

STRONG CONVERGENCE FOR m -PAIRWISE NEGATIVELY QUADRANT DEPENDENT RANDOM VARIABLES

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ABSTRACT. Complete convergence and the Marcinkiewicz-Zygmund strong law of large numbers for sequences of m -pairwise negatively quadrant dependent (m -PNQD) random variables is studied in this paper. The results obtained extend and improve the corresponding theorems of Choi and Sung ([4]) and Hu et al. ([9]). A version of the Kolmogorov strong law of large numbers for sequences of m -PNQD random variables is also obtained.

1. INTRODUCTION

The following concept of negatively quadrant dependent random variables was introduced by Lehmann ([12]).

DEFINITION 1.1. *Two random variables X and Y are said to be negatively quadrant dependent (NQD) if*

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y) \text{ for all } x \text{ and } y \in \mathbb{R}.$$

A finite or infinite sequence of random variables is said to be *pairwise NQD* (PNQD) if every two random variables in the sequence are NQD. It is well known and easy to show that random variables X and Y are NQD if and only if

$$P(X > x, Y > y) \leq P(X > x)P(Y > y) \text{ for all } x \text{ and } y \in \mathbb{R}.$$

In many stochastic models, an independence assumption among the random variables in the model is not a reasonable assumption since they may be “repelling” in the sense that increases in any of the random variables often

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correspond to decreases in the others. Thus the assumption of PNQD is often more suitable than the classical assumption of independence.

A major survey article concerning a general “theory of negative dependence” was prepared by Pemantle ([20]). That article discussed the relationship between various definitions of “negative dependence”, outlines some possible directions that the theory can take, and provides some interesting conjectures.

We now define a more general dependence structure which contains PNQD as a special case. This dependence structure was introduced by Anh ([2]) for a finite set of random variables. Definition 1.2 is the same as that of Anh ([2]) except that it is for a sequence of random variables rather than for a finite set.

DEFINITION 1.2. *Let $m \geq 1$ be a fixed integer. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be m -PNQD if for all positive integers j and k with $|j - k| \geq m$, X_j and X_k are NQD.*

Clearly, PNQD is the special case $m = 1$ of the concept of m -PNQD. Indeed, if $\{X_n, n \geq 1\}$ is m -PNQD for some $m \geq 1$, then $\{X_n, n \geq 1\}$ is m' -PNQD for all $m' > m$.

Li et al. ([13]) showed that for every sequence of continuous distribution functions $\{F_n, n \geq 1\}$, a sequence of PNQD random variables $\{X_n, n \geq 1\}$ can be constructed such that the distribution function of X_n is F_n , $n \geq 1$ and such that for all $k \geq 1$, $\{X_n, n \geq k\}$ is not a sequence of independent random variables.

We now provide two examples of sequences of m -PNQD random variables.

EXAMPLE 1.3. Let $\{Y_n, n \geq 1\}$ be a PNQD sequence of random variables and let $m \geq 2$. For $n \geq 1$, let $r \geq 1$ be such that $(r - 1)m + 1 \leq n \leq rm$ and let $X_n = Y_r$. Then $\{X_n, n \geq 1\}$ is a sequence of m -PNQD random variables.

EXAMPLE 1.4. Let $\{Y_n, n \geq 1\}$ be a PNQD sequence of random variables, let $m \geq 2$, and let $\{Z_{ij}, 1 \leq j \leq m - 1, i \geq 1\}$ be an array of random variables such that for all $i \neq i'$ and all $1 \leq j, j' \leq m - 1$, Y_i and $Z_{i'j'}$ are NQD and Z_{ij} and $Z_{i'j'}$ are NQD. For $n \geq 1$, let $r \geq 1$ be such that $(r - 1)m + 1 \leq n \leq rm$ and let

$$X_n = \begin{cases} Y_n, & \text{if } n = (r - 1)m + 1, \\ Z_{r, n - (r - 1)m - 1}, & \text{if } (r - 1)m + 2 \leq n \leq rm. \end{cases}$$

Then $\{X_n, n \geq 1\}$ is a sequence of m -PNQD random variables.

Years after the appearance of Lehmann ([12]), a large literature of investigation concerning the convergence properties of PNQD random variables has emerged. We refer the reader to Matula ([16]), Qi ([21]), Patterson and Taylor ([19]), Kim and Kim ([10]), Kim and Kim ([11]), Taylor et al. ([25]), Wu ([27]), Ordóñez Cabrera and Volodin ([18]), Li et al. ([13]), Gan and Chen ([6]), Li and Yang ([14]), Meng and Lin ([17]), Gerasimov ([7]), Xing ([31]),

Wu and Jiang ([28]), Wu and Guan ([29]), Wu and Wang ([30]), and Sung ([22]) among others.

The purpose of this article is to investigate complete convergence and the strong law of large numbers (SLLN) for sequences of m -PNQD random variables.

The concept of the complete convergence was introduced by Hsu and Robbins ([8]). A sequence of random variables $\{U_n, n \geq 1\}$ is said to *converge completely* to a constant θ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \varepsilon) < \infty \text{ for all } \varepsilon > 0.$$

In view of the Borel-Cantelli lemma, the above result implies that $U_n \rightarrow \theta$ almost surely (a.s.). Therefore, complete convergence is an important tool which is often used in establishing a.s. convergence of sums of random variables.

Choi and Sung ([4]) studied the SLLN for sequences of pairwise independent random variables and obtained the following Marcinkiewicz-Zygmund type result.

THEOREM 1.5. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise independent random variables. Suppose that $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable X (this technical definition is given in the next section). If*

$$(1.1) \quad E(|X|^r (\log^+ |X|)^2) < \infty \text{ for some } 1 < r < 2,$$

then

$$(1.2) \quad \lim_{n \rightarrow \infty} n^{-1/r} \sum_{k=1}^n (X_k - EX_k) = 0 \quad a.s.$$

Hu et al. ([9]) also studied the SLLN for sequences of PNQD random variables and obtained the following Marcinkiewicz-Zygmund type result.

THEOREM 1.6. *Let $\{X_n, n \geq 1\}$ be a sequence of PNQD random variables with $EX_n = 0$ for all $n \geq 1$. Suppose that $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable X . If there exist constants $1 \leq r < 2$ and $\alpha > r + 1$ such that*

$$(1.3) \quad E(|X|^r (\log^+ |X|)^\alpha) < \infty,$$

then

$$(1.4) \quad \lim_{n \rightarrow \infty} n^{-1/r} \sum_{k=1}^n X_k = 0 \quad a.s.$$

If we assume that $EX_k = 0$ for all $k \geq 1$ in Theorem 1.5, then (1.2) is exactly (1.4). To some extent, Theorem 1.6 extends Theorem 1.5 from the

pairwise independent case to the PNQD case. However, when $r > 1$, the moment condition (1.3) is not optimal since it is stronger than (1.1).

In the current work, we obtain some results on complete convergence and the Marcinkiewicz-Zygmund SLLN for sequences of m -PNQD random variables which improve and extend Theorems 1.5 and 1.6 to the m -PNQD case. We also establish the Kolmogorov SLLN for sequences of m -PNQD random variables. We point out that the method used in this article differs from those in Choi and Sung ([4]) and Hu et al. ([9]).

Throughout this paper, the symbol C is used to represent positive constants whose values may change from one place to another.

2. PRELIMINARIES

To prove our main results, we need the following technical lemmas.

LEMMA 2.1 ([12]). *Let X and Y be NQD random variables. Then*

- (i) $\text{Cov}(X, Y) \leq 0$,
- (ii) *If f and g are Borel functions which are both monotone increasing (or both monotone decreasing), then $f(X)$ and $g(Y)$ are NQD.*

LEMMA 2.2 ([27]). *Let $\{X_n, n \geq 1\}$ be a sequence of PNQD random variables with mean zero and $EX_n^2 < \infty$, $n \geq 1$, and let $T_j(k) = \sum_{i=j+1}^{j+k} X_i$, $j \geq 0$, $k \geq 1$. Then*

$$E\left(\max_{1 \leq k \leq n} (T_j(k))^2\right) \leq C(\log^2 n) \sum_{i=j+1}^{j+n} EX_i^2, \quad n \geq 1$$

where $\log n = \log_e \max\{e, n\}$, $n \geq 1$.

LEMMA 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of m -PNQD random variables with mean zero and $EX_n^2 < \infty$, $n \geq 1$. Then there exists a positive constant C depending only on m such that*

$$E\left(\max_{1 \leq j \leq n} \left(\sum_{k=1}^j X_k\right)^2\right) \leq C(\log^2 2n) \sum_{k=1}^n EX_k^2, \quad n \geq 1.$$

where $\log n = \log_e \max\{e, n\}$, $n \geq 1$.

PROOF. It is clear that exists a suitably chosen C satisfying the inequality in the conclusion for $1 \leq n \leq m$ since this set of values of n is finite. Therefore, we only need to consider the case $n > m$. Given any $1 \leq j \leq n$, let $\nu = [\frac{j}{m}]$ and $\tau = [\frac{n}{m}]$. Define

$$V_k^{(j)} = \begin{cases} X_k, & \text{if } 1 \leq k \leq j \\ 0, & \text{if } k > j \end{cases} \quad \text{for } 1 \leq j \leq n$$

and

$$T_{jl} = \sum_{i=0}^{\nu} V_{mi+l}^{(j)} \text{ for } 1 \leq l \leq m.$$

Clearly $\sum_{k=1}^j X_k = \sum_{l=1}^m T_{jl} = \sum_{l=1}^m \sum_{i=0}^{\nu} V_{mi+l}^{(j)}$. Hence by the C_r -inequality and Lemma 2.2,

$$\begin{aligned} & E\left(\max_{1 \leq j \leq n} \left(\sum_{k=1}^j X_k\right)^2\right) \\ & \leq E\left(\max_{1 \leq j \leq n} \left(\sum_{l=1}^m T_{jl}\right)^2\right) \leq m \sum_{l=1}^m E\left(\max_{1 \leq j \leq n} T_{jl}^2\right) \\ & \leq m \sum_{l=1}^m E\left(\max_{0 \leq \nu \leq \tau} \left(\sum_{i=0}^{\nu} V_{mi+l}^{(j)}\right)^2\right) \leq C m \sum_{l=1}^m (\log^2 2\tau) \sum_{i=0}^{\tau} E(V_{mi+l}^{(n)})^2 \\ & = C(\log^2 2n) \sum_{l=1}^m \sum_{i=0}^{\tau} E(V_{mi+l}^{(n)})^2 = C(\log^2 2n) \sum_{k=1}^n EX_k^2. \end{aligned}$$

The proof is completed. □

LEMMA 2.4 ([3]). Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative random variables with $\text{Var}(X_n) < \infty, n \geq 1$ and let $\{f(n), n \geq 1\}$ be a sequence such that $0 < f(n) \uparrow \infty$. Assume that

- (i) $\sup_{n \geq 1} f^{-1}(n) \sum_{k=1}^n EX_k < \infty$;
- (ii) there is a double sequence ρ_{ij} of nonnegative real numbers such that

$$\text{Var}\left(\sum_{k=1}^n X_k\right) \leq \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} \text{ for each } n \geq 1;$$

- (iii) $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\rho_{ij}}{f^2(i \vee j)} < \infty$, where $i \vee j = \max(i, j), i \geq 1, j \geq 1$.

Then

$$f^{-1}(n) \sum_{k=1}^n (X_k - EX_k) \rightarrow 0 \text{ a. s. as } n \rightarrow \infty.$$

The following concept of stochastic domination is a generalization of the concept of identical distributions. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *stochastically dominated* by a random variable X if there exists a constant $C > 0$ such that

$$(2.1) \quad \sup_{n \geq 1} P(|X_n| > x) \leq CP(|X| > x) \text{ for all } x \geq 0.$$

Stochastic dominance is of course automatic with $X = X_1$ and $C = 1$ if $\{X_n, n \geq 1\}$ is a sequence of identically distributed random variables. It

follows from Lemma 5.2.2 of Taylor ([24], p. 123) (or Lemma 3 of [26]) that stochastic dominance of a sequence of random variables can be accomplished by the random variables in the sequence having a bounded absolute r -th moment ($r > 0$). Specifically, if $\sup_{n \geq 1} E|X_n|^r < \infty$ for some $r > 0$, then there exists a random variable X with $E|X|^s < \infty$ for all $0 < s < r$ such that (2.1) holds with $C = 1$. (The provision that $r > 1$ in Lemma 5.2.2 of Taylor ([24], p. 123]) (or [26, Lemma 3]) is not needed as was pointed out by Adler et al. ([1]).

LEMMA 2.5. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X . Then there exists a constant C such that, for all $q > 0$ and $x > 0$,*

- (i) $E(|X_k|^q I(|X_k| \leq x)) \leq C\{E(|X|^q I(|X| \leq x)) + x^q P(|X| > x)\},$
- (ii) $E(|X_k|^q I(|X_k| > x)) \leq CE(|X|^q I(|X| > x)).$

This lemma can be easily proved by using integration by parts. We omit the details.

3. MAIN RESULTS

Now we present our main results and their proofs.

THEOREM 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of m -PNQD random variables which is stochastically dominated by a random variable X . If (1.1) holds, then for all $\varepsilon > 0$,*

$$(3.1) \quad \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} \left| \sum_{k=1}^j (X_k - EX_k) \right| > n^{1/r} \varepsilon\right) < \infty.$$

PROOF. For fixed $n \geq 1$, let

$$Y_{nk} = -n^{1/r} I(X_k < -n^{1/r}) + X_k I(|X_k| \leq n^{1/r}) + n^{1/r} I(X_k > n^{1/r}),$$

$$Z_{nk} = (X_k + n^{1/r}) I(X_k < -n^{1/r}) + (X_k - n^{1/r}) I(X_k > n^{1/r}).$$

Then $Y_{nk} + Z_{nk} = X_k$, and it follows by the definition of m -PNQD and Lemma 2.1(ii) that $\{Y_{nk}, k \geq 1\}$ is a sequence of m -PNQD random variables. Then

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} \left| \sum_{k=1}^j (X_k - EX_k) \right| > n^{1/r} \varepsilon\right) \\ & \leq \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} \left| \sum_{k=1}^j (Z_{nk} - EZ_{nk}) \right| > n^{1/r} \varepsilon/2\right) \\ & \quad + \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} \left| \sum_{k=1}^j (Y_{nk} - EY_{nk}) \right| > n^{1/r} \varepsilon/2\right) \\ & =: I_1 + I_2. \end{aligned}$$

To prove (3.1), it only needs to be shown that $I_1 < \infty$ and $I_2 < \infty$. Note that $|Z_{nk}| \leq |X_k|I(|X_k| > n^{1/r})$. By the Markov inequality, Lemma 2.5(ii) and (1.1), we have

$$\begin{aligned}
 I_1 &\leq C \sum_{n=1}^{\infty} n^{-1-1/r} \sum_{k=1}^n E|Z_{nk} - EZ_{nk}| \\
 &\leq C \sum_{n=1}^{\infty} n^{-1-1/r} \sum_{k=1}^n E(|X_k|I(|X_k| > n^{1/r})) \\
 &\leq C \sum_{n=1}^{\infty} n^{-1/r} \sum_{s=n}^{\infty} E(|X|I(s < |X|^r \leq s+1)) \\
 &= C \sum_{s=1}^{\infty} E(|X|I(s < |X|^r \leq s+1)) \sum_{n=1}^s n^{-1/r} \\
 &\leq C \sum_{s=1}^{\infty} s^{1-1/r} E(|X|I(s < |X|^r \leq s+1)) \leq CE|X|^r < \infty.
 \end{aligned}$$

Next we prove $I_2 < \infty$. By the Markov inequality, Lemma 2.3, and Lemma 2.5(i), we have

$$\begin{aligned}
 I_2 &\leq C \sum_{n=1}^{\infty} n^{-1-2/r} (\log^2 n) \sum_{k=1}^n EY_{nk}^2 \\
 &\leq C \sum_{n=1}^{\infty} n^{-1-2/r} (\log^2 n) \sum_{k=1}^n E(X_k^2 I(|X_k| \leq n^{1/r})) \\
 &\quad + C \sum_{n=1}^{\infty} n^{-1} (\log^2 n) \sum_{k=1}^n P(|X_k| > n^{1/r}) \\
 &\leq C \sum_{n=1}^{\infty} n^{-2/r} (\log^2 n) E(X^2 I(|X| \leq n^{1/r})) \\
 &\quad + C \sum_{n=1}^{\infty} (\log^2 n) P(|X| > n^{1/r}) \\
 &=: I_3 + I_4.
 \end{aligned}$$

Since the function $\log^2 x$ is slowly varying at ∞ , by applying Theorem VIII.9.1 of Feller ([5, p. 281]), we have

$$\begin{aligned} I_3 &= C \sum_{n=1}^{\infty} n^{-2/r} (\log^2 n) \sum_{s=1}^n E(X^2 I(s-1 < |X|^r \leq s)) \\ &= C \sum_{s=1}^{\infty} E(X^2 I(s-1 < |X|^r \leq s)) \sum_{n=s}^{\infty} n^{-2/r} \log^2 n \\ &\leq C \sum_{s=1}^{\infty} s^{1-2/r} (\log^2 s) E(X^2 I(s-1 < |X|^r \leq s)) \\ &\leq CE(|X|^r (\log^+ |X|)^2) < \infty \end{aligned}$$

and

$$\begin{aligned} I_4 &\leq C \sum_{n=1}^{\infty} n^{-1/r} (\log^2 n) E(|X| I(|X| > n^{1/r})) \\ &= C \sum_{n=1}^{\infty} n^{-1/r} (\log^2 n) \sum_{s=n}^{\infty} E(|X| I(s < |X|^r \leq s+1)) \\ &= C \sum_{s=1}^{\infty} E(|X| I(s < |X|^r \leq s+1)) \sum_{n=1}^s n^{-1/r} \log^2 n \\ &\leq C \sum_{s=1}^{\infty} s^{1-1/r} (\log^2 s) E(|X| I(s < |X|^r \leq s+1)) \\ &\leq CE(|X|^r (\log^+ |X|)^2) < \infty. \end{aligned}$$

The proof is completed. \square

The following corollary provides a Marcinkiewicz-Zygmund type SLLN for a sequence of m -PNQD random variables. Corollary 3.2 is indeed a weaker version of Corollary 2.2 in [2] (see Remark 3.7 below).

COROLLARY 3.2. *Under the conditions of Theorem 3.1, (1.2) holds.*

PROOF. Let $S_j = \sum_{k=1}^j (X_k - EX_k)$, $j \geq 1$. From (3.1), we have for arbitrary $\varepsilon > 0$,

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} |S_j| > n^{\frac{1}{r}} \varepsilon\right) \\ &= \sum_{i=0}^{\infty} \sum_{n=2^i}^{2^{i+1}-1} n^{-1} P\left(\max_{1 \leq j \leq n} |S_j| > n^{\frac{1}{r}} \varepsilon\right) \\ &\geq \frac{1}{2} \sum_{i=1}^{\infty} P\left(\max_{1 \leq j \leq 2^i} |S_j| > 2^{\frac{i+1}{r}} \varepsilon\right). \end{aligned}$$

Then by the Borel-Cantelli Lemma and the arbitrariness of $\varepsilon > 0$,

$$\lim_{i \rightarrow \infty} 2^{-\frac{i+1}{r}} \max_{1 \leq j \leq 2^i} |S_j| = 0 \quad \text{a.s.}$$

For every positive integer n , there exists a positive integer i_0 such that $2^{i_0-1} \leq n < 2^{i_0}$. Then $i_0 \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\begin{aligned} n^{-\frac{1}{r}} |S_n| &\leq \max_{2^{i_0-1} \leq j < 2^{i_0}} j^{-\frac{1}{r}} |S_j| \\ &\leq 2^{\frac{2}{r}} 2^{-\frac{i_0+1}{r}} \max_{1 \leq j < 2^{i_0}} |S_j| \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty. \end{aligned}$$

The proof is completed. □

REMARK 3.3. Since pairwise independence implies m -PNQD and since (3.1) implies (1.2) as has been shown in the proof of Corollary 3.2, Theorem 3.1 and Corollary 3.2 improve and extend Theorem 1.5. In addition, since PNQD implies m -PNQD and since (1.1) is weaker than (1.3) when $r > 1$, Theorem 3.1 and Corollary 3.2 also improve and extend Theorem 1.6.

REMARK 3.4. Martikainen ([15]) showed for a sequence of pairwise i.i.d. random variables that (1.2) holds under a weaker moment condition than that in Theorem 1.5. He proved that the condition (1.1) can be improved to $E|X_1|^r (\log |X_1|)^\tau < \infty$ for $1 < r < 2$ and $\tau > \max\{0, 4r - 6\}$. Sung ([23]) improved the result of Martikainen ([15]) under the much weaker moment condition $E|X_1|^r (\log \log |X_1|)^{2(r-1)} < \infty$ where $1 < r < 2$. We find that the truncation method used in Sung ([23]) is not suitable for obtaining PNQD or m -PNQD results. Therefore, it is an open problem as to whether or not Theorem 3.1 and Corollary 3.2 hold with Sung's ([23]) weaker moment condition.

The next theorem provides a Kolmogorov type SLLN for a sequence of m -PNQD random variables.

THEOREM 3.5. *Let $\{X_n, n \geq 1\}$ be a sequence of m -PNQD random variables which is stochastically dominated by a random variable X . If $E(|X| \log^+ |X|) < \infty$, then*

$$(3.2) \quad \lim_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n (X_k - EX_k) \rightarrow 0 \quad \text{a.s.}$$

PROOF. Without loss of generality, we may assume that $EX_k = 0, k \geq 1$. For $k \geq 1$, let

$$\begin{aligned} Y_k &= -kI(X_k < -k) + X_kI(|X_k| \leq k) + kI(X_k > k), \\ Y_k^* &= (X_k + k)I(X_k < -k) + (X_k - k)I(X_k > k). \end{aligned}$$

Then it follows by the definition of m -PNQD and Lemma 2.1(ii) that $\{Y_k, k \geq 1\}$ is a sequence of m -PNQD random variables. From $E|X| < \infty$, we have

$$\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(|X_n| > n) \leq C \sum_{n=1}^{\infty} P(|X| > n) \leq CE|X| < \infty.$$

Thus by the Borel-Cantelli lemma, we have

$$(3.3) \quad n^{-1} \sum_{k=1}^n (X_k - Y_k) \rightarrow 0 \quad \text{a.s. } n \rightarrow \infty.$$

Note that $|Y_k^*| \leq |X_k|I(|X_k| > k)$, $k \geq 1$. By $EX_k = 0$, $k \geq 1$ and Lemma 2.5(ii), we have

$$\begin{aligned} \sum_{k=1}^{\infty} k^{-1} |EY_k| &= \sum_{k=1}^{\infty} k^{-1} |EY_k^*| \leq C \sum_{k=1}^{\infty} k^{-1} E(|X|I(|X| > k)) \\ &= C \sum_{k=1}^{\infty} k^{-1} \sum_{j=k}^{\infty} E(|X|I(j < |X| \leq j+1)) \\ &= C \sum_{j=1}^{\infty} E(|X|I(j < |X| \leq j+1)) \sum_{k=1}^j k^{-1} \\ &\leq C \sum_{j=1}^{\infty} (\log j) E(|X|I(j < |X| \leq j+1)) \\ &\leq CE(|X| \log^+ |X|) < \infty. \end{aligned}$$

Hence by the Kronecker lemma, we have

$$n^{-1} \sum_{k=1}^n |EY_k| \rightarrow 0 \quad \text{a.s. } n \rightarrow \infty,$$

which implies

$$(3.4) \quad n^{-1} \sum_{k=1}^n EY_k \rightarrow 0 \quad \text{a.s. } n \rightarrow \infty.$$

Next, we will prove that

$$(3.5) \quad n^{-1} \sum_{k=1}^n (Y_k - EY_k) \rightarrow 0 \quad \text{a.s. } n \rightarrow \infty.$$

Let $Z_k = (Y_k - EY_k)^+$, $R_k = (Y_k - EY_k)^-$, $k \geq 1$, and $f(n) = n$, $n \geq 1$, where $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. By the definition of m -PNQD and Lemma 2.1(ii), $\{Z_k, k \geq 1\}$ and $\{R_k, k \geq 1\}$ are each m -PNQD sequences.

Now by Lemma 2.5(i), we have

$$\begin{aligned}
 \sup_{n \geq 1} n^{-1} \sum_{k=1}^n EZ_k &\leq \sup_{n \geq 1} n^{-1} \sum_{k=1}^n E|Y_k - EY_k| \\
 (3.6) \quad &\leq C \sup_{n \geq 1} n^{-1} \sum_{k=1}^n E(|X|I(|X| \leq k)) + C \sum_{k=1}^n P(|X| > k) \\
 &\leq CE|X| < \infty.
 \end{aligned}$$

Define

$$\rho_{ij} = \begin{cases} \text{Var}(Z_i), & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases}$$

Given any $1 \leq k \leq n$, take $\tau = \lfloor \frac{n}{m} \rfloor$. Let

$$Z_k^* = \begin{cases} Z_k, & \text{if } 1 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

Clearly $\sum_{k=1}^n Z_k = \sum_{j=1}^m \sum_{i=0}^{\tau} Z_{mi+j}^*$. Therefore,

$$\begin{aligned}
 \text{Var} \left(\sum_{k=1}^n Z_k \right) &= \text{Var} \left(\sum_{j=1}^m \sum_{i=0}^{\tau} Z_{mi+j}^* \right) \\
 &= \sum_{j=1}^m \text{Var} \left(\sum_{i=0}^{\tau} Z_{mi+j}^* \right) + 2 \sum_{1 \leq l < s \leq m} \text{Cov} \left(\sum_{i=0}^{\tau} Z_{mi+l}^*, \sum_{t=0}^{\tau} Z_{mt+s}^* \right) \\
 &=: A + B.
 \end{aligned}$$

For A , by the definition of m -PNQD and Lemma 2.1(i), we have

$$\begin{aligned}
 A &= \sum_{j=1}^m \left(\sum_{i=0}^{\tau} \text{Var}(Z_{mi+j}^*) + 2 \sum_{0 \leq u < v \leq \tau} \text{Cov}(Z_{mu+j}^*, Z_{mv+j}^*) \right) \\
 &\leq \sum_{j=1}^m \sum_{i=0}^{\tau} \text{Var}(Z_{mi+j}^*) = \sum_{k=1}^n \text{Var}(Z_k).
 \end{aligned}$$

For B , by a property of covariance, the definition of m -PNQD, and Lemma 2.1(i), we have

$$B = 2 \sum_{1 \leq l < s \leq m} \sum_{i=0}^{\tau} \sum_{t=0}^{\tau} \text{Cov}(Z_{mi+l}^*, Z_{mt+s}^*) \leq 0.$$

Hence we get

$$(3.7) \quad \text{Var} \left(\sum_{k=1}^n Z_k \right) \leq \sum_{k=1}^n \text{Var}(Z_k) = \sum_{k=1}^n \sum_{j=1}^n \rho_{kj}, \quad n \geq 1.$$

Note that

$$\begin{aligned}
 & \sum_{k=1}^{\infty} k^{-2} E(X^2 I(|X| \leq k)) \\
 &= \sum_{k=1}^{\infty} k^{-2} \sum_{s=1}^k E(X^2 I(s-1 < |X| \leq s)) \\
 &= \sum_{s=1}^{\infty} E(X^2 I(s-1 < |X| \leq s)) \sum_{k=s}^{\infty} k^{-2} \\
 &\leq \sum_{s=1}^{\infty} s^{-1} E(X^2 I(s-1 < |X| \leq s)) \leq E|X|.
 \end{aligned}$$

Then by the definitions of ρ_{ij} and $f(n)$, we have by Lemma 2.5(i) that

$$\begin{aligned}
 (3.8) \quad \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{\rho_{kj}}{f^2(k \vee j)} &= \sum_{k=1}^{\infty} \frac{\text{Var}(Z_k)}{k^2} \leq \sum_{k=1}^{\infty} \frac{EY_k^2}{k^2} \\
 &= \sum_{k=1}^{\infty} k^{-2} (E(X_k^2 I(|X_k| \leq k)) + k^2 P(|X_k| > k)) \\
 &\leq C \sum_{k=1}^{\infty} k^{-2} E(X^2 I(|X| \leq k)) + C \sum_{k=1}^{\infty} P(|X| > k) \\
 &\leq 2E|X| < \infty.
 \end{aligned}$$

In view of (3.6), (3.7) and (3.8), we get by Lemma 2.4 that

$$(3.9) \quad n^{-1} \sum_{k=1}^n (Z_k - EZ_k) \rightarrow 0 \quad \text{a.s. } n \rightarrow \infty.$$

An argument similar to that for establishing (3.9) gives

$$(3.10) \quad n^{-1} \sum_{k=1}^n (R_k - ER_k) \rightarrow 0 \quad \text{a.s. } n \rightarrow \infty.$$

By (3.9) and (3.10), we get (3.5). Combining (3.3), (3.4) and (3.5) yields (3.2). The proof is completed. \square

REMARK 3.6. If $m = 1$, then the moment condition $E(|X_1| \log^+ |X_1|) < \infty$ can be weakened to $E|X_1| < \infty$ for a sequence $\{X_n, n \geq 1\}$ of PNQD identically distributed random variables as was proved by Matula ([16]). Matula's ([16]) result was extended by Li et al. ([13]) to the case of weighted sums of PNQD identically distributed random variables as a consequence of a much more general result they obtained in the same article. It remains open as to whether or not the moment condition $E(|X| \log^+ |X|) < \infty$ can be weakened to $E|X| < \infty$ in Theorem 3.5.

REMARK 3.7. Some discussion is in order comparing our work with that of Anh ([2]). Theorem 2.1 of Anh ([2]) is a Marcinkiewicz-Zygmund type SLLN for a sequence of *blockwise* and m -PNQD random variables with respect to a sequence of blocks $\{B_k, k \geq 1\}$. The blocks are disjoint finite sets of positive integers and each block is an index set for a finite set of random variables. As we mentioned prior to Definition 1.2 above, it was Anh ([2]) who introduced the m -PNQD structure for a finite set of random variables. Anh ([2]) did not impose any dependence conditions between random variables with indices in different blocks. In our Theorem 3.1 and Corollary 3.2, we establish, respectively, a complete convergence theorem and a Marcinkiewicz-Zygmund type SLLN under a moment condition which is identical to that of Theorem 2.1 and Corollary 2.2 of Anh ([2]) when $1 < r < 2$. Since a sequence of m -PNQD random variables is automatically a sequence of blockwise and m -PNQD random variables with respect to the sequence of blocks $\{[2^{k-1}, 2^k), k \geq 1\}$, our Corollary 3.2 indeed follows immediately from Corollary 2.2 of Anh ([2]) but their proofs are entirely different. On the other hand, the Kolmogorov type SLLN provided by our Theorem 3.5 has a moment condition which is strictly weaker than that of Corollary 2.2 of Anh ([2]) when $r = 1$.

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