating d from these equations, we obtain the following system of generalized Pell equations

$$az^2 - cx^2 = a - c, \tag{3.1}$$

$$bz^2 - cy^2 = b - c. \tag{3.2}$$

The solutions of equations (3.1) and (3.2) can be respectively expressed as $z = v_m$ and $z = w_n$ with positive integers m and n, where

$$v_0 = z_0, \qquad v_1 = sz_0 + cx_0, \qquad v_{m+2} = 2sv_{m+1} - v_m, \tag{3.3}$$

$$w_0 = z_1, \qquad w_1 = tz_1 + cy_1, \qquad w_{n+2} = 2tw_{n+1} - w_n. \tag{3.4}$$

LEMMA 3.1. (cf. [7, Lemma 12]) *Let $N = abc$ and let θ_1, θ_2 be as in Theorem 2.2. Then all positive solutions of the system of Diophantine equations* (3.1) *and* (3.2) *satisfy*

$$\max\left\{\left|\theta_1 - \frac{sbx}{abz}\right|, \left|\theta_2 - \frac{tay}{abz}\right|\right\} < \frac{c}{2a} z^{-2}.$$

LEMMA 3.2. *Suppose that there exist integers $m \geq 3$ and $n \geq 2$ such that $z = v_{2m} = w_{2n}$ and $|z_0| = 1$, and that $c \geq 3.706a'b(b-a)^2/a$. Then, $\log z > n \log(4bc)$.*

PROOF. One can prove this lemma in the same way as [7, Lemma 25] (see also the proof of [12, Lemma 2.10]). □

Now we are ready to obtain an upper bound for the solution.

LEMMA 3.3. *Suppose that there exist integers $m \geq 3$ and $n \geq 2$ such that $z = v_{2m} = w_{2n}$ and $|z_0| = 1$, and that $c \geq 3.706a'b(b-a)^2/a$. Then,*

$$n < \frac{4\log(8.406 \cdot 10^{13} a^{1/2}(a')^{1/2}b^2c)\log(1.643a^{1/2}b^{1/2}(b-a)^{-1}c)}{\log(4bc)\log(0.2699a(a')^{-1}b^{-1}(b-a)^{-2}c)}.$$

PROOF. By Lemma 2.1, we may apply Theorem 2.2 with $q = abz$, $p_1 = sbx$, $p_2 = tay$, and $N = abc$. In view of Lemma 3.1, we have

$$z^{2-\lambda} < 0.7065 \cdot 10^{28} aa'b^4c^2 < (8.406 \cdot 10^{13} a^{1/2}(a')^{1/2}b^2c)^2.$$

From

$$\frac{1}{2-\lambda} = \frac{\log(2.699ab(b-a)^{-2}c^2)}{\log\frac{2.699a(b-a)^{-2}c}{10a'b}} < \frac{2\log(1.643a^{1/2}b^{1/2}(b-a)^{-1}c)}{\log(0.2699a(a')^{-1}b^{-1}(b-a)^{-2}c)}$$

and Lemma 3.2 we obtain the asserted inequality. □

We need a recent improvement on the lower bound for the solution. Its proof is similar to the proof of [23, Lemma 2].

LEMMA 3.4. ([6, Lemma 2.4]) *If there exist integers $m \geq 3$ and $n \geq 2$ such that $z = v_{2m} = w_{2n}$ and $|z_0| = 1$, then $m > 0.5b^{-1/2}c^{1/2}$.*

The following is useful to refine some estimates for the solutions.

LEMMA 3.5. *If $\{a, b\}$ is a Diophantine pair with $b > a + 2$, then*

$$b - a \geq 2\sqrt{a+1} + 1.$$

Proof. The assumption $b-a>2$ implies $a+b-2r\geq 1$, from which the assertion can be easily deduced. $\square$

Proof of Theorem 1.1. Assuming the contrary, we will show a contradiction. First of all, we note that if we replace "c" in (3.1), (3.2), (3.3), (3.4) by "d", then $z=v_{2m}=w_{2n}$ and $|z_0|=1$ hold for some $m\geq 3$ and $n\geq 2$ (see the part just before the proof of [12, Theorem 2.1]).

Suppose first that $b<2a$. Then $a'=a$ and $b-a<b/2$. Since

$$\frac{3.706\,a'b(b-a)^2}{a}<3.706\,b\cdot\frac{b^2}{4}=0.9265\,b^3,$$

$$d>4\,abc>4\cdot\frac{b}{2}\cdot b(a+b+2r)>2\left(\frac{1}{2}+1+\sqrt{2}\right)b^3>5.8284\,b^3,$$

we can apply Lemma 3.2 with c replaced by d. By Lemma 3.5 (which can be applied thanks to the main result in [14]) we have

$$a^{1/2}(b-a)^{-1}\leq\frac{\sqrt{a}}{2\sqrt{a+1}+1}<0.5,$$

which together with Lemma 3.3 implies

$$n<\frac{4\log(8.406\cdot 10^{13}b^3d)\log(0.8215\,b^{1/2}d)}{\log(4bd)\log(1.0796\,b^{-3}d)}.$$

Since the right-hand side is a decreasing function of d and $d>5.8284\,b^3$, we have

$$n<\frac{4\log(4.9\cdot 10^{14}b^6)\log(4.789\,b^{7/2})}{\log(23.3136\,b^4)\log(6.292)}<h_1(b),$$

where

$$h_1(b)=\frac{21\log(280.78\,b)\log(1.565\,b)}{\log(2.197\,b)\log(6.292)}.$$

On the other hand, we know by Lemma 3.4 with $m\leq 2n$ that

$$n>0.25b^{-1/2}d^{1/2}>0.6035\,b,$$

which implies $0.6035\,b<h_1(b)$. Therefore we obtain $b<200$, which contradicts Lemma 2.1.

Suppose secondly that $2a\leq b\leq 3a$, so that $a'=b-a\leq 2b/3$. Since

$$\frac{3.706\,a'b(b-a)^2}{a}=3.706\cdot\frac{b}{a}\cdot(b-a)^3<3.295\,b^3,$$

$$d>4abc\geq 4\cdot\frac{b}{3}\cdot b(a+b+2r)>3.317\,b^3,$$

we can apply Lemma 3.2 with d instead of c. Since $a' = b - a \leq 2b/3$, and $a^{1/2}(b-a)^{1/2} \leq b/2$ by the arithmetic mean-geometric mean inequality, we see from Lemma 3.3 that

$$n < \frac{4\log(4.203 \cdot 10^{13} b^3 d)\log(2.3236\, d)}{\log(4\, bd)\log(0.3036\, b^{-3} d)}.$$

In a similar way to the above, we have $d > 3.317\, b^3$ and $n < h_2(b)$, where

$$h_2(b) = \frac{18\log(227.712\, b)\log(1.976\, b)}{\log(1.908\, b)\log(1.007)},$$

which together with $n > 0.4553\, b$ shows that $0.4553\, b < h_2(b)$. Therefore we obtain $b < 97000$, which contradicts Lemma 2.1. This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. Once again, we argue by reduction to absurd. By [19, Lemma 4], if $c > a + b + 2r$ then actually $c > 4\, ab$. Having in view Theorem 1.1, we may also assume that $b > 3\, a$. Hence, $a' = b - a$.

First suppose $b \leq 21a$. Then we find

$$\frac{3.706\, a' b(b-a)^2}{a} = 3.706 \cdot \frac{b}{a} \cdot (b-a)^3 < 67.23\, b^3,$$
$$d > 4\, abc > 16\, a^2 b^2 \geq 0.0362\, b^4.$$

Since $b > 2000$ according to Lemma 2.1, the hypothesis of Lemma 3.3 is fulfilled.

From $3\, a < b$ it results $a^{1/2} b^{1/2} (b-a)^{-1} < \sqrt{3}/2$, while the upper bound on b in terms of a entails $ab^{-1}(b-a)^{-3} > 21^2\, b^{-3}/20^3$. Using these inequalities, Lemma 3.3 yields

$$n < \frac{4\log(4.203 \cdot 10^{13} b^3 d)\log(1.4229\, d)}{\log(4\, bd)\log(67.2123^{-1} b^{-3} d)}.$$

As the right side of the previous relation is a function decreasing in d, for $d > 0.0362\, b^4$ one obtains

$$n < \frac{22.4\log(54.996\, b)\log(0.4764\, b)}{\log(0.6794\, b)\log(1856.7^{-1} b)}.$$

This upper bound on n is compatible with the lower bound $n > b^{3/2}/21$ derived with Lemma 3.4 only for $b < 1980$. Thus, Lemma 2.1 leads us to a contradiction.

Suppose secondly $b \leq 2a^{3/2}$. Since

$$\frac{3.706 a' b(b-a)^2}{a} < \frac{3.706\, b^4}{(b/2)^{2/3}} < 5.883\, b^{10/3},$$
$$d > 16\, (b/2)^{4/3} b^2 > 6.349\, b^{10/3},$$

we may apply Lemma 3.3, and thereby obtain

$$n < \frac{4\log(4.203\cdot 10^{13}b^3 d)\log(1.4229\, d)}{\log(4\, bd)\log(0.17002\, b^{-10/3}d)}.$$

Similarly to the first case, $d > 6.349\, b^{10/3}$ implies

$$n < \frac{760\log(189.597\, b)\log(1.936\, b)}{39\log(2.109\, b)\log(1.0794)}.$$

Comparing this upper bound with $n > 2^{-2/3}b^{7/6}$ deduced from Lemma 3.4, we obtain $b < 1490$. As this range is prohibited by Lemma 2.1, the proof of Theorem 1.2 is complete. ∎

At the request of the referee, we close with a few remarks on the uniqueness of the extension of a Diophantine triple $\{a, b, c\}$ $(a < b < c)$ with a and b very close to each other, such as $b \leq 3\, a$ or $b \leq \max\{21\, a,\, 2\, a^{3/2}\}$, where we proved the non-extendibility to a Diophantine quintuple.

The main technical obstacle which presently prevents to obtain such a result is the non-availability of a lower bound on indices m and n for which $v_m = w_n$ as powerful as that given in Lemma 3.4 and which should be valid for all combinations odd-even. The best lower bounds proved so far for mn odd are of order $b^{-3/4}c^{1/4}$ (see [6, 13]). Clearly, such a bound is irrelevant in our problem unless $c \approx b^{\alpha}$ with α sensibly greater than 3. Moreover, a small gap between b and c does not even allow one to show that the indices m and n have the same parity.

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