GLASNIK MATEMATIČKI
Vol. 50(70)(2015), 35 – 41

A GENERALIZATION OF A PROBLEM OF MORDELL

Bo He, Ákos Pintér, Alain Togbé and Nóra Varga
Aba Normal University, P. R. China, Hungarian Academy of Sciences and University of Debrecen, Hungary and Purdue University North Central, USA

Abstract. In this paper, we use polygonal and pyramidal numbers Pol_x^m and Pyr_x^m to extend a problem of Mordell. Then we prove that if $m \geq 3, n \geq 3$ with $(m,n) \neq (50,3),(50,6)$, all the solutions x and y to the related equation verify $\max(x,y) < C$, where C is an effectively computable constant depending only on m and n.

1. Introduction

Mordell, in his classical book [13, Chapter 27], proposed the following Diophantine problem. Are the only integer solutions of the equation

$$\binom{x}{3} + \binom{x}{2} + \binom{x}{1} + \binom{x}{0} = y^2 \tag{1.1}$$

given by $x = -1, 0, 2, 7, 15, 74$? Ljunggren ([12]) and Bremner ([3]), independently, resolved this equation, showing that there exists exactly one additional solution, $x = 676$. Let Pol_x^m and Pyr_x^m denote the polygonal and pyramidal numbers, respectively, with integer parameters $x \geq 1$ and $m \geq 3$, that is

$$\mathrm{Pol}_x^m = \frac{x((m-2)x + 4 - m)}{2},$$

2010 *Mathematics Subject Classification.* 11D41, 11J86, 11B39, 11D61.
Key words and phrases. Diophantine equation, binomial coefficients, polygonal numbers, pyramidal numbers.
The first author was supported by Natural Science Foundation of China (Grant No. 11301363), and Sichuan provincial scientific research and innovation team in Universities (Grant No. 14TD0040), and the Natural Science Foundation of Education Department of Sichuan Province (Grant No. 13ZA0037 and No. 13ZB0036). The second author was supported in part by the Hungarian Academy of Sciences, OTKA grants K100339, NK101680, NK104208 and by the European Union and the European Social Fund through project Supercomputer, the national virtual lab (grant no.: TÁMOP-4.2.2.C-11/1/KONV-2012-0010). The fourth author was supported by the Hungarian Academy of Sciences.

and

$$\mathrm{Pyr}_x^m = \frac{x(x+1)((m-2)x+5-m)}{6}.$$

These numbers are special cases of the figurate numbers, and for their general properties we refer to [8, 9]. Further, for some Diophantine questions related to these combinatorial objects, see [5, 10, 11, 15]. The aim of this note is to generalize equation (1.1) to polygonal and pyramidal numbers. More precisely, we consider the Diophantine equation

$$\mathrm{Pyr}_{x-2}^m + \mathrm{Pol}_{x-1}^m + x + 1 = \mathrm{Pol}_y^n. \tag{1.2}$$

One can see that for $(m, n) = (3, 4)$ we get back equation (1.1), using the easy facts

$$\mathrm{Pol}_x^3 = \binom{x+1}{2}, \ \ \mathrm{Pyr}_x^3 = \binom{x+2}{3}, \ \ \mathrm{Pol}_x^4 = x^2.$$

Now we can prove

Theorem 1.1. *For fixed positive integers $m \geq 3, n \geq 3$ with $(m, n) \neq (50, 3), (50, 6)$, all the solutions x and y to (1.2) satisfy $\max(x, y) < C$, where C is an effectively computable constant depending only on m and n.*

In the exceptional cases $(m, n) = (50, 3)$ and $(50, 6)$, we have the curves

$$(16x + 1)(2x - 3)^2 = (2y + 1)^2$$

and

$$(16x + 1)(2x - 3)^2 = (4y - 1)^2,$$

respectively. It is trivial that there are infinitely many integer points (x, y) on these curves. Apart from these cases, one can transform equation (1.2) into an elliptic equation and using Baker's classical result concerning the solutions of elliptic equations (see Lemma 2.1), it is enough to guarantee that the discriminant of the corresponding cubic polynomial is nonzero except for the pairs $(m, n) = (50, 3)$ and $(50, 6)$. To prove this statement we apply a method different from that developed in [15].

2. Auxiliary result

In this section, we recall a result due to Baker ([1]). For generalizations, we refer the reader to [4, 6].

Lemma 2.1. *Let $f(x)$ be a cubic polynomial with rational integer coefficients and nonzero discriminant. The equation $f(x) = y^2$ implies $\max(|x|, |y|) < C_1$, where C_1 is an effectively computable constant depending only on the coefficients of f.*

Proof. See [1, Theorems 1 and 2]. ∎

3. Proof of Theorem 1.1

Equation (1.2) leads to the equation

$$F_{m,n}(x) = 8(n-2)\left(\mathrm{Pyr}^m_{x-2} + \mathrm{Pol}^m_{x-1} + x + 1\right) + (n-4)^2$$
$$= \left(2(n-2)y + 4 - n\right)^2.$$

A straightforward calculation gives that the discriminant $D(F_{m,n})$ of $F_{m,n}(x)$ in x is

$$D(F_{m,n}) = \frac{16}{81}(n-2)^2(64n^2m^4 - 256nm^4 + 256m^4 - 2240n^2m^3 - 8960m^3 + 8960nm^3 + 27456m^2 - 27456nm^2 - 243n^4m^2 + 15936n^2m^2 - 4536n^3m^2 - 20864m + 972n^4m + 20864nm + 23976n^3m - 53168n^2m - 4672n + 4672 - 31104n^3 + 63376n^2 - 972n^4)$$
$$= \frac{16}{81}(n-2)^2 \cdot D(m,n).$$

If the discriminant vanishes, then there is a rational multiple zero α of $F_{m,n}(x)$ and thus α is also a zero of the polynomial

$$F^{'}_{m,n}(x) = (3m-6)x^2 + (18-6m)x + 2m - 1.$$

However, the roots of the equation $(3m-6)x^2 + (18-6m)x + 2m - 1 = 0$ are

$$\alpha_{1,2} = \frac{3m - 9 \pm \sqrt{3m^2 - 39m + 75}}{3(m-2)},$$

so $3m^2 - 39m + 75$ must be a perfect square. Now, we have to consider the generalized Pell equation

$$3m^2 - 39m + 75 = k^2,$$

where m and k are integers. One can see that $3|k$. Let $k = 3k_1, k_1 \in \mathbb{Z}$. This gives

$$(2m-13)^2 - 3(2k_1)^2 = 69,$$

or

$$X^2 - 3Y^2 = 69, \tag{3.1}$$

where the new variables are $X = 2m - 13$ and $Y = 2k_1$.

From the general theory of Pell equations, if the Pell equation (3.1) has a fundamental solution (X_0, Y_0), all of integer solutions corresponding to this fundamental solution are given by

$$X + Y\sqrt{3} = (X_0 + Y_0\sqrt{3})(V_j + U_j\sqrt{3}) = (X_0 + Y_0\sqrt{3})\beta^j,\ j \in \mathbb{Z},$$

where $\beta = 2 + \sqrt{3}$ is the fundamental unit of the corresponding number field $\mathbb{Q}(\sqrt{3})$, and V_j, U_j are integer solutions to the Pell equation

$$V^2 - 3U^2 = 1. \tag{3.2}$$

In our case, there are two fundamental solutions $(X_0, Y_0) = (9, 2)$ and $(12, 5)$. Notice that $12 + 5\sqrt{3} = (9 - 2\sqrt{3})\beta$, thus all integer solutions to equation (3.1) are given by

$$X + Y\sqrt{3} = (9 \pm 2\sqrt{3})(V_j + U_j\sqrt{3}) = (9V_j \pm 6U_j) + (\pm 2V_j + 9U_j)\sqrt{3},\ j \in \mathbb{Z}.$$

We have $2m - 13 = X = 9V_j \pm 6U_j$ so $2m = 9V_j \pm 6U_j + 13$. As $2 \nmid V_j$, we get $2|j$. Put $j = 2t$. Then, we have

$$\begin{aligned} 2m &= 9V_{2t} \pm 6U_{2t} + 13 = 9(V_t^2 + 3U_t^2) \pm 6 \cdot 2V_tU_t + 13(V_t^2 - 3U_t^2) \\ &= 22V_t^2 \pm 12V_tU_t - 12U_t^2. \end{aligned}$$

Moreover, we get

$$2k_1 = Y = \pm 2V_{2t} + 9U_{2t} = \pm 2(V_t^2 + 3U_t^2) + 9 \cdot 2V_tU_t = \pm 2V_t^2 + 18V_tU_t \pm 6U_t^2.$$

Put $v = V_t$ and $u = \pm U_t = U_{\pm t}$. Thus, we have

$$m = 11v^2 + 6vu - 6u^2, \quad \pm k_1 = v^2 + 9vu + 3u^2. \tag{3.3}$$

Let $K = \pm k$. After substituting $\alpha_{1,2} = \frac{3m-9\pm k}{3(m-2)} = \frac{3m-9+K}{3(m-2)}$ into the equation $F_{m,n}(x) = 0$, we have quadratic equations for n with discriminant

$$\begin{aligned} \Delta = {} & 16(3m - 9 + K)(-3mK + 63m - 144 + 9K + K^2) \\ & (K^3 + 117mK - 9m^2K - 225K + 351m^2 - 1647m + 1944). \end{aligned} \tag{3.4}$$

Substituting (3.3) into (3.4), with $1 = v^2 - 3u^2$, we have

$$\Delta = 2^4 \cdot 3^{12}(v + u)^2(3v + 2u)^4(2v^2 - 4vu - u^2)(4v^4 - 13v^2u^2 - 6vu^3 - u^4).$$

Let

$$P = 2v^2 - 4vu - u^2, \qquad Q = 4v^4 - 13v^2u^2 - 6vu^3 - u^4.$$

One can check that both P and Q are negative for $(\pm t) \geq 1$ and positive for $(\pm t) \leq 0$.

If Δ is a square, then PQ is also a square. Consider the greatest common divisor of P and Q. One gets

$$\begin{aligned} 2Q &\equiv (16v + 3u)u^3 \pmod{P}, \\ 128P &\equiv -23u^2 \pmod{16v + 3u}. \end{aligned}$$

We have

$$D = (P, Q)|(P, 2Q) = ((16v + 3u)v^3, P).$$

Since $(v, P) = (v, 2v^2 - 4vu - u^2) = 1$, then

$$\begin{aligned} & D \mid (16v + 3u, P) \mid (16v + 3u, 128P) = (16v + 3u, 23u^2) \\ & (16v + 3u, 23u^2) \mid (16v + 3u, 23)(16v + 3u, u^2) \mid 2^8 \cdot 23. \end{aligned}$$

Hence, there exists an integer R such that

$$P = 6v^2 - (2v + u)^2 = (-1)^{\varepsilon}2^{\delta}23^{\eta}R^2, \quad \varepsilon, \delta, \eta \in \{0, 1\}, \tag{3.5}$$

where $\varepsilon = 0$ for $u \leq 0$, $\varepsilon = 1$ for $u \geq 1$.

If $P \equiv 0 \pmod{23}$, then we have $6v^2 \equiv (2v+u)^2 \pmod{23}$. This implies $\pm 11v \equiv 2v+u \pmod{23}$. We have $u \equiv 9v \pmod{23}$ or $u \equiv 10v \pmod{23}$. When $u \equiv 9v \pmod{23}$ holds, we get

$$1 = v^2 - 3u^2 \equiv -242v^2 \equiv 11v^2 \pmod{23}$$

and $\left(\frac{11}{23}\right) = -1$, which is a contradiction. When $u \equiv 10v \pmod{23}$, we have

$$1 = v^2 - 3u^2 \equiv -299v^2 \equiv 0 \pmod{23}.$$

It is impossible. Hence, we have $23 \nmid P$.

Therefore, we have to solve the equation

$$P = 6v^2 - (2v+u)^2 = (-1)^{\varepsilon} 2^{\delta} R^2, \quad \varepsilon, \delta, \in \{0,1\}. \tag{3.6}$$

We have $3 \nmid (2v+u)$. Otherwise, one has $3|R$, and so $9|6v^2$. The condition $3|v$ contradicts the fact that $v^2 - 3u^2 = 1$. By consideration modulo 3, the above equation gives

$$-1 = \left(\frac{-1}{3}\right) = \left(\frac{-(2v+u)^2}{3}\right) = \left(\frac{(-1)^{\varepsilon} 2^{\delta} R^2}{3}\right) = \left(\frac{-1}{3}\right)^{\varepsilon} \left(\frac{2}{3}\right)^{\delta} = (-1)^{\varepsilon+\delta}.$$

Thus, we have $\varepsilon + \delta = 1$. We divide equation (3.6) into two cases.

3.1. *Case I:* $\varepsilon = 1, \delta = 0$. In this case, equation (3.6) becomes

$$6v^2 - (2v+u)^2 = -R^2. \tag{3.7}$$

If $2|u$, then $2 \nmid v$. We have $-R^2 \equiv 2 \pmod 4$. It is impossible. Then we have $2 \nmid u$. This implies $2 \nmid R$. This and $3 \nmid R$ give $\gcd(2v+u, R) = 1$. From equation (3.7), we have

$$(2v+u+R)(2v+u-R) = 6v^2.$$

There exist integers G and H such that

$$2v+u+R = 2c_1G^2, \quad 2v+u-R = 2c_2H^2, \quad v = 2GH, \quad c_1c_2 = 6.$$

This implies

$$u = c_1G^2 + c_2H^2 - 4GH.$$

Substituting this into $v^2 - 3u^2 = 1$, we have

$$-3c^2G^4 + 24cG^3H - 80G^2H^2 + \frac{144}{c}H^3 - \frac{108}{c^2}H^4 = 1,$$

where $c \in \{1,2,3,6\}$. Put $(X,Y) = (G,H)$ for $c = 1$ or 2, $(X,Y) = (H,G)$ for $c = 3$ or 6. We have two quartic Thue equations

$$-3X^4 + 24X^3Y - 80X^2Y^2 + 144XY^3 - 108Y^4 = 1, \text{ if } c = 1,6; \tag{3.8}$$

and

$$-12X^4 + 48X^3Y - 80X^2Y^2 + 72XY^3 - 27Y^4 = 1, \text{ if } c = 2,3. \tag{3.9}$$

We use MAGMA (and also PARI/GP) to solve the two above Thue equations. There is no integer solution (X, Y) to the Thue equation (3.8). All the integer solutions to equation (3.9) are given by $(X, Y) = (1, 1)$ and $(-1, -1)$.

This implies $v = 2GH = 2XY = 2$ and $u = 2X^2 + 3Y^2 - 4XY = 1$. Substituting $m = 11v^2 + 6vu - 6u^2 = 50$ into equation $D(m, n) = 0$, we have

$$-1296(n - 3)(n - 6)(432n^2 + 11737n - 23474) = 0.$$

Hence, we have $(m, n) = (50, 3)$ and $(50, 6)$.

3.2. *Case II:* $\varepsilon = 0, \delta = 1$. In this case, equation (3.6) becomes

$$6v^2 - (2v + u)^2 = 2R^2. \tag{3.10}$$

One can see that $2|u$. We have

$$R^2 + 2(v + u/2)^2 = 3v^2.$$

The fact $2 \nmid v$ gives $2 \nmid R$. Since $3 \nmid v$, then $\gcd(R, v + u/2) = 1$. We have

$$(R + (v + u/2)\sqrt{-2}, R - (v + u/2)\sqrt{-2})|(2R, (v + u/2)\sqrt{-2}) = \sqrt{-2}.$$

But v is odd, therefore the common divisor of $R + (v + u/2)\sqrt{-2}$ and its conjugate is 1. The factorization of this equation over $\mathbb{Q}(\sqrt{-2})$ implies

$$R + (v + u/2)\sqrt{-2} = \pm(1 \pm \sqrt{-2})(G + H\sqrt{-2})^2,$$

for some integers G, H. Express it, then we have

$$v + u/2 = \pm(G^2 \pm 2GH - 2H^2)$$

and

$$v = G^2 + 2H^2.$$

Since $v^2 > 3u^2$ and $v > 0$, we have $v + u/2 > 0$. Put $(X, Y) = (G, \pm H)$, we have

$$u/2 = |X^2 + 2XY - 2Y^2| - (X^2 + 2Y^2), \quad v = X^2 + 2Y^2.$$

Substituting this into $v^2 - 3u^2 = 1$, we have two Thue equations

$$X^4 - 44X^2Y^2 + 192XY^3 - 188Y^4 = 1 \tag{3.11}$$

and

$$-47X^4 - 96X^3Y - 44X^2Y^2 + 4Y^4 = 1. \tag{3.12}$$

Using MAGMA (and checking by PARI/GP), we see that there is no integer solution (X, Y) to the Thue equation (3.12). All integer solutions to equation (3.11) are $(X, Y) = (\pm 1, 0)$. This implies that $u = 0$ and $v = 1$. We have $m = 11$. Substituting the value of m into the equation $D(m, n) = 0$, we get

$$-81(n^2 + 8n - 16)(243n^2 + 1960n - 3920) = 0.$$

There is no integer solution to the above equation.

Finally, if $(m,n) \neq (50,3),(50,6)$, one can see that the cubic polynomial $F_{m,n}(x)$ has integer coefficients and nonzero discriminant. Therefore, using Lemma 2.1, we complete the proof of Theorem 1.1.

Acknowledgements.

The work on this paper was completed during a visit of the first and third authors at the Institute of Mathematics of University of Debrecen. They thank this institution for the fruitful atmosphere of collaboration.

References

[1] A. Baker, *Bounds for the solutions of the hyperelliptic equation*, Math. Proc. Cambridge Philos. Soc. **65** (1969), 439–444.

[2] W. Bosma, J. Cannon and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[3] A. Bremner, *An equation of Mordell*, Math. Comp. **29** (1975), 925–928.

[4] B. Brindza, *On S-integral solutions of the equation $y^m = f(x)$*, Acta Math. Hungar. **44** (1984), 133–139.

[5] B. Brindza, Á. Pintér and S. Turjányi, *On equal values of pyramidal and polygonal numbers*, Indag. Math. (N.S.) **9** (1998), 183–185.

[6] Y. Bugeaud, *Bounds for the solutions of superelliptic equations*, Compos. Math. **107** (1997), 187–219.

[7] J. Cremona and D. Rusin, *Efficient solution of rational conics*, Math. Comp. **72** (2003), 1417–1441.

[8] E. Deza and M. M. Deza, Figurate numbers, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[9] L. E. Dickson, History of the theory of numbers. Vol. II: Diophantine analysis, Chelsea Publishing Co., New York, 1966.

[10] M. Kaneko and K. Tachibana, *When is a polygonal pyramid number again polygonal?* Rocky Mountain J. Math., **32** (2002), 149–165.

[11] L. Hajdu, Á. Pintér, Sz. Tengely and N. Varga, *Equal values of figurate numbers*, J. Number Theory, **137** (2014), 130–141.

[12] W. Ljunggren, *A diophantine problem*, J. London Math. Soc. (2), **3** (1971), 385–391.

[13] L. J. Mordell, Diophantine equations, Pure and Applied Mathematics **30**, Academic Press, London-New York, 1969.

[14] The PARI Group. PARI/GP version `2.7.0`, 2014. Bordeaux, available from `http://pari.math.u-bordeaux.fr`.

[15] Á. Pintér and N. Varga, *Resolution of a nontrivial Diophantine equation without reduction methods*, Publ. Math. Debrecen **79** (2011), 605–610.

B. He
Institute of Mathematics
Aba Normal University
Wenchuan, Sichuan 623000
P. R. China
E-mail: `bhe@live.cn`