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ON QUADRATIZATIONS OF HOMOGENEOUS POLYNOMIAL SYSTEMS OF ODES

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Abstract. The quadratizations of a (homogeneous nonquadratic) nonlinear polynomial system of ODEs introduced by Myung and Sagle in [17] is considered. The 1-1 correspondence between homogeneous quadratic systems of ODEs and nonassociative algebras is used to prove a special structure of the algebra corresponding to a general homogeneous quadratic systems being a quadratization. Every homogeneous solution-preserving map (corresponding to a quadratization) determines the so called essential set which turns out to be crucial for preserving the (in)stability of the origin from homogeneous nonquadratic systems to their quadratizations and vice versa. In particular the quadratizations of homogeneous systems $x' = f_\alpha(x)$ (of order $\alpha > 2$) and cubic planar systems are considered. In the main result we prove that for quadratizations of cubic planar systems the (in)stability of the origin is preserved from the original system $\vec{x}' = f_\alpha(\vec{x})$, $\alpha > 2$ to the quadratization (regarding the essential set of the corresponding solution-preserving map) and vice versa.

1. Introduction

We consider autonomous homogeneous polynomial systems of ODEs and their relations to homogeneous quadratic systems

$$\vec{x}' = Q(\vec{x}), \tag{1.1}$$

where $Q : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree two in every component. It was Markus idea [12] to define algebra multiplication

$$\vec{x} * \vec{y} = B(\vec{x}, \vec{y}) = \frac{1}{2}\left(Q(\vec{x} + \vec{y}) - Q(\vec{x}) - Q(\vec{y})\right) \tag{1.2}$$

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in order to equip $\mathbb{R}^n$ with a structure of a (nonassociative) commutative algebra $(\mathcal{A}, *)$. In the corresponding algebra $(\mathcal{A}, *)$, where the multiplication is defined by (1.2), the system $\vec{x}' = Q(\vec{x})$ obviously becomes a Riccati equation $\vec{x}' = \vec{x} * \vec{x} = \vec{x}^2$ and many interesting relations follow (cf. [3–6, 9, 10, 17, 20]). Among many we mention just few (see [9] or [20] for proofs):

- a system of homogeneous quadratic ODEs has ray solutions iff there exists a nonzero idempotent in the corresponding algebra;
- a system of homogeneous quadratic ODEs has a line of critical points iff there exist a nonzero nilpotent (of index two) in the corresponding algebra;
- a system of homogeneous quadratic ODEs can be solved by reduction iff the corresponding algebra contains a nontrivial ideal;
- if $\vec{x}' = K_1(\vec{x})$ and $\vec{x}' = K_2(\vec{x})$ are two homogeneous quadratic systems of ODEs (on vector spaces V_1 and V_2, respectively) and $\mathcal{A}_1$ and $\mathcal{A}_2$ the corresponding algebras, then a linear map $\Phi : V_1 \to V_2$ is solution-preserving iff Φ is a homomorphism from $\mathcal{A}_1$ into $\mathcal{A}_2$.

Because of the autonomization ([1]), homogenization ([20]) and finally quadratization ([17]), almost every polynomial finite dimensional dynamical system can be treated in terms of this homogeneous quadratic system theory. Unfortunately, all this processes extend the dimension of the system. Therefore, one should have a sensible reason to apply them. The author (cf. [13–16]) already considered the stability of the origin of (1.1) (which is a total degenerate critical point in any dimension) in 2D and in 3D. For cubic systems (in 1D and 2D) and their quadratizations (in 2D and 4D or 5D) we prove a complete preserving of the stability of the origin by the process of the quadratization. In the next Section the solution preserving maps and the quadratization process are considered. Using Markus idea (1.2) we prove a special algebraic structure of the quadratized systems and consider briefly the conditions of optimizing the dimension of a quadratized system. Next two sections are on solution-preserving maps, quadratizations and preserving the (in)stability of the origin. The paper ends with some conclusions.

2. Solution-preserving maps and quadratizations

In this section we consider some results concerning (homogeneous) nonlinear solution-preserving maps and quadratizations. We follow the Myung-Sagle (cf. [17]) definition of the quadratization.

Definition 2.1 ([20]). *Let $I \subset \mathbb{R}$ and V_1, V_2 be real vector spaces with nonempty open subsets U_i of V_i and $f_i : I \times U_i \to V_i$ continuous for $i = 1, 2$. A map $h : U_1' \to U_2$ (where $U_1' \subset U_1$ is open and nonempty) is called a solution-preserving map from $\vec{x}' = f_1(t, \vec{x})$ into $\vec{x}' = f_2(t, \vec{x})$, if it maps parametrized solutions of the former to parametrized solutions of the latter equation.*

In this article we deal with nonlinear solution-preserving maps. The following lemma is instrumental to determine whether a map is solution-preserving or not without actually solving both ODEs.

LEMMA 2.2 ([20]). *Assume that h is a C^1-map. Then h is solution-preserving map if and only if*

$$h'(\vec{x})f_1(t,\vec{x}) = f_2(t,h(\vec{x})) \text{ for all } \vec{x} \in U_1 \text{ and } t \in I. \tag{2.1}$$

Since we are dealing with autonomous systems, equation (2.1) gets the simpler form

$$h'(\vec{x})f_1(\vec{x}) = f_2(h(\vec{x})) \text{ for all } \vec{x} \in U_1. \tag{2.2}$$

Let us demonstrate the power of Lemma 2.2 on the following example.

EXAMPLE 2.3. By (2.2) the following identity

$$\begin{bmatrix} 4(x^2)(xy) \\ 2(x^2)^2 + 3(xy)^2 \end{bmatrix} = \begin{bmatrix} 2x & 0 \\ y & x \end{bmatrix} \begin{bmatrix} 2x^2y \\ 2x^3 + xy^2 \end{bmatrix}$$

implies that the map $h(x,y) = (x^2, xy)$ is a solution-preserving map

$$\text{from the system } \begin{array}{l} x' = 2x^2y \\ y' = 2x^3 + xy^2 \end{array} \text{ into the system } \begin{array}{l} X' = 4XY \\ Y' = 2X^2 + 3Y^2 \end{array}.$$

EXAMPLE 2.4. By introducing the (new) variables $y_1 := x_1^2$, $y_2 := x_1x_2$ into the homogeneous system (of the above example)

$$\begin{array}{l} x_1' = 2x_1^2x_2 \\ x_2' = 2x_1^3 + x_1x_2^2 \end{array}, \tag{2.3}$$

we get the following quadratic system in $\mathbb{R}^4$:

$$\begin{array}{l} x_1' = 2y_1x_2 \\ x_2' = 2x_1y_1 + x_2y_2 \\ y_1' = 2x_1x_1' = 4y_1y_2 \\ y_2' = x_1'x_2 + x_1x_2' = 2y_1^2 + 3y_2^2 \end{array} \tag{2.4}$$

Note, that every solution restricted to the old variables x_1 and x_2 of the quadratic system (2.4) is also a solution to system (2.3). This gives rise to the following definition introduced in [17]:

DEFINITION 2.5 ([17]). *If the solutions to some homogeneous system $\vec{x}' = f_\alpha(\vec{x})$ of degree $\alpha > 2$, $\vec{x} \in \mathbb{R}^n$, can be obtained in terms of the solutions to some quadratic system*

$$\begin{array}{l} \vec{x}' = f_2(\vec{x},\vec{y}) \\ \vec{y}' = q(\vec{x},\vec{y}) \end{array}, \ \vec{x} \in \mathbb{R}^n, \ \vec{y} \in \mathbb{R}^m, \tag{2.5}$$

we say that the system $\vec{x}' = f_\alpha(\vec{x})$ can be quadratized in $\mathbb{R}^{n+m}$. The system (2.5) is called a quadratization of system $\vec{x}' = f_\alpha(\vec{x})$. The variables $\vec{x} = (x_1, x_2, ..., x_n)$ and $\vec{y} = (y_1, y_2, ..., y_m)$ are called the old and the new variables, respectively.

Obviously, the system (2.4) from Example 2.4 is a quadratization of the system (2.3). In [17] it is proven that every polynomial homogeneous system $\vec{x}' = f_\alpha(\vec{x})$ of degree $\alpha > 2$, $\vec{x} \in \mathbb{R}^n$, can be quadratized: one simply *add (for the new variables) all possible monomials of degree* $\alpha - 1$ in order to achieve

$$\deg\left(\frac{d(\text{any old variable})}{dt}\right) = 2.$$

Then the (first) time-derivative of any new variable can be expressed in terms of a quadratic function: actually in the form $\sum_{i,j} \alpha_{ij} y_i y_j$, where y_i and y_j are (some) new variables. Namely, for a general new variable y (i.e. a monomial of order $\alpha - 1$):

$$y = x_1^{\beta_1} x_2^{\beta_2} \cdot \dots \cdot x_n^{\beta_n}$$

where

$$\beta_1 + \beta_2 + \dots + \beta_n = \alpha - 1 \tag{2.6}$$

the degree of $\frac{dy}{dt}$ (in old variables) is $\deg\left(\frac{dy}{dt}\right) = (\alpha - 2) + \alpha = 2(\alpha - 1)$, implying $\deg\left(\frac{d(\text{any new variable})}{dt}\right) = 2$ in the new variables, and if all possible monomials of degree $\alpha - 1$ are added, the quadratization succeeds. Furthermore, one obtains also (see the proof of Theorem 2.7) that $q(\vec{x}, \vec{y})$ from (2.5) depends only on $\vec{y}$ (i.e. $q(\vec{x}, \vec{y}) = q(\vec{y})$). The rest of the proof are technical details (cf. [17, p. 664-666]).

It is a well known combinatorial result that the number of solutions of the equation (2.6) in nonnegative integers is $\binom{\alpha+n-2}{n-1}$. Thus, the system in $\mathbb{R}^n$ of degree α, can be quadratized in at least $\mathbb{R}^{n+\binom{\alpha+n-2}{n-1}}$.

Note, that the Myung-Sagle quadratization can naturally be applied also for the nonhomogeneous polynomial systems. If terms of degree $\alpha_1 > 2$ and $\alpha_2 > 2$ are present in some non-quadratic system, then in the worst case one must add $\binom{\alpha_1+n-2}{n-1} + \binom{\alpha_2+n-2}{n-1}$ new variables to get a proper quadratization.

The quadratization process is not unique, as shown in the following example.

Example 2.6. The system $x_1' = 2x_1y_2$, $x_2' = 2x_1y_1 + x_2y_2$, $y_1' = 4y_1y_2$, $y_2' = 2y_1^2 + 3y_2^2$ is (another) quadratization of (2.3) (which is different from (2.4)).

Note, that even though we have introduced the same new variables as in Example 2.4, different quadratization occurs. However, the new part of the quadratization (i.e. $y_1' = 4y_1y_2$, $y_2' = 2y_1^2 + 3y_2^2$) is obviously the same, since the new variables are unchanged. In the sequel it will turn out that this part of the quadratization is very important for the stability of the origin. We assume that

$$\vec{x}' = f_\alpha(\vec{x}) \tag{2.7}$$

is a homogeneous system of degree $\alpha > 2$, where $\vec{x} \in \mathbb{R}^n$. Let

$$\begin{array}{l} \vec{x}' = \widetilde{f}(\vec{x}, \vec{y}) \\ \vec{y}' = \widetilde{g}(\vec{y}) \end{array}, \text{ where } \vec{y} \in \mathbb{R}^m, \tag{2.8}$$

be a quadratization of (2.7). We shall see that the functional dependence $\vec{y} = h(\vec{x})$, as well as the identity $f(\vec{x}) = \widetilde{f}(\vec{x}, h(\vec{x}))$ plays the crucial role in the proof of the following theorem.

Theorem 2.7. *If a homogeneous system (2.7) of order $\alpha > 2$ in $\mathbb{R}^n$ has a quadratization in $\mathbb{R}^{n+m}$, where $m \leq \binom{\alpha+n-2}{n-1}$, then there exists a (nonlinear) homogeneous solution preserving map $h_{\alpha-1} : \mathbb{R}^n \to \mathbb{R}^m$ of order $\alpha - 1$, which preserves solutions from the (nonquadratic) system (2.7) to a certain quadratic system in $\mathbb{R}^m$.*

Proof. By assumption the system (2.7) has a quadratization in $\mathbb{R}^{n+m}$. Let (2.8) be that quadratization. Thus, all the functions $\widetilde{f}_i$ and $\widetilde{g}_j$ ($i \in \{1, 2, .., n\}$, $j \in \{1, 2, .., m\}$) are homogeneous of order 2. Note that (2.7) is homogeneous of order $\alpha > 2$ and (2.8) is quadratic. Let us denote by $\vec{y} = h(\vec{x})$ the functional dependence of new variables upon the old ones, where $h = (h_1, h_2, \ldots, h_n)$ and h_j are monomials. Since $\widetilde{f}(\vec{x}, \vec{y})$ contains only quadratic factors of type $y_k \cdot y_l$, $x_i \cdot y_k$ or $x_i \cdot x_j$, one can compare orders of monomials in the equality

$$f_\alpha(\vec{x}) = \underbrace{\widetilde{f}(\vec{x}, h(\vec{x}))}_{\deg(\widetilde{f})=2}$$

to obtain $\deg(h_k(\vec{x})) = \alpha - 1$ or $\deg(h_k(\vec{x})) = \frac{\alpha}{2}$ (if α is even). Let us first consider the latter case. Using the chain rule $y_t' = \sum_{k=1}^n \frac{\partial h_t(\vec{x})}{\partial x_k} \cdot x_k'$, one obtains $\deg(y_t') = \frac{3}{2}\alpha - 1$. Comparing this with $\deg\left(\widetilde{g}_t\left(\vec{x}, \vec{y}\right)\right)$ we have:

$$\begin{array}{l} \deg(y_k y_l) = \frac{\alpha}{2} + \frac{\alpha}{2} = \frac{3\alpha}{2} - 1 = \deg(y_t'), \\ \deg(y_k x_i) = \frac{\alpha}{2} + 1 = \frac{3\alpha}{2} - 1 = \deg(y_t'), \end{array}$$

yielding $\alpha = 2$ which contradicts the assumption. Thus $\deg(h_k(\vec{x})) = \alpha - 1$ for all k and $\vec{y} = h(\vec{x})$ is homogeneous of order $\alpha - 1$. From $\deg(h_k) = \alpha - 1$ (using the chain rule again) one obtains $\deg(y_t') = \alpha - 2 + \alpha = 2\,(\alpha - 1)$. Comparing this with $\deg(y_k y_l)$, $\deg(y_k x_i)$ and $\deg(x_i x_j)$ one obtains the identity in case $\widetilde{g}_t(\vec{y}) = \sum_{m,n} g^{(t)}_{(m,n)} \cdot y_m y_n$ and a contradiction in other two cases. Thus

$$\widetilde{g}(\vec{x}, \vec{y}) = \widetilde{g}(\vec{y}) \text{ and } \widetilde{f}_i(\vec{x}, \vec{y}) = \sum_{u,v} f_{(u,v)} \cdot x_u y_v$$

and (according to Lemma 2.2 and the chain rule)

$$h(\vec{x}) := (h_1(\vec{x}), h_2(\vec{x}), ..., h_m(\vec{x}))$$

is the solution-preserving map, as stated. $\square$

Example 2.8. By introducing $y_1 := h_1(\vec{x}) = x_1^2$ and $y_2 := h_2(\vec{x}) = x_1 x_2$ into system (2.3) one can quadratize it to system (2.4), as stated in Example 2.4. By Theorem 2.7 the map

$$(y_1, y_2) = \vec{y} = h(\vec{x}) = (x_1^2, x_1 x_2)$$

is a solution-preserving map from the system

$$\text{(C1)}\quad \begin{array}{l} x_1' = 2x_1^2 x_2 \\ x_2' = 2x_1^3 + x_1 x_2^2 \end{array} \quad \text{into the system} \quad \begin{array}{l} y_1' = 4y_1 y_2 \\ y_2' = 2y_1^2 + 3y_2^2 \end{array} \quad \text{(Q1)}$$

which can easily be verified by the solution-preserving map lemma.

Theorem 2.9. *Let $\vec{x}' = f_\alpha(\vec{x})$ for $\vec{x} \in \mathbb{R}^n$ be homogeneous of order $\alpha > 2$ and let $h : \mathbb{R}^n \to \mathbb{R}^m$ be a homogeneous solution-preserving map from $\vec{x}' = f_\alpha(\vec{x})$ into some quadratic system $\vec{y}' = q(\vec{y})$ for $\vec{y} \in \mathbb{R}^m$. If for every $j \in \{1, 2, .., n\}$ the function $f_{\alpha,j}(\vec{x})$ can be written as a sum of exclusively mixed terms: $\sum_{(s,t)\in S} \gamma_{(s,t)} \cdot x_s y_t$ for some real constants $\gamma_{(s,t)}$, where $S \subseteq \{1, 2, .., n\} \times \{1, 2, .., m\}$ (i.e. we have the functional identity $f_j(\vec{x}) = \sum_{(s,t)\in S} f_{(s,t)}^{(j)} \cdot x_s y_t$), then the system $\vec{x}' = f_\alpha(\vec{x})$ can be quadratized in $\mathbb{R}^{n+m}$.*

Proof. Since h is homogeneous and $\vec{y}' = q(\vec{y})$ is quadratic, we have: $\deg(h) := \alpha - 1$, as seen in the proof of Theorem 2.7. To show that

$$\begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}' = \begin{bmatrix} \widetilde{f}(\vec{x}, \vec{y}) \\ q(\vec{y}) \end{bmatrix} \text{ (where } \vec{y} = h(\vec{x}))$$

is a quadratization of $\vec{x}' = f_\alpha(\vec{x})$, recall that $\vec{x}' = \widetilde{f}(\vec{x}, h(\vec{x}))$ and $\widetilde{f}(\vec{x}, \vec{y})$ is quadratic by assumption. The result follows from Lemma 2.2:

$$y_i' = h_i'(\vec{x}) = \sum_{j=1}^{n} \frac{\partial h_i}{\partial x_j} x_j' = q_i(h(\vec{x})) = q_i(\vec{y}) \text{ for all } \vec{x} \in \mathbb{R}^n.$$

$\square$

Let $(\mathbb{R}^{n+m}, *)$ be the algebra corresponding to the quadratizated system $\vec{X}' = \vec{X} * \vec{X}$ where $\vec{X} = (\vec{x}, \vec{y})$; $\vec{x} \in X = \mathbb{R}^n$ and $\vec{y} \in Y = \mathbb{R}^m$. Denote $X = \operatorname{span}\{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$ and $Y = \operatorname{span}\left\{\vec{E}_1, \vec{E}_2, ..., \vec{E}_m\right\}$. The above theorem implies directly the following two results.

Corollary 2.10. *Let $\vec{X}' = \vec{X} * \vec{X}$, where $\vec{X} = (\vec{x}, \vec{y})$ ($\vec{x} \in X = \mathbb{R}^n$ and $\vec{y} \in Y = \mathbb{R}^m$) be a quadratization in sense of the above theorem. Then the* *corresponding algebra* $(\mathbb{R}^{n+m}, *)$ *admits the following structure:*

(2.9)

$*$	$\vec{e}_1$	$\cdots$	$\vec{e}_n$	$\vec{E}_1$	$\cdots$	$\vec{E}_m$
$\vec{e}_1$	0	$\cdots$	0	$\sum_k \alpha_k^{11}\vec{e}_k$	$\cdots$	$\sum_k \alpha_k^{1m}\vec{e}_k$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\vec{e}_n$	0	$\cdots$	0	$\sum_k \alpha_k^{n1}\vec{e}_k$	$\cdots$	$\sum_k \alpha_k^{nm}\vec{e}_k$
$\vec{E}_1$	$\sum_k \alpha_k^{11}\vec{e}_k$	$\cdots$	$\sum_k \alpha_k^{n1}\vec{e}_k$	$\sum_k \beta_k^{11}\vec{E}_k$	$\cdots$	$\sum_k \beta_k^{1m}\vec{E}_k$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\vec{E}_m$	$\sum_k \alpha_k^{1m}\vec{e}_k$	$\cdots$	$\sum_k \alpha_k^{nm}\vec{e}_k$	$\sum_k \beta_k^{m1}\vec{E}_k$	$\cdots$	$\sum_k \beta_k^{mm}\vec{E}_k$

Obviously X *is an ideal generated by a null-subalgebra and* Y *is a subalgebra (regardless of the dimension* $n+m$*).*

COROLLARY 2.11. *Every quadratization can be solved by a reduction: first solving a subsystem*

$$\frac{d\vec{y}}{dt} = q(\vec{y}) \text{ in } Y,$$

then solving a first order linear (nonautonomous) system

$$\frac{d\vec{x}}{dt} = \widetilde{f}(\vec{x}, \vec{y}(t)) \text{ in } X.$$

From the structure in (2.9) it can be seen that $X = \operatorname{span}\{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$ is an ideal in (2.9) and $Y = \operatorname{span}\left\{\vec{E}_1, \vec{E}_2, ..., \vec{E}_m\right\}$ is a (complementary) subalgebra. The above corollary follows directly from the result that a system of homogeneous quadratic ODEs can be solved by reduction iff the corresponding algebra contains a nontrivial ideal (cf. [9] or [20]).

Note that in $\mathbb{R}^2$ every algebra (2.9) is a quadratization. However, this is not true in $\mathbb{R}^n$ for $n \geq 3$, as shown in the next example.

EXAMPLE 2.12. Systems

$$Ex_- : \begin{array}{l} x' = 2\alpha xy + 2\beta xz \\ y' = -z^2 \\ z' = 2yz \end{array} \tag{2.10}$$

correspond to algebras with multiplication table of the form (2.9). Since in this case the ideal X is one dimensional, the only possibility is that Ex_- is a quadratization of $x' = Kx^n$ (for some n, K). However, Ex_- can not be a quadratization of $x' = Kx^n$, since from $y = Ax^{n-1}$ and $z = Bx^{n-1}$

(after differentiating and equating the coefficients of x^{2n-2}) one obtains the following condition for real parameters A and B:

$$\begin{array}{l} 2A\,(n-1)\,(\alpha A+\beta B)=-B^2 \\ 2B\,(n-1)\,(\alpha A+\beta B)=2AB \end{array} \Longrightarrow \frac{2A}{-B}=\frac{B}{A} \Longrightarrow 2A^2+B^2=0. \tag{2.11}$$

Next the necessity of the condition $f_j(x)=\sum_{(s,t)\in S} f^{(j)}_{(s,t)}\cdot x_s y_t$ in Theorem 2.9 is considered.

EXAMPLE 2.13. The map $h(x,y,u)=(y^2u+xu^2+\frac{1}{2}u^3,\,yu^2)$ is a solution-preserving map

$$\text{from}\quad \begin{array}{l} x'=-2y^4-2xy^2u \\ y'=y^3u+xyu^2+\frac{1}{2}yu^3 \\ u'=0 \end{array} \quad\text{into}\quad \begin{array}{l} X'=Y^2 \\ Y'=XY \end{array}.$$

But the new variables $X:=y^2u+xu^2+\frac{1}{2}u^3$, $Y:=yu^2$ (cf. Theorem 2.9) are not sufficient to obtain a quadratization of the former system, because neither $x'=-2y^4-2xy^2u$ nor $y'=y^3u+xyu^2+\frac{1}{2}yu^3$ can be expressed in quadratic terms using only x, y, u, and X, Y.

3. QUADRATIZATIONS AND THE STABILITY OF THE ORIGIN

Recall that the origin is a total degenerate critical point in every homogeneous (polynomial) system $\vec{x}'=f_\alpha(\vec{x})$; $\alpha\geq 2$. We consider the stability of the origin in sense of Lyapunov, as stated in the following definition.

DEFINITION 3.1. *The solution $\vec{x}=0$ of system $\vec{x}'=f(\vec{x})$ is called stable if for any given $\varepsilon>0$ and $t_0>0$, there is a $\delta>0$ such that for all $\vec{x}_0$ with $\|\vec{x}_0\|<\delta$ and for all $t>t_0$ (for which the solution is defined) one has*

$$\|\vec{x}(t)\|<\varepsilon.$$

The stability in homogeneous quadratic systems in $\mathbb{R}^2$ and partially in $\mathbb{R}^3$ are considered in [6, 9, 13–16], etc. In [5, 18, 19] homogeneous cubic systems in $\mathbb{R}^2$ are considered and classified. In [8, page 50] the following theorem is proved.

THEOREM 3.2 ([8]). *Every real finite dimensional algebra contains at least one nonzero idempotent or nonzero nilpotent of rank two.*

Note that this result implies that every system of quadratic ODEs in $\mathbb{R}^n$ has either a ray solution or a line of critical points. The result due to Kinyon and Sagle concerns the sufficient conditions for stability of the origin.

THEOREM 3.3 ([9]). *If a real algebra, corresponding to a system of quadratic ODEs in $\mathbb{R}^n$, contains a nonzero idempotent, then the origin is unstable.*

The proof of Theorem 3.3 is just a special case of next similar (but even more powerfull) result.

Theorem 3.4. *If there exists a real non zero fixed point of a homogeneous vector field $f_\alpha(\vec{x})$, then the origin is unstable.*

Proof. Assume $\alpha \geq 2$ and $\vec{x}' = f_\alpha(\vec{x})$ (where $f_\alpha(u\vec{x}) = u^\alpha f_\alpha(\vec{x}), \forall u$). Let $\vec{x}_0 \in \mathbb{R}^n$ be such that $f_\alpha(\vec{x}_0) = \vec{x}_0$. The proof follows directly from the Definition 3.1 and the blow-up of the solution $\vec{x}(t) := b_\alpha(t) \cdot \vec{x}_0$, where

$$b_\alpha(t) = \frac{1}{\sqrt[\alpha-1]{\varepsilon^{1-\alpha} - (\alpha-1)\,t}}.$$

∎

Note that if $\vec{x}' = \widetilde{f_\alpha}(\vec{x}, \vec{y})$, $\vec{y}' = q(\vec{y})$ is a quadratization of $\vec{x}' = f_\alpha(\vec{x})$ we have the following corollary.

Corollary 3.5. *If a homogeneous system $\vec{x}' = f_\alpha(\vec{x})$ ($\alpha > 2$, $\vec{x} \in \mathbb{R}^n$) contains a nontrivial fixed point, and if $\vec{y}' = q(\vec{y})$ is a (sub)system corresponding to the new variables (of the corresponding quadratization), then the (sub)algebra corresponding to $\vec{y}' = q(\vec{y})$ contains an idempotent, which means that the origin of the corresponding quadratization is unstable.*

Proof. Let a homogeneous nonquadratic system $\vec{x}' = f_\alpha(\vec{x})$ (i.e. $\alpha > 2$, $\vec{x} \in \mathbb{R}^n$) contain a fixed point $\vec{x}_0 \neq \vec{0}$ and let $\vec{y}' = q(\vec{y})$ be a quadratic (sub)system of the corresponding quadratization (i.e. subsystem corresponding to the subalgebra generated by the new variables $\vec{y}$ consisting possibly of all possible components of $h_{\alpha-1}(\vec{x})$). Denote by $(h(\vec{x}))_i$ ($(f_\alpha(\vec{x}))_i$) the $i-$th component of $h_{\alpha-1}(\vec{x})$ ($f_\alpha(\vec{x})$), respectively. This means: $y_1 = h_1(\vec{x}) = x_1^{\alpha-1}$, $y_2 = h_2(\vec{x}) = x_1^{\alpha-2}x_2$, $\ldots$ etc. Set $(y_0)_i := h_i(\vec{x}_0)$. Then $y_1' = (\alpha-1)\,x_1^{\alpha-2}x_1' = (\alpha-1)\,x_1^{\alpha-2}(f_\alpha(\vec{x}))_1$, $y_2' = (\alpha-2)\,x_1^{\alpha-3}(f_\alpha(\vec{x}))_1\,x_2 + x_1^{\alpha-2}(f_\alpha(\vec{x}))_2, \ldots$ etc. On the fixed-point-line (where $f_\alpha(\vec{x}_0) = \vec{x}_0$) we obtain the following equation(s):

$$\begin{aligned}
y_1' &= (\alpha-1)\,(x_0)_1^{\alpha-2}\,(x_0)_1 = (\alpha-1)\,(x_0)_1^{\alpha-1}\\
&= (\alpha-1)\,(y_0)_1\,,\\
y_2' &= (\alpha-2)\,(x_0)_1^{\alpha-3}\,(x_0)_1\,(x_0)_2 + (x_0)_1^{\alpha-2}\,(x_0)_2 = (\alpha-1)\,(x_0)_1^{\alpha-2}\,(x_0)_2\\
&= (\alpha-1)\,(y_0)_2\\
&\vdots
\end{aligned}$$

Similar we obtain $y_i' = (\alpha-1)\,(y_0)_i$ for every $n \leq i \leq \binom{\alpha+n-2}{n-1}$. Thus, from $\vec{y}' = q(\vec{y})$ for $\vec{y} = \vec{y}_0$ we have:

$$q(\vec{y}_0) = (\alpha-1)\,\vec{y}_0$$

and the result follows immediately - the corresponding idempotent is:

$$\vec{p} := \frac{1}{\alpha-1} h_\alpha(\vec{x}_0) = \frac{1}{\alpha-1} \cdot \vec{y}_0 \tag{3.1}$$

□

Example 3.6. Consider the cubic system (C1) and quadratic system (Q1) from Example 2.8. Obviously $(\widetilde{x}_1, \widetilde{x}_2) = \left(2^{-\frac{3}{4}}, 2^{-\frac{1}{4}}\right)$ is a fixed point of system (C1) and by (3.1)

$$\vec{p} = \frac{1}{2}(\widetilde{y}_1, \widetilde{y}_2) = \frac{1}{2}\left(\left(2^{-\frac{3}{4}}\right)^2, \left(2^{-\frac{3}{4}}\right)\left(2^{-\frac{1}{4}}\right)\right) = \left(\frac{\sqrt{2}}{8}, \frac{1}{4}\right)$$

is the idempotent (corresponding to the fixed point $(\widetilde{x}_1, \widetilde{x}_2)$) of the algebra corresponding to system (Q1), which can easily be verified by a direct computation.

In the sequel we consider first the matching of stability types in a generic case in 1D (which can always be quadratized in 2D). The matching of stability types turns out to be true also in the case of quadratizing the planar systems of homogeneity three (Example 3.13) which will be completely (case-by-case) discussed in a separate paper. Note that cubic systems are classified (see [6]) and the stability of the origin in planar cubic systems can be obtained directly as a case-by-case analysis.

3.1. *Motivation and the essential sets.* The simplest nonquadratic homogeneous 1D case is $x' = ax^3$, $a \neq 0$. Generally, one has $x' = ax^n$, $a \neq 0$, $n \in \{3, 4, 5, ...\}$. Because of the time change $\tau = \frac{t}{a}$ it is enough to consider just two cases: $q_\pm : \frac{dx}{d\tau} = \pm x^3$. Obviously $q_+ : x' = x^3$ is unstable, whilst $q_- : x' = -x^3$ is stable.

In the following example we consider the corresponding (Myung-Sagle) quadratizations of $q_\pm$ and their stability.

Example 3.7. Quadratic system $Q : x' = xy_1$, $y_1' = 2y_1^2$ (where $y_1 = \pm x^2$) is a quadratization of $q_\pm : x' = \pm x^3$. Obviously, the origin is unstable in Q, since $\vec{p} = \left(0, \frac{1}{2}\right)$ is an idempotent of the algebra corresponding to the quadratization Q. At the first glance it seems that the stability properties of the origin in Q and $q_\pm$ are not related at all (since q_+ is unstable and q_- is stable), whilst the origin in Q is always unstable. But, when considering just the stability on the upper half plane, $\mathbb{R}^2_+$, for q_+ (since $y_1 = x^2 > 0$) and on the lower half plane, $\mathbb{R}^2_-$, for q_- (since $y_1 = -x^2 < 0$)

$$\mathbb{R}^2_+ := \left\{ (x, y) \in \mathbb{R}^2 \middle|\ y > 0 \right\},\ \mathbb{R}^2_- := \left\{ (x, y) \in \mathbb{R}^2 \middle|\ y < 0 \right\}$$

one obtains the complete matching of stability of the origin in the "essential phase space".

Remark 3.8. Concerning Example 3.7, one might argue, that the stability property of the original system(s) $q_\pm$ coincides "only" on a special solution $y_1 = \pm x^2$ of quadratized system Q. However, there is a solution-preserving map $y_1 = cx^2$ for every $c > 0$, implying the same type of stability in the essential (half)plane. Namely, beginning with $x' = \pm x^3$, using $y_1 = \pm cx^2$ and applying $t = c\tau$ $(c > 0)$ we are left with the following quadratization:

$$\begin{array}{l} x' = \frac{dx}{dt} = \frac{xy_1}{c} \\ y_1' = \frac{dy_1}{dt} = \frac{2y_1^2}{c} \end{array} \xrightarrow{t=c\tau} \begin{array}{l} \dot{x} = \frac{dx}{d\tau} = xy_1 \\ \dot{y}_1 = \frac{dy_1}{d\tau} = 2y_1^2 \end{array},$$

implying that the "essential sets" $\mathbb{R}^2_+$ $(\mathbb{R}^2_-)$ are "filled up" with solutions $y_1 = cx^2$ $(y_1 = -cx^2)$, $c > 0$, of the same type of stability as in q_+ (q_-), respectively. The only half-line not obtained by quadratization for $y_1 = cx^2$ is

$$l_+ = \left\{ (x, y_1) \in \mathbb{R}^2 \middle|\ x = 0, y_1 > 0 \right\} \subset \mathbb{R}^2_+.$$

Thus,

$$h_{c,\alpha-1} : (x; c) \mapsto \left(x, cx^2\right) \ ; \ \ x \in \mathbb{R},\, c > 0$$

is a surjection from $\mathbb{R} \times \mathbb{R}_+$ (where $\mathbb{R}_+ = \{c \in \mathbb{R}|\ c > 0\}$) to $(\mathbb{R}\setminus\{0\}) \times \mathbb{R}_+ = \mathbb{R}^2_+ \setminus l_+$. This yields that the stability of $x = 0$ in $x' = \pm x^3$ and $(x, y_1) = (0, 0)$ in $\dot{x} = xy_1$, $\dot{y}_1 = 2y_1^2$ (for $\pm y > 0$) coincides completely in the set $\mathbb{R}^2_+ \setminus l_+$.

This gives rise to the following definition.

Definition 3.9 (Essential set: 1D). *Let $X = \mathbb{R}$ and $Y = \mathbb{R}$ and let $h^{\pm}_{c,\alpha-1} : X \to Y$ be defined by $h^{\pm}_{c,\alpha-1}(x) = \pm cx^{\alpha-1}$ $(\alpha > 2)$. Then for any $c > 0$ and for any $\alpha = 2k + 1$ the mapping*

$$h^{\pm}_{\alpha-1} : (x; c) \mapsto \pm cx^{\alpha-1}$$

as a mapping of vector spaces $\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defines a nonsurjective function. The subset $Y_E = h^{\pm}_{c,\alpha-1}(X)$ of vector space Y, for which $h^{\pm}_{\alpha-1}$ is surjective is called the essential set of $h^{\pm}_{c,\alpha-1}$. Let $X_E = X\setminus\{0\}$ then $\vec{h}^{\pm}_{\alpha-1} : X \times \mathbb{R}_+ \to X_E \times Y_E$ defined by

$$\vec{h}^{\pm}_{\alpha-1} : (x; c) \mapsto \left(x, \pm cx^{\alpha-1}\right)$$

is surjective and $X_E \times Y_E \subset X \times Y$ is called the complete essential set of $h^{\pm}_{c,\alpha-1}$. Similar, for $\alpha = 2k$ we have $Y_E = Y\setminus\{0\}$ (and $X_E = X\setminus\{0\}$).

Example 3.10. Quadratic system $Q : x' = xy_1$, $y_1' = 3y_1^2$ (where $y_1 = \pm cx^3$) is a quadratization of systems $q_\pm : x' = \pm x^4$. The origin in $q_\pm$ is unstable. The corresponding quadratization Q contains an idempotent $\vec{p} = \left(0, \frac{1}{3}\right)$. Note that in this case the essential set of $h^{\pm}_{c,3}(x) = \pm cx^3$ coincides with the original space, yielding again a complete matching of the stability at the origin. The complete essential set is $X_E \times Y_E = (\mathbb{R}\setminus\{0\})^2$.

Definition 3.11 (Essential set: 2D). *Let $X = \mathbb{R}^2$ and $Y = \mathbb{R}^3$ and $H^{\pm}_{c,2} : X \to Y$ be defined like in Theorem 2.7 (i.e. by all/some monomials of order 2, multiplied by $c > 0$). The subset $Y_E = H^{\pm}_{c,2}(X)$ of Y for which*

$$H^{\pm}_2 : (x_1, x_2; c) \mapsto \pm\left(cx_1^2, cx_1x_2, cx_2^2\right)$$

is surjective is called the essential set of $H^{\pm}_{c,2}$. *Let* $Y_E = \mathbb{R}_{\pm} \times \mathbb{R} \times \mathbb{R}_{\pm}$ *and* $X_E = (\mathbb{R} \setminus \{0\})^2$ *then* $\vec{H}^{\pm}_2 : X \times \mathbb{R}_+ \to X_E \times Y_E$ *defined by*

$$\vec{H}^{\pm}_2 : (x_1, x_2; c) \mapsto \left(x_1, x_2, \pm c x_1^2, \pm c x_1 x_2, \pm c x_2^2\right) \tag{3.2}$$

is surjective and $X_E \times Y_E \subset X \times Y$ *is called the complete essential set of* $H^{\pm}_{c,2}$.

Examples 3.7 and 3.10 can readily be generalized to $q_n : x' = \pm x^n$, $n = 4, 5, \ldots$ Quadratic system $Q_n : x' = xy_1,\ y_1' = (n-1)\, y_1^2$ (where $y_1 = \pm c x^{n-1}$) is readily the quadratization of q_n. For every $n > 4$ one can obtain either Example 3.7–like or Example 3.10–like stability matching (depending on n being odd or even) and we have:

Proposition 3.12. *The Myung-Sagle quadratization of a homogeneous polynomial system* $x' = f_\alpha(x)$, $x \in \mathbb{R}$, *preserves the stability of the origin from the essential set of* $h^{\pm}_{c,\alpha-1}$ *(corresponding to the qudratization of* $x' = f_\alpha(x)$*) to the original system* $x' = f_\alpha(x)$ *and vice versa.*

3.2. *The quadratization of a 2D homogeneous cubic system and the stability of the origin.* The quadratization of a 2D homogeneous cubic system is a natural generalization of the above examples. The quadratizations "live" in $\mathbb{R}^4$ and/or in $\mathbb{R}^5$. Cubic systems in the plane are classified in [18,19] and considered also in [5,11]. The complete case-by-case (stability matching) analysis of quadratization(s) of

$$\begin{array}{l} x' = a_1 x^3 + b_1 x^2 y + c_1 x y^2 + d_1 y^3 \\ y' = a_2 x^3 + b_2 x^2 y + c_2 x y^2 + d_2 y^3 \end{array}, \ a_i, b_i, c_i, d_i \in \mathbb{R} \tag{3.3}$$

is a very time consuming job and will be done in a separate paper. Note, that in general

$$H^{\pm}_2(x, y; c) = \pm\left(cx^2, cxy, cy^2\right); \ \ c > 0$$

is the proper solution-preserving map and $(X, Y, Z) = \pm\left(cx^2, cxy, cy^2\right)$ are new variables which definitely yield to the quadratization. In the following $\mathbb{R}^2$ to $\mathbb{R}^4-$example we consider just some topologically different cases.

Example 3.13. A special case of (3.3) is

$$\begin{array}{l} x' = c_1 x y^2 \\ y' = b_2 x^2 y \end{array}, \ c_1 \neq 0, \ b_2 \neq 0.$$

The solutions of the original cubic system are on curves $y^2 - \frac{b_2}{c_1} x^2 = C$. Case $b_2 c_1 < 0$ is stable, case $b_2 >$, $c_1 > 0$ is unstable, while case $b_2 < 0$, $c_1 < 0$ is stable again. The above system can be quadratized by adding $X = cx^2$ and $Z = cy^2$ (and introducing $c\tau = t$; $c > 0$) like this

$$\begin{array}{l} x' = c_1 x Z \\ y' = b_2 X y \\ X' = 2c_1 X Z \\ Z' = 2b_2 X Z. \end{array}$$

The corresponding algebra has the multiplication table of the form (2.9):

$*$	e_1	e_2	E_1	E_2
e_1	0	0	0	$\frac{1}{2}c_1e_1$
e_2	0	0	$\frac{1}{2}b_2e_2$	0
E_1	0	$\frac{1}{2}b_2e_2$	0	$c_1E_1+b_2E_2$
E_2	$\frac{1}{2}c_1e_1$	0	$c_1E_1+b_2E_2$	0

This algebra contains subalgebra $Y = \operatorname{span}(E_1, E_2)$ containing two nilpotents of rank two: E_1 and E_2. Since $b_2 \neq 0$ and $c_1 \neq 0$, the only nontrivial idempotent $\vec{p}$ is

$$\begin{array}{l} X = 2c_1XZ \\ Z = 2b_2XZ \end{array} \Longrightarrow \vec{p} = \left(\frac{1}{2b_2}, \frac{1}{2c_1}\right).$$

The (sub)algebra span (E_1, E_2) is isomorphic to algebra no. 3 from [12, Theorem 6]. Typical phase portraits of the subsystem for different values of c_1 and b_2 are shown in Fig. 1 and 2 (containing the idempotent in 1^{st} and 2^{nd} quadrant, respectively). On Fig. 1 - 6 the dots stand for a line of critical points. Note, that if $b_2 < 0$ and $c_1 < 0$ the idempotent lies in 3^{rd} quadrant and, if $b_2 > 0$ and $c_1 < 0$ the idempotent lies in 4^{th} quadrant.

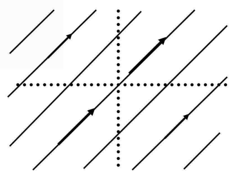

FIGURE 1. $\vec{p}$ in 1^{st} quadrant.

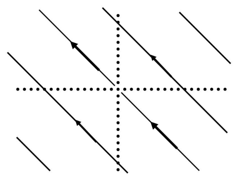

FIGURE 2. $\vec{p}$ in 2^{nd} quadrant.

Now, considering the essential sets (without loss of generality one can assume $c > 0$ to obtain the essential set $X > 0, Z > 0$) one obtains again

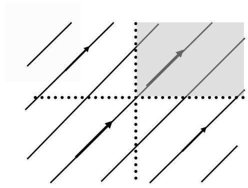

FIGURE 3. ES b_2, $c_1 > 0$.

(see Figs. 3 - 6) the complete matching of the stability on the essential sets (coloured light-gray) with the original space.

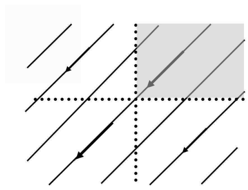

FIGURE 4. $b_2 < 0$, $c_1 < 0$.

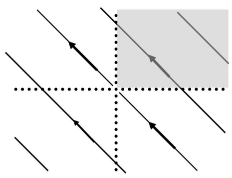

FIGURE 5. $b_2 < 0$, $c_1 > 0$.

The above examples suggest the following theorem.

THEOREM 3.14. *The Myung-Sagle quadratization of a homogeneous polynomial system $\vec{x}' = f_3(\vec{x})$, $\vec{x} \in \mathbb{R}^2$ preserves the (in)stability of the origin from the essential set of the solution-preserving map corresponding to qudratization $\vec{x}' = \widetilde{f}_3(\vec{x}, \vec{y})$, $\vec{y}' = \widetilde{g}(\vec{y})$ to the original system $\vec{x}' = f_3(\vec{x})$ and vice versa.*

Before proving Theorem 3.14, let us consider some Lemmas.

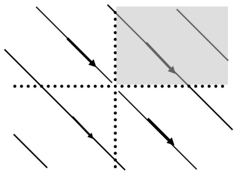

FIGURE 6. $b_2 > 0$, $c_1 < 0$.

LEMMA 3.15. *For any* $(x, y) \in \mathbb{R}^2$ *and any* $\mu > 1$, $r > 0$ *we have*

$$x^2 + y^2 < r^2 \Longrightarrow x^4 + y^4 + x^2y^2 < \mu r^4. \tag{3.4}$$

PROOF. Let $x^2 + y^2 < r^2$ then

$$x^4 + y^4 + x^2y^2 = \left(x^2 + y^2\right)^2 - x^2y^2 < \left(r^2\right)^2 - x^2y^2 = r^4 - x^2y^2 < r^4 < \mu r^4,$$

since $\mu > 1$. □

Obviously, one can set $\mu = 2$ in (3.4) to obtain the following

COROLLARY 3.16. *Let* $(x, y) \in \mathbb{R}^2$ *and* $\left(cx^2, cxy, cy^2\right) \in \mathbb{R}^3$. *Let* $x^2 + y^2 < r^2$, *then*

$$x^2 + y^2 + \left(cx^2\right)^2 + (cxy)^2 + \left(cy^2\right)^2 < r^2 + 2c^2r^4. \tag{3.5}$$

LEMMA 3.17. *Let (3.5) be fulfilled and* $c \neq 0$, *then*

$$x^2 + y^2 < \frac{\sqrt{1 + 2c^2r^2 + 4c^4r^4} - 1}{c^2}. \tag{3.6}$$

REMARK 3.18. One of the simplest real analysis inequalities $2ab \leq (a + b)^2$ yields (for $a = x^2$ and $b = y^2$):

$$x^2y^2 \leq \frac{1}{2}\left(x^2 + y^2\right)^2 \qquad \forall (x, y) \in \mathbb{R}^2. \tag{3.7}$$

PROOF. Let (3.5) be fulfilled, then

$$c^2\left(\left(x^2 + y^2\right)^2 - x^2y^2\right) + x^2 + y^2 < r^2 + 2c^2r^4$$
$$\Updownarrow$$
$$c^2\left(x^2 + y^2\right)^2 + x^2 + y^2 < r^2 + 2c^2r^4 + c^2x^2y^2.$$

According to (3.7) we have

$$c^2\left(x^2 + y^2\right)^2 + x^2 + y^2 < r^2 + 2c^2r^4 + c^2x^2y^2 < r^2 + 2c^2r^4 + \frac{c^2}{2}\left(x^2 + y^2\right)^2,$$

yielding

$$\frac{c^2}{2}\left(x^2 + y^2\right)^2 + x^2 + y^2 < r^2 + 2c^2r^4.$$

Now for $c \neq 0$, set $M := x^2 + y^2$ and solve $c^2M^2 + 2M - \left(2r^2 + 4c^2r^4\right) < 0$ to obtain

$$x^2 + y^2 = M < \frac{\sqrt{1 + 2c^2r^2 + 4c^4r^4} - 1}{c^2},$$

as asserted. ∎

LEMMA 3.19. *Let $c \neq 0$ and*

$$x^2 + y^2 + c^2\left(x^4 + x^2y^2 + y^4\right) > r^2,$$

then

$$x^2 + y^2 > \frac{\sqrt{1 + 4c^2r^2} - 1}{2c^2}. \tag{3.8}$$

PROOF. Rewriting $x^4 + x^2y^2 + y^4$ we get

$$x^2 + y^2 + c^2\left(\left(x^2 + y^2\right)^2 - x^2y^2\right) > r^2$$
$$\Downarrow$$
$$x^2 + y^2 + c^2\left(x^2 + y^2\right)^2 > r^2 + c^2x^2y^2 > r^2.$$

Setting again $M := x^2 + y^2$ and solving $M + c^2M^2 > r^2$ yields (3.8), as asserted. ∎

LEMMA 3.20. *For any $r > 0$ we have*

$$x^4 + y^4 + x^2y^2 < r^4 \Longrightarrow x^2 + y^2 < \sqrt{2}r^2 \qquad \forall\,(x, y) \in \mathbb{R}^2. \tag{3.9}$$

PROOF. From $x^4 + y^4 + x^2y^2 < r^4$ using (3.7) we obtain

$$\left(x^2 + y^2\right)^2 < r^4 + x^2y^2 < r^4 + \frac{1}{2}\left(x^2 + y^2\right)^2,$$

yielding

$$\frac{1}{2}\left(x^2 + y^2\right)^2 < r^4 \Longrightarrow x^2 + y^2 < \sqrt{2}r^2,$$

as asserted. ∎

COROLLARY 3.21. *Let $(x, y) \in \mathbb{R}^2$ and $\left(cx^2, cxy, cy^2\right) \in \mathbb{R}^3$. Let $x^4 + y^4 + x^2y^2 < r^4$, then*

$$x^2 + y^2 + c^2x^4 + c^2y^4 + c^2x^2y^2 < \sqrt{2}r^2 + c^2r^4. \tag{3.10}$$

REMARK 3.22. The result of Lemma 3.20 can also be stated as

$$x^2 + y^2 > \sqrt{2}r^2 \Longrightarrow x^4 + y^4 + x^2y^2 > r^4. \tag{3.11}$$

Now we are ready to prove Theorem 3.14. We use $\vec{x} = (x_1, x_2)$ ($\vec{y} = (y_1, y_2, y_3) = \left(cx_1^2, cx_1x_2, cx_2^2\right) = H_{c,2}(\vec{x})$) for old (new) variables of the corresponding quadratization, respectively. Without loss of generality, we can assume that $c > 0$.

PROOF. ($\Longrightarrow$) Suppose first that the origin in $\vec{x}' = f_3(\vec{x})$, $\vec{x} \in \mathbb{R}^2$ is stable: i.e. for every $\varepsilon > 0$, there is a $\sqrt[4]{2}\delta > 0$ such that

$$x_{10}^2 + x_{20}^2 < \sqrt{2}\delta^2 \Longrightarrow x_1^2(t) + x_2^2(t) < \varepsilon^2.$$

Let $c > 0$ be fixed. Then by (3.9) and (3.10) we have:

$$y_{10}^2 + y_{20}^2 + y_{30}^2 < c^2\delta^4 \Longrightarrow x_{10}^2 + x_{20}^2 + y_{10}^2 + y_{20}^2 + y_{30}^2 < \sqrt{2}\delta^2 + c^2\delta^4 = \Delta^2.$$

On the other hand by (3.5) we have:

$$x_1^2(t) + x_2^2(t) < \varepsilon^2 \Longrightarrow x_1^2(t) + x_2^2(t) + y_1^2(t) + y_2^2(t) + y_3^2(t) < \varepsilon^2 + 2c^2\varepsilon^4 = \epsilon^2.$$

Thus: for every $\epsilon > 0$ there is a $\Delta > 0$ such that

$$x_{10}^2 + x_{20}^2 + y_{10}^2 + y_{20}^2 + y_{30}^2 < \Delta^2 \Longrightarrow x_1^2(t) + x_2^2(t) + y_1^2(t) + y_2^2(t) + y_3^2(t) < \epsilon^2,$$

yielding the stability of the origin $(\vec{0}, \vec{0}) = (\vec{x}, \vec{y})$ for the essential set of $H_{c,2}$ corresponding to qudratization $\vec{x}' = \widetilde{f}_3(\vec{x}, \vec{y})$, $\vec{y}' = \widetilde{g}(\vec{y})$.

If the origin in $\vec{x}' = f_3(\vec{x})$, $\vec{x} \in \mathbb{R}^2$ is unstable (i.e. for every $\delta > 0$ there is a $\epsilon_* = \sqrt[4]{2}\varepsilon_* > 0$ such that $x_{10}^2 + x_{20}^2 < \delta^2$ and $x_1^2(t) + x_2^2(t) > \epsilon_*^2$) then by (3.5)

$$x_{10}^2 + x_{20}^2 + y_{10}^2 + y_{20}^2 + y_{30}^2 < \delta^2 + 2c^2\delta^4 = \Delta^2$$

and by (3.11)

$$x_1^2(t) + x_2^2(t) + y_1^2(t) + y_2^2(t) + y_3^2(t) > \sqrt{2}\varepsilon_*^2 + c^2\varepsilon_*^4 = E_*^2,$$

yielding the instability of the origin on the essential set of $H_{c,2}$.

($\Longleftarrow$) First, let the quadratized system, $(\vec{x}, \vec{y})' = \left(\widetilde{f}_3(\vec{x}, \vec{y}), q(\vec{y})\right)$, contain an unstable origin and let the initial condition

$$(\vec{x}_0, \vec{y}_0) = \left(x_{10}, x_{20}, cx_{10}^2, cx_{10}x_{20}, cx_{20}^2\right)$$

contradicts the stability for some E_*. Thus: for every $\delta^2 + 2c^2\delta^4 = \Delta^2 > 0$ and a E_* we have (for all $t \geq t_0$):

$$\begin{aligned} &x_{10}^2 + x_{20}^2 + c^2x_{10}^4 + c^2x_{10}^2x_{20}^2 + c^2x_{20}^4 \\ &\qquad < \Delta^2 \wedge x_1^2(t) + x_2^2(t) + y_1^2(t) + y_2^2(t) + y_3^2(t) > E_*^2. \end{aligned}$$

Then by (3.6) we have $x_{10}^2 + x_{20}^2 < (\sqrt{1 + 2c^2\delta^2 + 4c^4\delta^4} - 1)/c^2$ on one hand, and by (3.8) $x_1^2(t) + x_2^2(t) > (\sqrt{1 + 4c^2E_*^2} - 1)/(2c^2)$ on the other hand, yielding instability of the origin for $\vec{x}' = f_3(\vec{x})$.

Finally, let for $\vec{y} = H_{c,2}(\vec{x})$ the quadratized system

$$(\vec{x}, \vec{y})' = \left(\widetilde{f}(\vec{x}, \vec{y}), q(\vec{y})\right)$$

be stable on the essential set $Y_E \subset \mathbb{R}^3$. Thus, for every $\epsilon > 0$ there is a $\delta^2 + 2c^2\delta^4 = \Delta^2 > 0$ such that $x_{10}^2 + x_{20}^2 + y_{10}^2 + y_{20}^2 + y_{30}^2 < \Delta^2$ implies

$x_1^2(t)+x_2^2(t)+y_1^2(t)+y_2^2(t)+y_3^2(t)<\epsilon^2$. Let δ be a solution of $\Delta^2=\delta^2+2c^2\delta^4$. Then for any $\delta>0$ by (3.5)

$$x_{10}^2+x_{20}^2<\delta^2\Rightarrow x_{10}^2+x_{20}^2+y_{10}^2+y_{20}^2+y_{30}^2<\Delta^2$$

which by assumption implies $x_1^2(t)+x_2^2(t)+y_1^2(t)+y_2^2(t)+y_3^2(t)<\epsilon^2$. By (3.6) we have

$$x_1^2(t)+x_2^2(t)<\frac{\sqrt{1+2c^2\epsilon^2+4c^4\epsilon^4}-1}{c^2}$$

which completes the proof. □

Example 2.12 showed that the quadratization does not exist for every system/algebra of the form (2.9). In the following example we see that system Ex_- from example 2.12 (which is not a quadratization, but of the form (2.9) and contains a stable subalgebra) is not necessary stable in the whole.

Example 3.23. System Ex_- (2.10) corresponds to algebra with the multiplication table of the form (2.9). The corresponding subalgebra $Y=\operatorname{span}\left\{\vec{E}_1,\vec{E}_2\right\}$ is obviously stable, but the whole system (2.10) can be stable or unstable (for different values of α,β). The solutions

$$y(t)=C\frac{y_0+C+(y_0-C)e^{4Ct}}{y_0+C-(y_0-C)e^{4Ct}},\qquad z(t)=\frac{2Cz_0e^{2Ct}}{C+y_0+(C-y_0)e^{4Ct}}$$

and

$$x(t)=x_0e^{4\beta\left(\arctan\left(\frac{2y_0}{z_0}\right)+\arctan\left(\frac{2(C\tanh(Ct)-y_0)}{z_0}\right)\right)+\alpha\left(2Ct+\ln\frac{2C}{C+y_0+(C-y_0)e^{4Ct}}\right)},$$

(where $C=\sqrt{y_0^2+\frac{1}{2}z_0^2}$) are yielding for instance for $\alpha=\beta=1$ a stable origin in (2.10), while for $\alpha=-1$, $\beta=1$ an unstable one.

4. Conclusions

The one-to-one correspondence between homogeneous quadratic systems and (binary) algebras solves many algebraic and dynamical problems in both areas ([9, 10]). The stability theory based on this theory provided some general and partial results and is still developing (cf. [13, 15, 16], etc.). In this article it is shown that the quadratization process might play an important role when considering the stability of the origin in homogeneous systems in $\mathbb{R}^n$. The connections between solution-preserving maps and quadratizations are important also from the dimensional point of view. The examples in this article and Theorem 3.14 are showing that the stability matching may be regarded as two-sided: one can use it in order to determine the stability of a quadratic system, but on the other hand, if the stability of the quadratized system is known, one can determine the stability of the corresponding nonquadratic system which might also appear in higher dimensions. In this context the so called problem of "dequadratization" of a (special) quadratic system appears (i.e. given a quadratic system of possibly large dimension, when can one decrease the dimension of the system by, possibly, increasing the degree of homogeneity).

In Theorem 2.7 the optimization (in sense of increasing the dimension $n+m$) of the process of quadratization is briefly considered. It would be interesting to know what is a minimal integer m providing a possible quadratization. Theorem 2.7 shows that m depends on f_α and on the existence of a solution preserving map from $\vec{x}' = f_\alpha(\vec{x})$ to some quadratic system.

The Myung-Sagle quadratization preserves the stability (on the essential set) for all quadratizations in $\mathbb{R}^2$. Example 3.13 motivated the author to prove Theorem 3.14 where the complete matching of the stability of the origin in $\vec{x}' = f_3(\vec{x})$ ($\vec{x} \in \mathbb{R}^2$), and the stability of the origin on the essential set of $H_{c,2}(\vec{x})$, where $\vec{x}' = \widetilde{f_3}(\vec{x},\vec{y})$, $\vec{y}' = q(\vec{y})$ ($\vec{y} = H^{\pm}_{c,2}(\vec{x})$) is the corresponding quadratization. Probably a similar result for $\vec{x}' = \widetilde{f_\alpha}(\vec{x},\vec{y})$, $\vec{y}' = q(\vec{y})$ and $\vec{x}' = f_\alpha(\vec{x})$ for any $\alpha > 3$ and any dimension n of vector $\vec{x}$ holds.

Finally note that the quadratization process (in particular the quadratized systems) might be potentially useful at searching the Darboux polynomials [7], as well as at the blow up technique [2].

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