GLASNIK MATEMATIČKI
Vol. 50(70)(2015), 207 – 221

MINIMAL SURFACES IN $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ GEOMETRY

Zlatko Erjavec
University of Zagreb, Croatia

Abstract. In this paper some geometric properties of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry are considered, the minimal surface equation is derived and fundamental examples of minimal surfaces are given.

1. Introduction

The $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry is one of eight homogeneous Thurston 3-geometries

$$E^3, S^3, H^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, \widetilde{\mathrm{SL}(2,\mathbb{R})}, \mathrm{Nil}, \mathrm{Sol}.$$

The Riemannian manifold (M, g) is called homogeneous if for any $x, y \in M$ there exists an isometry $\Phi : M \to M$ such that $y = \Phi(x)$. The two- and three-dimensional homogeneous geometries are discussed in detail in [16].

In 1997 Emil Molnár proposed (see [9]) a projective spherical model of homogeneous geometries as a unified geometrical model. He believed that this model could be a starting point for possible attack on the Thurston conjecture. He also introduced the hyperboloid model of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry and determined the corresponding metric tensor. The hyperboloid model of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry is described in details in [3,11,12] where geodesics, the fibre translation group and translation curves (with corresponding spheres) are given. Generally, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry, because of its specificity, represents a rich area for future investigation. Recently, there are several papers that discuss ball packing and regular prism tilings in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry (see [13, 14, 17]).

2010 *Mathematics Subject Classification.* 53A40.
Key words and phrases. $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry, minimal surfaces.

Main goal of this paper is to determine the minimal surface equation and fundamental examples of minimal surfaces using the hyperboloid model of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry.

Minimal surfaces in real special linear group $\mathrm{SL}(2,\mathbb{R})$ are already studied in [7] by Kokubu. He classified CMC rotational surfaces and minimal conoids using different metric, and he also gave two examples of minimal surfaces in $\mathrm{SL}(2,\mathbb{R})$ geometry. Minimal surfaces in other twisted product homogeneous geometries, Nil and Sol, are examined in [2, 4, 5, 8]. Considering the number and types of minimal surfaces, results obtained in this article are comparable with results given in aforementioned papers.

This paper is organized as follows. In Section 2, a short description of the hyperboloid model of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry is given. In Section 3, we give some geometric properties of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry. Riemannian curvature, sectional curvature and Ricci curvature are computed explicitly. In Section 4, the minimal surface equation in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry is derived and corresponding examples are determined. Finally, further examples of minimal surfaces in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ are given and illustrated in Section 5.

2. Preliminaries

In this section we shortly recall the hyperboloid model of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry, introduced by E. Molnár in [9] and described in [11] and [12].

The idea is to start with the collineation group which acts on projective 3-space $\mathcal{P}^3(\mathbb{R})$ and projective sphere $\mathcal{PS}^3(\mathbb{R})$ and preserves a hyperboloid polarity, i.e. a scalar product of signature $(--++)$. Using the one-sheeted hyperboloid solid

$$\mathcal{H}: \ -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 < 0,$$

with an appropriate choice of a subgroup of the collineation group of $\mathcal{H}$ as an isometry group, the universal covering space $\widetilde{\mathcal{H}}$ of our hyperboloid $\mathcal{H}$ will give us the so-called hyperboloid model of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry.

In the Molnár's approach, one start with the one parameter group of matrices

$$\begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix} \tag{2.1}$$

which acts on $\mathcal{P}^3(R)$ and leaves the polarity of signature $(--++)$ and the hyperboloid solid $\mathcal{H}$ invariant. By right action of this group on the point

$(x^0; x^1; x^2; x^3)$ we obtain its orbit

$$\begin{aligned}(x^0\cos\varphi - x^1\sin\varphi; x^0\sin\varphi + x^1\cos\varphi;\\ x^2\cos\varphi + x^3\sin\varphi; -x^2\sin\varphi + x^3\cos\varphi)\end{aligned} \tag{2.2}$$

which is the unique line (fibre) through the given point. This action is called *fibre translation* and φ is called fibre coordinate.

By usual inhomogeneous E^3 coordinates $x = \frac{x^1}{x^0}, y = \frac{x^2}{x^0}, z = \frac{x^3}{x^0}$, $x^0 \neq 0$ the fibre (2.2) is given by

$$(1, x, y, z) \mapsto (1, \frac{x+\tan\varphi}{1 - x\cdot\tan\varphi}, \frac{y + z\cdot\tan\varphi}{1 - x\cdot\tan\varphi}, \frac{z - y\cdot\tan\varphi}{1 - x\cdot\tan\varphi}), \tag{2.3}$$

where $\varphi \neq \frac{\pi}{2} + k\pi$.

The mentioned subgroup of collineations that acts transitively on the points of $\widetilde{\mathcal{H}}$ and maps the origin $E_0(1;0;0;0)$ onto $X(x^0;x^1;x^2;x^3)$ is represented by the matrix

$$\mathbf{T} : (t_i^j) := \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^2 & x^3 & x^0 & x^1 \\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix} \tag{2.4}$$

Therefore by using pull-back transform on the base differential forms, the Riemannian metric is obtained. In $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, the metric is given by

$$(ds)^2 = (dr)^2 + \cosh^2 r \sinh^2 r (d\vartheta)^2 + \big((d\varphi) + \sinh^2 r (d\vartheta)\big)^2, \tag{2.5}$$

where (r, ϑ) are polar coordinates of the intersection point of a fiber and the hyperbolic base plane and φ is a fiber coordinate. One can easily see that the metric is invariant under rotations about a fiber through the origin and translations along fibers.

Therefore, the symmetric metric tensor field g is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2 r \cosh 2r & \sinh^2 r \\ 0 & \sinh^2 r & 1 \end{pmatrix}. \tag{2.6}$$

The Euclidean coordinates, corresponding to the hyperboloid coordinates (r, ϑ, φ), are given by

$$\begin{aligned} x &= \tan\varphi, \\ y &= \tanh r \cdot \frac{\cos(\vartheta - \varphi)}{\cos\varphi}, \\ z &= \tanh r \cdot \frac{\sin(\vartheta - \varphi)}{\cos\varphi}, \end{aligned} \tag{2.7}$$

where $r \in [0,\infty)$, $\vartheta \in [-\pi,\pi)$ and $\varphi \in (-\frac{\pi}{2},\frac{\pi}{2})$ with extension to $\mathbb{R}$ for the universal covering. This formulas are important for later visualization of surfaces in E^3.

3. Some geometric properties of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry

In [9] the fibre translation rule in spatial projective coordinates in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry (see rule (2.3) here), the right action of the collineation group and the metrics in Nil and Sol geometry was given, but it wasn't given an explicit formula for the multiplication law which would correspond to the similar laws in Nil and Sol geometry. This law is important because it allows determination of an orthonormal basis which corresponds to the metric (2.5).

Therefore, by comparing the right action of the collineation groups and the well known multiplication laws in Nil and Sol geometry

$$\begin{aligned}
&\mathrm{Nil}\ldots \qquad (x,y,z)*(a,b,c) = \big(x+a, y+b, z+c+\frac{1}{2}(bx-ay)\big)\\
&\mathrm{Sol}\ldots \qquad (x,y,z)*(a,b,c) = (x+e^{-z}a, y+e^{z}b, z+c)
\end{aligned}$$

we determine multiplication law in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ using hyperboloid coordinates.

If we assume that the origin $(1,0,0,0)$ in homogeneous coordinates is an identity for multiplication, then we can reconstruct the pull-back transform of the basis differential forms and the right action of transformation group simultaneous. We obtain

$$\begin{aligned}
&(1,\tilde{r},\tilde{\vartheta},\tilde{\varphi})\begin{pmatrix} 1 & r & \vartheta & \varphi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{\sinh 2r} & -\tanh r \\ 0 & 0 & 0 & 1 \end{pmatrix}\\
&\qquad = (1, r+\tilde{r}, \vartheta + \frac{2\tilde{\vartheta}}{\sinh 2r}, \varphi + \tilde{\varphi} - \tilde{\vartheta}\tanh r).
\end{aligned} \tag{3.1}$$

and

$$(0,dr,d\vartheta,d\varphi)\begin{pmatrix} 1 & -r & -\frac{1}{2}\vartheta\sinh 2r & -\varphi-\vartheta\sinh^2 r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sinh 2r & \sinh^2 r \\ 0 & 0 & 0 & 1 \end{pmatrix} = (0, d\bar{r}, d\bar{\vartheta}, d\bar{\varphi}),$$

Therefore, the multiplication law is defined by

$$(r,\vartheta,\varphi)*(\tilde{r},\tilde{\vartheta},\tilde{\varphi}) := (r+\tilde{r}, \vartheta + \frac{2\tilde{\vartheta}}{\sinh 2r}, \varphi + \tilde{\varphi} - \tilde{\vartheta}\tanh r). \tag{3.2}$$

The right identity is $(0,0,0)$ and the right inverse of (r,ϑ,φ) is

$$(-r, -\cosh r\sinh r\cdot\vartheta, -\varphi - \sinh^2 r\cdot\vartheta).$$

Furthermore, from the right action of the collineation group we can find (just read it from the rows of matrix in (3.1)) the vector fields which form an orthonormal basis $\varepsilon = (e_1, e_2, e_3)$ of tangential space in an arbitrary point of the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ (except the origin).

$$e_1 = \frac{\partial}{\partial r}, \quad e_2 = \frac{2}{\sinh 2r}\frac{\partial}{\partial \vartheta} - \tanh r \frac{\partial}{\partial \varphi}, \quad e_3 = \frac{\partial}{\partial \varphi}. \tag{3.3}$$

One can easily check that $g(e_i, e_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta.

The dual co-frame field $\theta = (\theta^1, \theta^2, \theta^3)$ associated to $\varepsilon = (e_1, e_2, e_3)$, satisfying the condition $\theta^i(e_j) = \delta_{ij}$, is given by

$$\theta^1 = dr, \quad \theta^2 = \frac{1}{2}\sinh 2r\; d\vartheta, \quad \theta^3 = \sinh^2 r\; d\vartheta + d\varphi. \tag{3.4}$$

It is obvious that $ds^2 = \sum_{i=1}^{3} (\theta^i)^2$ (see (2.5)).

Remark 3.1. The orthonormal base could be find in another way, noting the product (3.2) as $X_1 * X_2 = X_1 + X_2 + L(X_1)\; X_2$ i.e.

$$\begin{pmatrix} r \\ \vartheta \\ \varphi \end{pmatrix} * \begin{pmatrix} \tilde{r} \\ \tilde{\vartheta} \\ \tilde{\varphi} \end{pmatrix} = \begin{pmatrix} r \\ \vartheta \\ \varphi \end{pmatrix} + \begin{pmatrix} \tilde{r} \\ \tilde{\vartheta} \\ \tilde{\varphi} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{\sinh 2r} - 1 & 0 \\ 0 & -\tanh r & 0 \end{pmatrix} \begin{pmatrix} \tilde{r} \\ \tilde{\vartheta} \\ \tilde{\varphi} \end{pmatrix}.$$

The columns of the matrix

$$I + L(X_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sinh 2r} & 0 \\ 0 & -\tanh r & 1 \end{pmatrix}$$

gives us the vectors of the orthonormal frame.

The Cristoffel symbols Γ^k_{ij} and the Levi-Civita connection ∇ are explicitly given by

$$\begin{aligned}
\Gamma^1_{ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2}\sinh 2r(1 + 4\sinh^2 r) & -\frac{1}{2}\sinh 2r \\ 0 & -\frac{1}{2}\sinh 2r & 0 \end{pmatrix}, \\
\Gamma^2_{ij} &= \begin{pmatrix} 0 & \frac{2(1+3\sinh^2 r)}{\sinh 2r} & \frac{2}{\sinh 2r} \\ \frac{2(1+3\sinh^2 r)}{\sinh 2r} & 0 & 0 \\ \frac{2}{\sinh 2r} & 0 & 0 \end{pmatrix}, \\
\Gamma^3_{ij} &= \begin{pmatrix} 0 & -2\sinh^2 r \tanh r & -\tanh r \\ -2\sinh^2 r \tanh r & 0 & 0 \\ -\tanh r & 0 & 0 \end{pmatrix}
\end{aligned} \tag{3.5}$$

and

$$
\begin{array}{llll}
 & \nabla_{e_1} e_1 = 0 & \nabla_{e_1} e_2 = -e_3 & \nabla_{e_1} e_3 = e_2 \\
(3.6) & \nabla_{e_2} e_1 = 2\coth 2r\ e_2 + e_3 & \nabla_{e_2} e_2 = -2\coth 2r\ e_1 & \nabla_{e_2} e_3 = -e_1 \\
 & \nabla_{e_3} e_1 = e_2 & \nabla_{e_3} e_2 = -e_1 & \nabla_{e_3} e_3 = 0.
\end{array}
$$

Hence we have the commutators

$$
\begin{aligned}
&[e_1, e_2] = -2\coth 2r\ e_2 - 2\ e_3, \\
&[e_i, e_i] = [e_1, e_3] = [e_2, e_3] = 0 \quad \forall i \in \{1, 2, 3\}.
\end{aligned} \tag{3.7}
$$

Knowing the metric (2.5), we can determine the Riemannian curvature, the Ricci curvature and the sectional curvature of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry.

The components of the Riemannian curvature R^s_{ijk} are expressed by

$$
R^s_{ijk} = \Gamma^l_{ik}\Gamma^s_{jl} - \Gamma^l_{jk}\Gamma^s_{il} + \partial_j \Gamma^s_{ik} - \partial_i \Gamma^s_{jk}. \tag{3.8}
$$

By direct calculations we get 22 non-vanishing components of the Riemannian curvature tensor:

$$
\begin{aligned}
&R^1_{313} = R^2_{323} = R^3_{131} = -R^1_{133} = -R^2_{233} = -R^3_{311} = 1, \\
&R^2_{211} = -R^2_{121} = 7, \quad R^3_{121} = -R^3_{211} = 8\sinh^2 r, \\
&R^3_{232} = -R^3_{322} = \cosh 2r \sinh^2 r, \\
&R^1_{312} = R^1_{213} = R^2_{322} = R^3_{233} = -R^1_{132} \\
&\qquad = -R^1_{123} = -R^2_{232} = -R^3_{323} = \sinh^2 r, \\
&R^1_{122} = -R^1_{212} = (4 + 3\cosh 2r)\sinh^2 r.
\end{aligned} \tag{3.9}
$$

The Ricci curvature tensor, defined as the trace of the Riemannian curvature tensor

$$
Ric_{ij} := R^l_{ilj}, \tag{3.10}
$$

is given by

$$
Ric = \begin{pmatrix} -6 & 0 & 0 \\ 0 & -2(2+\cosh 2r)\sinh^2 r & 2\sinh^2 r \\ 0 & 2\sinh^2 r & 2 \end{pmatrix}. \tag{3.11}
$$

The scalar curvature is defined as the contraction of the Ricci tensor

$$
K := Ric_{ij} g^{ij} \tag{3.12}
$$

and it follows

$$
K = -10. \tag{3.13}
$$

The sectional curvatures with respect to the plane generated by ∂_i, ∂_j, defined by the formula

$$
K_{ij} := \frac{R_{ijij}}{g_{ii}g_{jj} - (g_{ij})^2} \tag{3.14}
$$

are given by

$$K_{12} = -3 - 4\operatorname{sech} 2r, \qquad K_{23} = 1, \qquad K_{31} = 1. \tag{3.15}$$

4. Minimal surface equation in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry

Let S be an immersed surface in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ given as the graph of the function $\varphi = f(r,\vartheta)$. Hence, the position is described by $\mathbb{X}(r,\vartheta) = (r,\vartheta,f(r,\vartheta))$ and the tangent vectors $\mathbb{X}_r = \frac{\partial \mathbb{X}}{\partial r}$ and $\mathbb{X}_\vartheta = \frac{\partial \mathbb{X}}{\partial \vartheta}$ in terms of the orthonormal frame $\varepsilon = (e_1, e_2, e_3)$ are described by

$$\mathbb{X}_r = \frac{\partial}{\partial r} + f_r \frac{\partial}{\partial \varphi} = e_1 + U\ e_3 \tag{4.1}$$

$$\mathbb{X}_\vartheta = \frac{\partial}{\partial \vartheta} + f_\vartheta \frac{\partial}{\partial \varphi} = Z\ e_2 + V\ e_3 \tag{4.2}$$

where the functions U, V, Z are defined by

$$U := f_r, \qquad V := \sinh^2 r + f_\vartheta \qquad \text{and} \qquad Z := \sinh r \cosh r. \tag{4.3}$$

The coefficients of the first fundamental form

$$I = E\ dr^2 + 2F\ drd\vartheta + G\ d\vartheta^2$$

of S, defined as $E = g(\mathbb{X}_r, \mathbb{X}_r),\ F = g(\mathbb{X}_r, \mathbb{X}_\vartheta),\ G = g(\mathbb{X}_\vartheta, \mathbb{X}_\vartheta)$ are given by

$$E = 1 + U^2, \qquad F = U\ V, \qquad G = V^2 + Z^2. \tag{4.4}$$

Requesting that a unit normal vector fulfill the conditions $g(\mathbb{X}_r, \mathbb{N}) = g(\mathbb{X}_\vartheta, \mathbb{N}) = 0$ and $g(\mathbb{N}, \mathbb{N}) = 1$, we obtain

$$\mathbb{N} = \frac{-UZ \cdot e_1 - V \cdot e_2 + Z \cdot e_3}{W}, \quad W := \sqrt{V^2 + Z^2 + U^2 Z^2}. \tag{4.5}$$

The coefficients of the second fundamental form

$$II = L\ dr^2 + 2M\ drd\vartheta + N\ d\vartheta^2,$$

given by $L = g(\nabla_{\mathbb{X}_r} \mathbb{X}_r, \mathbb{N})$, $M = g(\nabla_{\mathbb{X}_\vartheta} \mathbb{X}_r, \mathbb{N}),\ N = g(\nabla_{\mathbb{X}_\vartheta} \mathbb{X}_\vartheta, \mathbb{N})$ are

$$\begin{aligned} L &= \frac{1}{W}(Z f_{rr} - 2UV), \\ M &= \frac{1}{W}(Z^2 + Z^2U^2 - V^2 - V\cosh 2r + Z f_{r\vartheta}), \\ N &= \frac{1}{W}(2UVZ^2 + UZ^2\cosh 2r + Z f_{\vartheta\vartheta}). \end{aligned} \tag{4.6}$$

A surface is *minimal* if its mean curvature, computed by the formula

$$\mathbf{H} = \frac{EN - 2FM + GL}{2W^2},$$

vanishes identically.

Similarly, a surface is called *flat* if the Gaussian curvature

$$\mathbf{K} = \frac{LN - M^2}{EG - F^2}$$

is equal to zero.

The differential equation $\mathbf{H} = 0$ for a surface given as the graph of the function $\varphi = f(r, \vartheta)$ is called the *minimal surface equation.*

Using the expressions (4.4) and (4.6) we obtain the following theorem.

THEOREM 4.1. *The minimal surface equation in* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *geometry is given by*

$$\begin{aligned} Z(V^2 + Z^2)f_{rr} - 2UVZ(f_{r\vartheta} + Z) + Z(1 + U^2)f_{\vartheta\vartheta} \\ + \sqrt{1 + 4Z^2}\, U(2V^2 + Z^2 + U^2Z^2) = 0 \end{aligned} \tag{4.7}$$

where U, V, Z are the functions defined in (4.3).

PROPOSITION 4.2. *The Gaussian curvature of a surface* $\mathbb{X}(r, \vartheta) = (r, \vartheta, f(r, \vartheta))$ *in* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *geometry is given by*

$$\begin{aligned} K = \frac{1}{W^4} \big(&(Zf_{rr} - 2UV)(Zf_{\vartheta\vartheta} + 2UVZ^2 + UZ^2 \cosh 2r) \\ &-(Zf_{r\vartheta} + Z^2 - V^2 + U^2Z^2 - V \cosh 2r)^2 \big)\,, \end{aligned}$$

where U, V, Z are functions defined in (4.3) and W in (4.5).

EXAMPLE 4.3. It is easy to observe that the surface $\mathbb{X}(r, \vartheta) = (r, \vartheta, f(r, \vartheta))$, where $f(r, \vartheta) = c \in \mathbb{R}$ is a solution of the equation (4.7).

Note that this surface, which is an analogue of a plane parallel to the base yz-plane in Euclidean 3-space $\mathbf{E^3}$ (see (2.7)), is not flat in $\mathrm{SL}(2, \mathbb{R})$, moreover $K = \frac{-4\sinh^4 r}{(\cosh^2 2r)} < 0$.

REMARK 4.4. All figures represent surfaces in $\mathrm{SL}(2, R)$ space because it is not possible to illustrate surfaces in covering space $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

EXAMPLE 4.5. Now, we solve the minimal surface equation (4.7) for the function

$$f(r, \vartheta) = v(\vartheta).$$

In this case the equation (4.7) becomes $v_{\vartheta\vartheta} = 0$. Hence we get

$$v(\vartheta) = a\vartheta + b\,, \qquad a, b \in \mathbb{R}. \tag{4.8}$$

Therefore, we proved the following proposition.

PROPOSITION 4.6. *The surface in* $\mathrm{SL}(2, \mathbb{R})$ *space, given as the graph of a function $f(r, \vartheta) = a\vartheta + b$, is minimal for arbitrary $a, b \in \mathbb{R}$.*

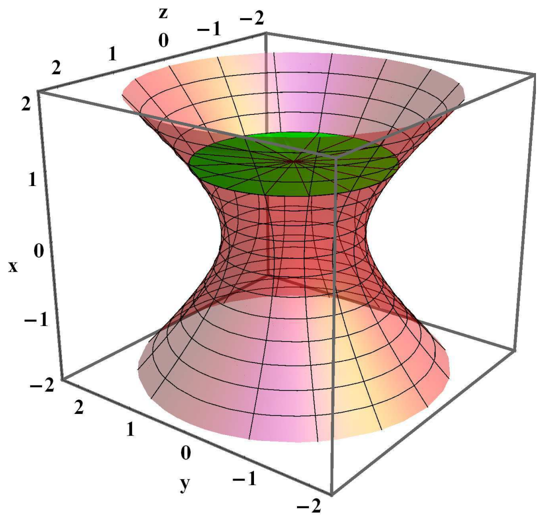

FIGURE 1. $\mathbb{X}(r,\vartheta)=(r,\vartheta,\pi/4)$

The Gaussian curvature is again negative,

$$K=-\frac{(3-4a+4a^2+(8a-4)\cosh 2r+\cosh 4r)^2}{16(a^2+(2a+\cosh 2r)\sinh^2 r)^2},$$

and although it is not obvious, for $a=0$ this expression becomes the expression for Gaussian curvature from Example 1.

Note that $(0,\vartheta,v(\vartheta))*(r,0,0)=(r,\vartheta,v(\vartheta))$, so the obtained surface is a $*$-right-translation surface in the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry and may be thought as an analogue of helicoid in $\mathbf{E^3}$. It is also worth to mention that we could obtain the same surface assuming $f=g(\cot\vartheta)$ and using approach from [1].

It is also worth to mention that the particular cases $v(\vartheta)=\vartheta+\frac{\pi}{2}$ and $v(\vartheta)=\vartheta+\pi$ give us the Euclidean planes $y=0$ and $z=0$, respectively.

EXAMPLE 4.7. Here we solve the minimal surface equation (4.7) for the function

$$f(r,\vartheta)=u(r).$$

In this case the equation (4.7) becomes

$$u_{rr}+4\left(\frac{\cosh^2 2r-\sinh^2 r}{\sinh 4r}\right)u_r+\coth r(u_r)^3=0,$$

where $u_r=\frac{\partial u}{\partial r}$ and $u_{rr}=\frac{\partial^2 u}{\partial r^2}$.

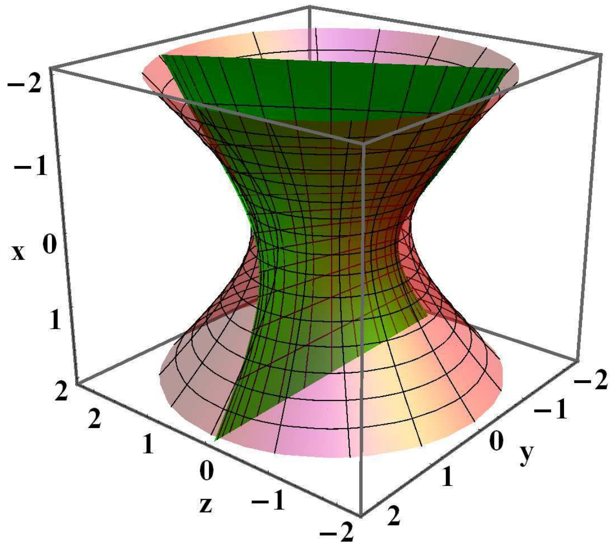

FIGURE 2. $\mathbb{X}(r,\vartheta) = (r,\vartheta,2\vartheta)$

Reducing the order $(u_r = y(r))$ we obtain the following Bernoulli differential equation

$$y_r + 4\left(\frac{\cosh^2 2r - \sinh^2 r}{\sinh 4r}\right) y + \coth r\; y^3 = 0 \tag{4.9}$$

which after substitution $z = y^{-2}$ becomes the linear equation

$$z_r - 8\left(\frac{\cosh^2 2r - \sinh^2 r}{\sinh 4r}\right) z = -2\coth r.$$

After solving this equation, we have

$$z(r) = \coth r \sinh 4r\left(-\frac{1}{4} + c\tanh^2 2r\right), \qquad c \in \mathbb{R},$$

and hence

$$y(r) = \frac{2\sqrt{\tanh r}}{\sqrt{\sinh 4r(4c\tanh^2 2r - 1)}}.$$

Therefore,

$$u(r) = 2\int \frac{2\sqrt{\tanh r}}{\sqrt{\sinh 4r(4c\tanh^2 2r - 1)}}dr, \qquad c \in \mathbb{R}. \tag{4.10}$$

If the condition

$$c > \frac{1}{4\tanh^2 2r}$$

is fulfilled (which is true e.g. for $c = 0,5$ and $r > 0,5$, or $c = 7$ and $r > 0,1$), then we obtain a real integrand and using *Mathematica* we can find numerical solution. On the other hand, we have complex solution e.g. for $c = \frac{1}{4}$ we even obtain an explicit (but complex) solution,

$$\begin{aligned} u(r) \quad &= \quad \frac{1+e^{2r}}{2\sqrt{(1+e^{4r})(-1-\operatorname{sech} 2r)}}\Big(\sqrt{2}\operatorname{arsinh} e^{2r} - 2\ln(1+e^{2r}) + \\ (4.11) \quad + \quad & 2\ln(-\sqrt{2}+\sqrt{2}e^{2r} - 2\sqrt{1+e^{4r}}) + \sqrt{2}\left(2r - \ln(1+\sqrt{1+e^{4r}})\right)\Big). \end{aligned}$$

PROPOSITION 4.8. *The surface given as the graph of the function*

$$f(r) = 2\int \frac{2\sqrt{\tanh r}}{\sqrt{\sinh 4r(4c\tanh^2 2r - 1)}}dr, \quad c \in \mathbb{R} \tag{4.12}$$

is minimal for $c \in \mathbb{R}, r \in [0,\infty)$ *and* $c > \frac{1}{\tanh^2 2r}$.

This surface is an example of axially symmetric minimal surface, so it is an analogue of a catenoid in $\mathbf{E^3}$.

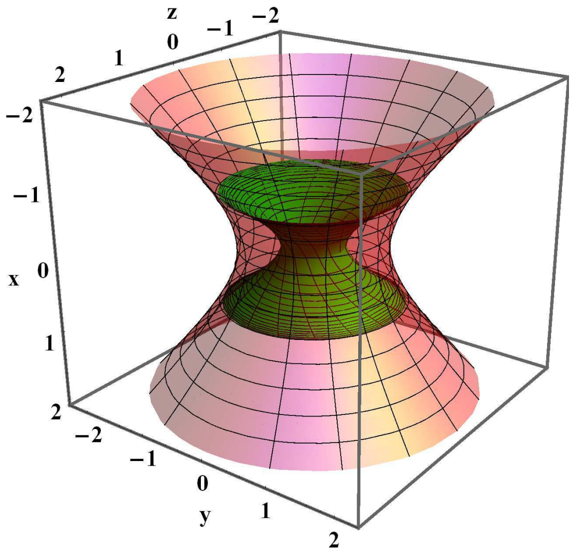

FIGURE 3. $\mathbb{X}(r,\vartheta) = (r, \vartheta, 2\int \frac{2\sqrt{\tanh r}}{\sqrt{\sinh 4r(2\tanh^2 2r - 1)}}dr)$

5. SOME OTHER MINIMAL SURFACES IN $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ GEOMETRY

In this section we consider a surface of the form $\mathbb{X}(u,v) = (r(u), \vartheta(u), v)$ which is on the first look an analogue of a cylindric surface in $\mathbf{E^3}$ but having in mind equations (2.7), it follows that it is an analogue of a conoidal surface. The tangential vectors, in terms of the orthonormal frame are described by

$$\mathbb{X}_u = r_u e_1 + \frac{1}{2}\sinh 2r\ \vartheta_u e_2 + \sinh^2 r \vartheta_u e_3, \qquad \mathbb{X}_v = e_3. \tag{5.1}$$

The first fundamental form, the unit normal and the second fundamental form are given by

$$I = (r_u^2 + \sinh^2 r \cosh 2r\ \vartheta_u^2)\ du^2 + \sinh^2 r\ \vartheta_u\ dudv + dv^2, \tag{5.2}$$

$$N = \frac{1}{\mathbf{w}}(\frac{1}{2}\sinh 2r\ \vartheta_u\ e_1 - r_u\ e_2),\ \text{where } \mathbf{w} = \sqrt{r_u^2 + \sinh^2 r \cosh^2 r\ \vartheta_u^2} \tag{5.3}$$

$$\begin{aligned} II = \frac{1}{\mathbf{w}}\Bigg(\bigg(&\frac{1}{2}\sinh 2r(r_{uu}\vartheta_u - r_u\vartheta_{uu}) - \frac{1}{4}\sinh^2 2r(1 + 4\sinh^2 r)\vartheta_u^3 \\ &- 2(1 + 3\sinh^2 r)\ r_u^2\vartheta_u\bigg)\ du^2 - 2\mathbf{w}\ dudv\Bigg). \end{aligned} \tag{5.4}$$

Proposition 5.1. *The mean curvature of a surface of the form* $\mathbb{X}(u, v) = (r(u), \vartheta(u), v)$ *in* $\mathrm{SL}(2, \mathbb{R})$ *is given by*

$$H = \frac{1}{2\mathbf{w}^3}\left(\frac{1}{2}\sinh 2r\ (r_{uu}\vartheta_u - r_u\vartheta_{uu}) - \cosh 2r\,\vartheta_u(2r_u^2 + \frac{1}{4}\sinh^2 2r\ \vartheta_u^2)\right), \tag{5.5}$$

where $\mathbf{w}$ *is given in (5.3). The Gauss curvature is constant and equal to* -1.

Proof. It follows directly from definitions of the mean and Gauss curvature and the formulas (5.2), (5.3) and (5.4). ∎

Example 5.2. Obviously from (5.5) it follows that surface $\mathbb{X}(u, v) = (r(u), c, v)$, for an arbitrary $r(u)$ and $c \in \mathbb{R}$ is minimal in $\mathrm{SL}(2, \mathbb{R})$ space. This surface is an analogue of hyperbolic paraboloid in $\mathbf{E^3}$.

Note that if $r = const.$, then from (5.5) follows $\vartheta = const.$ So, possible circular cylindric minimal surface in $\mathrm{SL}(2, \mathbb{R})$ space degenerate in a fibre.

Example 5.3. If we suppose that $r(u) = u$, then from (5.1) follows that surface $\mathbb{X}(u, v) = (r(u), \vartheta(u), v)$ will be minimal if $\vartheta(u)$ fulfills the following differential equation

$$\vartheta_{uu} + 4\coth 2r\ \vartheta_u^2 + \frac{1}{4}\sinh 4r\ \vartheta_u^3 = 0. \tag{5.6}$$

After reducing the order, we have Bernoullie equation which is similar to the equation (4.9). After solving, we obtain

$$\begin{aligned} \vartheta(u) = \sqrt{2}\arctan\left(\frac{\sqrt{2}\cosh 2u}{\sqrt{-1 - 4c + (4c - 1)\cosh 4u}}\right) \\ \frac{\sqrt{-1 - 4c + (4c - 1)\cosh 4u}\sinh 2u}{\sqrt{16c\sinh^4 2u - \sinh^2 4u}} \end{aligned} \tag{5.7}$$

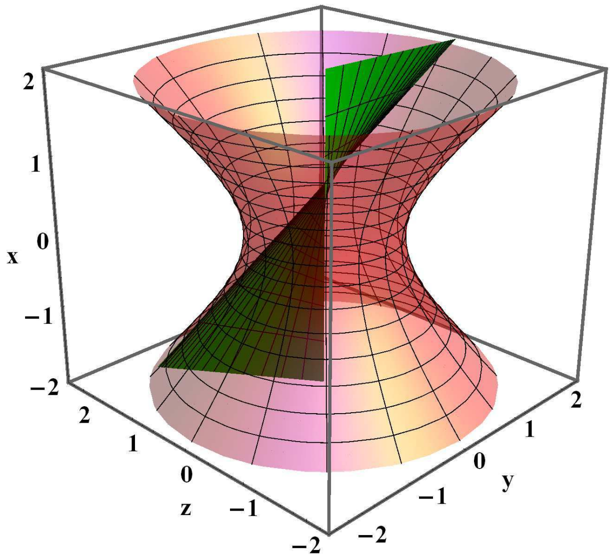

FIGURE 4. $\mathbb{X}(u,v) = (u, \frac{\pi}{3}, v)$

which is real again for

$$c > \frac{1}{\tanh^2 2r}.$$

PROPOSITION 5.4. *The surface* $\mathbb{X}(u,v) = (u, \vartheta(u), v)$, *where*

$$\vartheta(u) = \sqrt{2}\arctan\left(\frac{\sqrt{2}\cosh 2u}{\sqrt{-1-4c+(4c-1)\cosh 4u}}\right)$$
$$\frac{\sqrt{-1-4c+(4c-1)\cosh 4u}\sinh 2u}{\sqrt{16c\sinh^4 2u - \sinh^2 4u}}$$

for $c \in \mathbb{R}, u \in [0,\infty)$ *and* $c > \frac{1}{\tanh^2 2r}$ *is minimal in* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *space.*

This surface is an analogue of a conoid in $\mathbf{E}^3$.

REMARK 5.5. Analogically to the previous case, if we suppose that $\vartheta(u) = u$, then from Proposition 5.1 follows the surface $\mathbb{X}(u,v) = (r(u), \vartheta(u), v)$ will be minimal if $r(u)$ fulfills the following differential equation,

$$r_{uu} - 4\coth 2r\ r_u^2 = \frac{1}{4}\sinh 4r. \tag{5.8}$$

After reducing the order ($r_u = y(u)$), this equation becomes Riccati equation,

$$y_u = 4\coth 2r\ y^2 + \frac{1}{4}\sinh 4r. \tag{5.9}$$

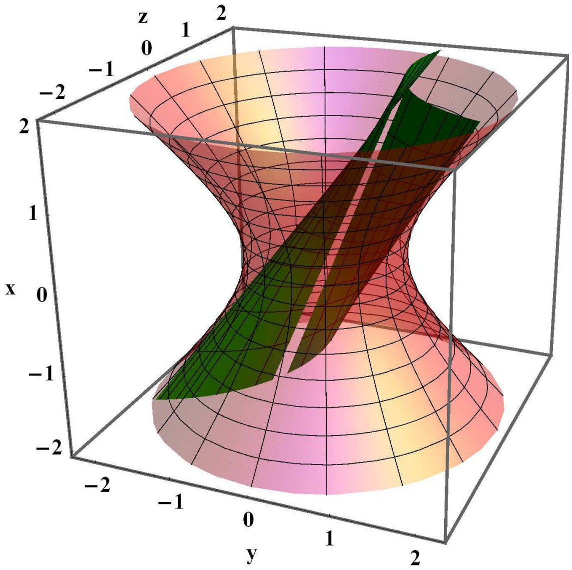

FIGURE 5. $\mathbb{X}(u,v) = \left(u, \sqrt{2}\arctan\left(\frac{\sqrt{2}\cosh 2u}{\sqrt{\cosh 4u - 3}}\right)\frac{\sqrt{\cosh 4u-3}\,\sinh 2u}{\sqrt{8\sinh^4 2u - \sinh^2 4u}}, v\right)$

for which, at this moment, we don't know is there a solution or not.

ACKNOWLEDGEMENTS.

The author is thankful to the referee for valuable suggestions. The author was supported by the Ministry of Science, Education and Sports (grant 016-0372785-0892).

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Faculty of Organization and Informatics
University of Zagreb
Pavlinska 2, HR-42000 Varaždin
Croatia
E-mail: `zlatko.erjavec@foi.hr`

Received: 27.2.2014.
Revised: 24.6.2014.