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SUMS OF ZEROS OF SOLUTIONS TO SECOND ORDER DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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Abstract. We consider the equation $u'' = P(z)u$, where $P(z)$ is a polynomial. Let $z_k(u), k = 1, 2, \ldots$ be the zeros of a solution $u(z)$ to that equation. Inequalities for the sums $\sum_{k=1}^{j} \frac{1}{|z_k(u)|}$ $(j = 1, 2, \ldots)$ are derived. They considerably improve the previous result of the author. Some applications of the obtained bounds are also discussed. An illustrative example is presented. It shows that the suggested results are sharp.

1. Introduction and statement of the main result

In the present paper we consider linear differential equations with polynomial coefficients in the complex domain. The literature devoted to the zeros of solutions of such equations is very rich. Here the main tool is the Nevanlinna theory. An excellent exposition of the Nevanlinna theory and its applications to differential equations is given in the book [22]. In that book, in particular, the well-known results of Bank ([3]), Brűggemann ([5]), Hellerstein ([18]) and other mathematicians are featured. The classical comparison principle for zeros of ODE in the complex plane is presented in [20]. The real zeros of solutions to equations with polynomial coefficients were investigated in the very interesting papers by Gundersen ([17]), Eremenko and Merenkov ([9]), and by C. Z. Huang ([21]). In connection with recent results see also the papers [4, 10, 11], [23]-[27]. In particular, in the paper [26], the authors study the convergence of the zeros of a non-trivial (entire) solution to the linear

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differential equation

$$f'' + \{Q_1(z)e^{P_1(z)} + Q_2(z)e^{P_2(z)} + Q_3(z)e^{P_3(z)}\}f = 0$$

where P_j are polynomials of degree $n \geq 1$ and $Q_j (\not\equiv 0)$ are entire functions of order less than n $(j = 1, 2, 3)$. In the paper [11], by certain separation and comparison results, estimates for the counting functions of the zeros of solutions to nth-order linear differential equations are deduced. It is proved that these estimates are in any possible case the sharpest ones, and they generalize known results for the zeros of solutions to third- and fourth-order linear differential equations. The remarkable results on the zeros of a wide class of ordinary differential equations with polynomial coefficients whose solutions are classical orthogonal polynomials were established by N. Anghel ([1]). Besides, he had derived results connected with the equations of mathematical physics. In addition, N. Anghel ([2]), investigated the following question: when is an entire function of finite order, solution to a complex 2nd order homogeneous linear differential equation with polynomial coefficients? He gives two (equivalent) answers to this question, one of which involves certain Stieltjes-like relations for the zeros of solutions, the second one requires the vanishing of all but finitely many suitable expressions constructed via the relations of the sums of the zeros of the function derived in [13].

Certainly, we could not survey the whole subject here and refer the reader to the above listed publications and references given therein. In the above cited works mainly the asymptotic distributions of zeros and counting functions of zeros are investigated. At the same time, bounds for the zeros of solutions are very important in various applications. But to the best of our knowledge, they have been investigated considerably less than the asymptotic distributions. In the paper [14] the author has established bounds for the sums of the zeros of solutions for the second order equations with polynomial coefficients. In the interesting paper [6], some results from [14] have been extended to the equation $u^{(m)} = P(z)u$, where P is a polynomial and $m > 2$. In [16] the main result from [14] is extended to the second order ODE with non-polynomial coefficients. Perturbations of the zeros of solutions to second order differential equations with polynomial coefficients were investigated in the paper [16].

In the present paper, we considerably refine the main result from [14]. Note that, the proof of the main result of the present paper-Theorem 1.1 is considerably different from the proof of the one from [14]. In addition, we estimate the zero free domains. That estimation supplements the well-known results from [20] as well as the results from the paper of Eloe and Henderson [8] on the positivity of solutions for higher order ordinary differential equations, since the positivity of solutions implies the absence of zeros.

Consider the equation

$$u'' = P(z)u, \quad u(0) = u_0, u'(0) = u_1 \quad (u_0, u_1 \in \mathbb{C}; u_0 \neq 0), \tag{1.1}$$

where

$$P(z) = \sum_{k=0}^{n} c_k z^k \ \ (c_n \neq 0)$$

is a polynomial with complex coefficients. As it is well-known [22], the zeros of solutions to (1.1) are simple. Enumerate the zeros $z_k(u)$ of u in order of increasing modulus: $|z_k(u)| \leq |z_{k+1}(u)|$ $(k = 1, 2, \ldots)$. Put

$$s(P) = \left(\sum_{k=0}^{n} |c_k|\right)^{1/2}, \ \ \beta(P) = ((1 + n/2)s(P)e)^{2/(n+2)}$$

and

$$C_0(P) := \beta(P)e^{s(P)} + \frac{|u_1|}{|u_0|}\left(1 + 4e^{s(P)}\right).$$

THEOREM 1.1. *If $u_0 \neq 0$, then*

$$\sum_{k=1}^{j} \frac{1}{|z_k(u)|} \leq C_0(P) + \sqrt{2}\beta(P) \sum_{k=1}^{j} \frac{1}{(k+1)^{\frac{2}{n+2}}} \ \ (j = 1, 2, \ldots).$$

The proof of this theorem is presented in the next two sections.

To estimate the sharpness of the theorem, consider the equation

$$u'' + u = 0, u(0) = 1, u'(0) = 0.$$

Then $u(z) = \cos(z)$ and its zeros are $\pi(k + 1/2)$ $(k = 0, \pm 1, \pm 2, \ldots)$. So

$$\sum_{k=1}^{2j+1} \frac{1}{|z_k(u)|} = \frac{1}{\pi}\left(\frac{1}{j + 1/2} + 2\sum_{k=0}^{j-1} \frac{1}{k + 1/2}\right). \tag{1.2}$$

In the considered case $n = 0, s(P) = 1, \beta(P) = e$, $C_0(P) = e^2$. Therefore, Theorem 1.1 gives us the inequality

$$\sum_{k=1}^{2j+1} \frac{1}{|z_k(u)|} \leq e^2 + \sqrt{2}e \sum_{k=1}^{2j+1} \frac{1}{k+1} \ (j = 1, 2, \ldots). \tag{1.3}$$

We can see that (1.2) and (1.3) are asymptotically equivalent.

2. ESTIMATES FOR SOLUTIONS

Consider the equation

$$\frac{d^2 u}{dz^2} = Q(z)u, \ \ u(0) = u_0, u'(0) = u_1 \ \ (u_0, u_1 \in \mathbb{C}), \tag{2.1}$$

where

$$Q(z) = \sum_{k=0}^{\infty} c_k z^k \ (c_0 \neq 0)$$

is an entire function. Put $M_f(r) = \sup_{|z|\leq r} |f(z)|$ for a function $f(z)$.

LEMMA 2.1. *A solution $u(z)$ of equation (2.1) satisfies the inequality*

$$M_u(r) \leq (|u_0| + r|u_1|)\cosh(r\sqrt{q(r)}\,) \quad (r \geq 0; \cosh(r) = \frac{1}{2}(e^r + e^{-r})),$$

where

$$q(r) = \sum_{k=0}^{\infty} |c_k| r^k.$$

PROOF. For a fixed $t \in [0, 2\pi)$ and $z = re^{it}$ we have

$$e^{-2it}\frac{d^2 u(re^{it})}{dr^2} = Q(re^{it})u(re^{it}).$$

Integrating twice this equation in r, we obtain

$$e^{-2it}u(re^{it}) = e^{-2it}(u_0 + ru_1) + \int_0^r (r-s)Q(se^{it})u(se^{it})ds.$$

Hence,

$$|u(re^{it})| \leq |u_0| + r|u_1| + \int_0^r (r-s)q(s)|u(se^{it})|ds. \tag{2.2}$$

Due to the comparison lemma [7, Lemma III.2.1], we have $|u(re^{it})| \leq v(r)$, where $v(r)$ is a solution of the equation

$$v(r) = w(r) + \int_0^r (r-s)q(s)v(s)ds = w(r) + Vv(r), \tag{2.3}$$

where $w(r) = |u_0| + r|u_1|$ and V is the Volerra operator defined by

$$(Vv)(r) = \int_0^r (r-s)q(s)v(s)ds.$$

So

$$v = \sum_{k=0}^{\infty} V^k w. \tag{2.4}$$

But for any positive nondecreasing $h(r)$ we have

$$Vh(r) = \int_0^r (r-s)q(s)h(s)ds \leq h(r)q(r)\int_0^r (r-s)ds.$$

Hence,

$$V^m h(r) \leq q^m(r)h(r)\int_0^r \int_0^{r_1} \dots \int_0^{r_{m-1}} (r-r_1)\dots(r_{m-1}-r_m)dr_1\dots dr_m$$
$$= q^m(r)h(r)\frac{r^{2m}}{(2m)!}.$$

Thus from (2.4) it follows

$$v(r) \leq (|u_0| + r|u_1|)\sum_{k=0}^{\infty}\frac{q^k(r)r^{2k}}{(2k)!}.$$

But

$$\sum_{k=0}^{\infty} \frac{q^k(r) r^{2k}}{(2k)!} = \cosh(r\sqrt{q(r)}).$$

This implies the required result. ∎

It should be noted that the inequality stated in Lemma 2.1 can be proved by Herold's comparison theorem [19].

Consider again the equation

$$\frac{d^2u}{dz^2} = P(z)u, \quad u(0) = u_0, u'(0) = u_1 \quad (u_0, u_1 \in \mathbb{C}). \tag{2.5}$$

In this case

$$q(r) = \hat{p}(r) := \sum_{k=0}^{n} |c_k| r^k.$$

In addition

$$r\sqrt{\hat{p}(r)} \le \sqrt{\hat{p}(1)}(1 + r^{n/2+1}) \quad (r > 0) \tag{2.6}$$

and

$$\cosh(r\sqrt{\hat{p}(r)}) \le \exp\,[\sqrt{\hat{p}(1)}(1 + r^{n/2+1})].$$

Sine $\hat{p}(1) = s^2(P)$, Lemma 2.1 yields

Corollary 2.2. *A solution of equation (2.5) satisfies the inequality*

$$M_u(r) \le (|u_0| + r|u_1|)e^{s(P)(1+r^{n/2+1})}.$$

The previous corollary is sharp: as it is well-known a solution of equation (2.5) is an entire function of order no more than $(n+2)/2$, see, e.g. [22, Proposition 5.1]. Besides, our proof is absolutely different.

3. Proof of Theorem 1.1

Lemma 3.1. *Let an entire function*

$$f(z) = \sum_{k=0}^{\infty} f_k z^k$$

satisfy the inequality

$$M_f(r) \le (D_1 + D_2 r) exp[B r^\rho]$$

$$(D_1, D_2 = \text{const} \ge 0; B = \text{const} > 0; \rho \ge 1; r > 0). \tag{3.1}$$

Then its Taylor coefficients are subject to the inequalities

$$|f_j| \le D_1 \frac{(eB\rho)^{j/\rho}}{(j!)^{1/\rho}} + D_2 \frac{(eB\rho)^{(j-1)/\rho}}{[(j-1)!]^{1/\rho}} \quad (j \ge 2). \tag{3.2}$$

PROOF. By the well-known inequality for the coefficients of a power series

$$|f_j| \leq \frac{M_f(r)}{r^j} \leq (D_1 + D_2 r)\frac{e^{Br^\rho}}{r^j}.$$

Employing the usual method for finding extrema it is easy to see that the function $r^{-j}e^{Br^\rho}$ $(j \geq 1)$ takes its smallest value in the range $r > 0$ for $r_j = (\frac{j}{B\rho})^{1/\rho}$ and therefore

$$|f_j| \leq \frac{M_f(r_j)}{r_j^j} \leq D_1\left(\frac{eB\rho}{j}\right)^{j/\rho} + D_2\left(\frac{eB\rho}{j-1}\right)^{(j-1)/\rho} \quad (j \geq 2).$$

Hence, due to the well known inequality, $j^j \geq j!$ $(j \geq 1)$, we have (3.2), as claimed. □

Note that Lemma 3.1 can be proved also by the classical Valiron-Wiman theory, cf. [25, page 11,Q. 67 (1)].

Now let us consider the entire function

$$f(z) = \sum_{k=0}^{\infty} \frac{d_k z^k}{(k!)^{1/\rho}} \quad (\rho \geq 1,\ \lambda \in \mathbb{C},\ d_0 = 1, d_k \in \mathbb{C}). \tag{3.3}$$

Assume that

$$\theta(f) := [\sum_{k=1}^{\infty} |d_k|^2]^{1/2} < \infty. \tag{3.4}$$

THEOREM 3.2. *Let f be defined by (3.3) and condition (3.4) hold. Then*

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|} \leq \theta(f) + \sum_{k=1}^{j} \frac{1}{(k+1)^{1/\rho}} \quad (j = 1, 2, \ldots).$$

This theorem is proved in [12] (see also [13, Section 5.1]).

Denote

$$b := (eB\rho)^{1/\rho} \text{ and } \tau_\rho := \left(\sum_{j=2}^{\infty} \frac{j^{2/\rho}}{2^j}\right)^{1/2}.$$

LEMMA 3.3. *Let an entire function f satisfy the conditions (3.1) and $f(0) = 1$. Then*

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|} \leq |f_1| + D_1 b + \sqrt{2}D_2\tau_\rho + \sqrt{2}b\sum_{k=1}^{j} \frac{1}{(k+1)^{1/\rho}} \quad (j = 1, 2, \ldots).$$

PROOF. Due to Lemma 3.1

$$|f_k| \leq \frac{D_1 b^k}{(k!)^{1/\rho}} + \frac{D_2 b^{k-1}}{[(k-1)!]^{1/\rho}} = \frac{1}{(k!)^{1/\rho}}(D_1 b^k + D_2 b^{k-1}k^{1/\rho}) \quad (k \geq 2).$$

Consider the function

$$f_\chi(z) = f(\chi z) = 1 + \sum_{k=1}^{\infty} \frac{f_k \chi^k z^k}{(k!)^{1/\rho}}$$

with $\chi = \frac{1}{b\sqrt{2}}$. We have

$$\theta^2(f_\chi) = \sum_{j=1}^{\infty} |\chi^j f_j|^2 \le |f_1|^2\chi^2 + \sum_{j=2}^{\infty} \frac{1}{(b\sqrt{2})^{2j}} (D_1 b^j + D_2 b^{j-1} j^{1/\rho})^2$$
$$= |f_1|^2\chi^2 + \sum_{j=2}^{\infty} \frac{1}{2^j} (D_1 + \frac{D_2}{b} j^{1/\rho})^2.$$

Hence, by the triangle inequality

$$\theta(f_\chi) \le |f_1|\chi + D_1 (\sum_{j=2}^{\infty} \frac{1}{2^j})^{1/2} + \frac{D_2 \tau_\rho}{b} = |f_1|\chi + \frac{D_1}{\sqrt{2}} + \frac{D_2 \tau_\rho}{b}.$$

Thus due to Theorem 3.2 we obtain

$$\sum_{k=1}^{j} \frac{1}{|z_k(f_\chi)|} \le |f_1|\chi + \frac{D_1}{\sqrt{2}} + D_2 \frac{\tau_\rho}{b} + \sum_{k=1}^{j} \frac{1}{(k+1)^{1/\rho}} \ (j = 1, 2, \ldots).$$

Since $\chi z_k(f_\chi) = z_k(f)$, the lemma is proved. ∎

Proof of Theorem 1.1. Delete (1.1) by $u(0)$. According to Corollary 2.2 we take $B = s(P)$, $\rho = 1 + n/2$,

$$D_1 = \frac{1}{|u_0|} e^{s(P)} \text{ and } D_2 = \frac{|u_1|}{|u_0|} e^{s(P)}.$$

Then $b = \beta(P)$. In addition,

$$\tau_{1+n/2}^2 = \sum_{j=2}^{\infty} \frac{j^{4/(n+2)}}{2^j} \le \frac{1}{2} \sum_{j=2}^{\infty} \frac{j(j-1)}{2^{j-2}} = 8 \ (n \ge 0).$$

Now Lemma 3.3 implies

$$\sum_{k=1}^{j} \frac{1}{|z_k(u)|} \le \beta(P) e^{s(P)} + \frac{|u_1|}{|u_0|} \left(1 + 4e^{s(P)}\right) + \sqrt{2}\beta(P) \sum_{k=1}^{j} \frac{1}{(k+1)^{2/(n+2)}}$$
$$= C_0(P) + \sqrt{2}\beta(P) \sum_{k=1}^{j} \frac{1}{(k+1)^{2/(2+n)}} \ \ (j = 1, 2, \ldots),$$

as claimed. ∎

4. Applications of Theorem 1.1

Again $u(z)$ is a solution of equation (1.1). Everywhere in this section we assume that $u(0) \neq 0$ and for the brevity let

$$\hat{\beta} = \sqrt{2}\beta(P) \text{ and } \gamma = \frac{2}{n+2}.$$

Since $|z_k(u)| \leq |z_{k+1}(u)|$, Theorem 1.1 implies that

$$j|z_j(u)|^{-1} \leq C_0(P) + \hat{\beta} \sum_{k=1}^{j} \frac{1}{(k+1)^\gamma} \quad (j = 1, 2, \ldots).$$

But

$$\sum_{k=1}^{j} (k+1)^{-\gamma} \leq \int_1^{j+1} \frac{dx}{x^\gamma} = \frac{(1+j)^{1-\gamma} - 1}{1-\gamma} \quad (0 < \gamma < 1).$$

Denote by $\nu(f, a)$ $(a > 0)$ the counting function of the zeros of f in the circle $|z| \leq a$. We thus get

Corollary 4.1. *Let $u(0) \neq 0$ and $n > 0$. Then with the notation*

$$\eta_j(u) := \frac{j}{C_0(P) + \hat{\beta}\, \frac{(1+j)^{1-\gamma}-1}{1-\gamma}},$$

the inequality $|z_j(u)| \geq \eta_j(u)$ holds and thus $\nu(u, a) \leq j$ for any positive $a \leq \eta_j(u)$ $(j = 1, 2, \ldots)$.

Furthermore, put

$$\vartheta_1 = C_0(P) + \frac{\hat{\beta}}{2^\gamma} \text{ and } \vartheta_k = \frac{\hat{\beta}}{(k+1)^\gamma} \quad (k = 2, 3, \ldots).$$

Theorem 1.1 and Lemma 1.2.1 from [13] yield

Corollary 4.2. *Let $\phi(t)$ $(0 \leq t < \infty)$ be a continuous convex scalar-valued function, such that $\phi(0) = 0$. Then*

$$\sum_{k=1}^{j} \phi(|z_k(u)|^{-1}) \leq \sum_{k=1}^{j} \phi(\vartheta_k) \quad (j = 1, 2, \ldots).$$

In particular, for any $p \geq 1$ and $j = 2, 3, \ldots$, we have

$$\sum_{k=1}^{j} \frac{1}{|z_k(u)|^p} \leq \sum_{k=1}^{j} \vartheta_k^p$$

and therefore

$$\sum_{k=1}^{\infty} \frac{1}{|z_k(u)|^p} < \infty,$$

provided that $p > n + 1/2$.

In addition, making use Theorem 1.1 and [13, Lemma 1.2.2], we obtain our next result.

COROLLARY 4.3. *Let* $\Phi(t_1, t_2, \ldots, t_j)$ *be a function defined on domain Introduce a scalar-valued function* $\Phi(t_1, t_2, \ldots, t_j)$ *with an integer* j *defined on the domain*

$$-\infty < t_j \leq t_{j-1} \ldots \leq t_2 \leq t_1 < \infty$$

and satisfying the condition

$$\frac{\partial\Phi}{\partial t_1} > \frac{\partial\Phi}{\partial t_2} > \ldots > \frac{\partial\Phi}{\partial t_j} > 0 \text{ for } t_1 > t_2 > \ldots > t_j > -\infty.$$

Then

$$\Phi(\frac{1}{|z_1(u)|}, \ldots, \frac{1}{|z_j(u)|}) \leq \Phi(\vartheta_1, \ldots, \vartheta_j).$$

In particular, let $\{d_k\}_{k=1}^{\infty}$ be a decreasing sequence of positive numbers with $d_1 = 1$. Then the previous corollary and Theorem 1.1 yield the inequality

$$\sum_{k=1}^{j} \frac{d_k}{|z_k(u)|} \leq C_0(P) + \hat{\beta} \sum_{k=1}^{j} \frac{d_k}{(k+1)^{\gamma}} \quad (j = 1, 2, \ldots).$$

Finally we improve Corollary 4.1 in the case $j = 1$. Besides, $n \geq 0$. Due to [13, Theorem 5.12.1], we can write

$$\inf_j |z_j(u)| \geq 1/(1 + \max_j |a_j|),$$

where a_j $(j = 1, 2, \ldots)$ are the Taylor coefficients at zero of $v(z) = u(z)/u(0)$. We have $|a_j| \leq M_v(1)$. Now Corollary 2.2 implies

$$\inf_j |z_j(u)| \geq \frac{1}{(1 + \frac{|u_1|}{|u_0|})e^{2s(P)}}. \tag{4.1}$$

The just obtained result gives us a bound for the zero-free domain.

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