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On derivation operators with respect to the Duhamel convolution in the space of analytic functions

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Abstract. We solve the Leibniz operator equation with respect to the Duhamel convolution in the class of linear continuous operators that act in the space of analytic functions in an arbitrary starlike domain of the complex plane with respect to the origin.

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1. Introduction

Let G be an arbitrary domain of the complex plane. Let $\mathcal{H}(G)$ denote the space of all analytic functions in G equipped with the topology of compact convergence [1]. By $\mathcal{L}(\mathcal{H}(G))$ we denote the set of all linear continuous operators on the space $\mathcal{H}(G)$.

Let G be a starlike domain with respect to the origin. By $\mathcal{J}$ we shall denote the integration operator in the space $\mathcal{H}(G)$ defined by the formula $(\mathcal{J}f)(z) = \int\limits_0^z f(t)dt$. The operator $\mathcal{J}$ acts linearly and continuously in the space $\mathcal{H}(G)$. The Duhamel convolution of functions $f, g \in \mathcal{H}(G)$, is defined by the rule

$$(f * g)(z) = \int\limits_0^z f(z-t)g(t)dt,$$

$z \in G$. The Duhamel convolution $* : \mathcal{H}(G) \times \mathcal{H}(G) \longrightarrow \mathcal{H}(G)$ is a bilinear, commutative and associative operation. Properties of the Duhamel convolution in spaces of analytic functions in domains were studied in [2]. In particular, it was shown in [2] that the integration operator $\mathcal{J}$ is a multiplier of the Duhamel convolution, i.e., the equality $\mathcal{J}(f*g) = (\mathcal{J}f)*g$ holds for arbitrary functions $f, g \in \mathcal{H}(G)$. Herewith $\mathcal{J}f = 1 * f$ for any $f \in \mathcal{H}(G)$. This relation allows to obtain explicit representation of the commutants of the integration operator $\mathcal{J}$ in the space $\mathcal{H}(G)$. Namely, in order to have an operator $T \in \mathcal{L}(\mathcal{H}(G))$ commute with the integration operator $\mathcal{J}$ in $\mathcal{H}(G)$ it is necessary and sufficient that $Tf = D(\varphi * f)$, where $D = \frac{d}{dz}$ and φ is some function of $\mathcal{H}(G)$.

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Let $\tilde{*}$ be an arbitrary convolution on the space $\mathcal{H}(G)$, i.e., a bilinear, commutative and associative operation on $\mathcal{H}(G)$. An operator T of the class $\mathcal{L}(\mathcal{H}(G))$ is called a derivation operator with respect to convolution $\tilde{*}$ if the equality $T(f\tilde{*}g) = (Tf)\tilde{*}g + f\tilde{*}(Tg)$ holds for any functions $f, g \in \mathcal{H}(G)$. When the convolution of two functions coincides with the product of these functions, then the respective derivation equation

$$T(fg) = (Tf)g + f(Tg) \tag{1}$$

is called the Leibnitz's operator equation [3]. An operator $T \in \mathcal{L}(\mathcal{H}(G))$ satisfies (1) if and only if it can be represented in the form $T = U_\psi D$, where ψ is an arbitrary function of the space $\mathcal{H}(G)$ and U_ψ is the multiplication operator induced by ψ (see [4, 5]). All derivations operators in spaces of analytic functions of several variables were described in [6]. Note that all solutions of (1) in the classes of operators that act in spaces of functions of real variables were described in [7, 8].

The purpose of this paper is to determine all derivation operators with respect to the Duhamel convolution $*$ in the space $\mathcal{H}(G)$.

2. The main result

Let G be a starlike domain of the complex plane with respect to the origin. Assume that an operator $T \in \mathcal{L}(\mathcal{H}(G))$ is a derivation operator with respect to the Duhamel convolution. Then the following equality holds for all $z \in G$:

$$[T(f * g)](z) = [(Tf) * g](z) + [f * (Tg)](z), \tag{2}$$

for any $f, g \in \mathcal{H}(G)$. By $t(\lambda, z) = (Tf_\lambda)(z)$, where $f_\lambda(z) = e^{\lambda z}$, we denote the characteristic function of the operator T (see [9]). The function $t(\lambda, z)$ is entire with respect to λ and analytic with respect to z in G. Let $T1 = \varphi$, $\varphi \in \mathcal{H}(G)$. Since $\lambda(f_\lambda * 1)(z) = \lambda(\mathcal{J} f_\lambda)(z) = f_\lambda(z) - 1$ and $((Tf_\lambda) * 1)(z) = (\mathcal{J}(Tf_\lambda))(z) = \mathcal{J} t(\lambda, z)$, where $\lambda \in \mathbb{C}$ and $z \in G$, setting $f = f_\lambda$ and $g = 1$ in (2) we get

$$t(\lambda, z) - \lambda \mathcal{J} t(\lambda, z) = \varphi(z) + \lambda((f_\lambda * \varphi)(z)), \tag{3}$$

for any $\lambda \in \mathbb{C}$ and $z \in G$. The operator $E - \lambda\mathcal{J}$ is an isomorphism of the space $\mathcal{H}(G)$ for each $\lambda \in \mathbb{C}$ (by E we denote the identity operator of $\mathcal{L}(\mathcal{H}(G))$). Herewith the inverse operator $(E - \lambda\mathcal{J})^{-1}$ can be rewritten in the form of the sum of the Neumann series $(E - \lambda\mathcal{J})^{-1} = \sum\limits_{n=0}^{\infty} \lambda^n \mathcal{J}^n$, and obviously, this series converges pointwise in the space $\mathcal{H}(G)$. Then it follows from (3) that

$$t(\lambda, z) = \sum_{n=0}^{\infty} \lambda^n (\mathcal{J}^n \varphi)(z) + \left(\sum_{n=0}^{\infty} \lambda^{n+1} \mathcal{J}^n \right) (f_\lambda * \varphi)(z). \tag{4}$$

Applying Dirichlet's formula

$$(\mathcal{J}^n f)(z) = \int\limits_0^z \frac{(z-t)^{n-1}}{(n-1)!} f(t)dt,$$

where $f \in \mathcal{H}(G)$, $n \in \mathbb{N}$, $z \in G$, to the first term on the right-hand side of (4) we get

$$\begin{aligned}\sum_{n=0}^{\infty} \lambda^n (\mathcal{J}^n \varphi)(z) &= D\left(\sum_{n=0}^{\infty} \lambda^n (\mathcal{J}^{n+1}\varphi)(z)\right)\\ &= D\left(\int\limits_0^z \left(\sum_{n=0}^{\infty} \frac{\lambda^n (z-t)^n}{n!}\right)\varphi(t)dt\right)\\ &= D\left(\int\limits_0^z e^{\lambda(z-t)}\varphi(t)dt\right) = D(f_\lambda * \varphi)(z), \quad \lambda \in \mathbb{C}, z \in G.\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} \lambda^n (\mathcal{J}^n \varphi)(z) = D(f_\lambda * \varphi)(z). \tag{5}$$

Using (5), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \lambda^{n+1} \mathcal{J}^{n+1} f_\lambda(z) = \sum_{n=1}^{\infty} \lambda^n \mathcal{J}^n f_\lambda(z) &= \sum_{n=0}^{\infty} \lambda^n \mathcal{J}^n f_\lambda(z) - f_\lambda(z)\\ &= D(f_\lambda * f_\lambda)(z) - f_\lambda(z) = D(zf_\lambda(z)) - f_\lambda(z)\\ &= zDf_\lambda(z) = (UDf_\lambda)(z),\end{aligned}$$

where U is the operator of multiplication by the independent variable.

We now transform the second term on the right-hand side of (4). Since the Duhamel convolution is continuous and the operator $\mathcal{J}$ is a multiplier of the Duhamel convolution, we have

$$\begin{aligned}\left(\sum_{n=0}^{\infty} \lambda^{n+1} \mathcal{J}^n\right)(f_\lambda * \varphi)(z) &= D\left(\sum_{n=0}^{\infty} \lambda^{n+1} \mathcal{J}^{n+1}\right)(f_\lambda * \varphi)(z)\\ &= D\left(\left(\sum_{n=0}^{\infty} \lambda^{n+1} \mathcal{J}^{n+1} f_\lambda\right) * \varphi\right)(z)\\ &= D\left((UDf_\lambda) * \varphi\right)(z).\end{aligned}$$

Thus,

$$\begin{aligned}t(\lambda, z) &= D((f_\lambda + UDf_\lambda) * \varphi)(z) = D((DUf_\lambda) * \varphi)(z)\\ &= D(D\mathcal{J})((DUf_\lambda) * \varphi)(z) = D^2((\mathcal{J}DUf_\lambda) * \varphi)(z) = D^2((Uf_\lambda) * \varphi)(z).\end{aligned}$$

Hence, by the definition of characteristic functions of operators and using that the system of functions $\{f_\lambda : \lambda \in \mathbb{C}\}$ is complete in $\mathcal{H}(G)$, we obtain that

$$Tf = D^2((Uf) * \varphi) \tag{6}$$

for an arbitrary function $f \in \mathcal{H}(G)$. Thus, we have proved the necessary part of the following theorem.

Theorem 1. *Let G be a starlike domain of the complex plane with respect to the origin. In order to have an operator T of the class $\mathcal{L}(\mathcal{H}(G))$ as a derivation operator with respect to the Duhamel convolution it is necessary and sufficient that this operator is of the form* (6)*, where φ is a function of the space $\mathcal{H}(G)$.*

Proof. Sufficiency. Assume that $\varphi \in \mathcal{H}(G)$. Then formula (6) defines an operator T from the class $\mathcal{L}(\mathcal{H}(G))$. To verify that T is a derivation operator with respect to the Duhamel convolution, we prove that the following equality

$$D^2((U(f*g))*\varphi) = (D^2((Uf)*\varphi))*g + f*(D^2((Ug)*\varphi)) \tag{7}$$

holds for arbitrary functions $f, g \in \mathcal{H}(G)$.

Let us consider the function $F = (Uf) * g$, where $f, g \in \mathcal{H}(G)$. Since $F(0) = F'(0) = 0$, we have

$$\mathcal{J}^2 D^2((Uf)*g) = (Uf)*g. \tag{8}$$

The following equality also holds for any $f, g \in \mathcal{H}(G)$:

$$U(f*g) = (Uf)*g + f*(Ug). \tag{9}$$

Using (8) and (9) we get

$$\begin{aligned}
\left(D^2\left((Uf)*\varphi\right)\right)*g + f*\left(D^2\left((Ug)*\varphi\right)\right) \\
&= D^2\left[\left(\mathcal{J}^2 D^2\left((Uf)*\varphi\right)\right)*g + f*\left(\mathcal{J}^2 D^2\left((Ug)*\varphi\right)\right)\right] \\
&= D^2\left[((Uf)*\varphi)*g + f*((Ug)*\varphi)\right] \\
&= D^2\left[((Uf)*g + f*(Ug))*\varphi\right] \\
&= D^2\left((U(f*g))*\varphi\right),
\end{aligned}$$

for arbitrary functions $f, g \in \mathcal{H}(G)$. Thus, we conclude that (7) holds true. □

Note that the formula

$$(f \circledast g)(z) = D\left(\int_0^z f(z-t)g(t)dt\right) \tag{10}$$

also defines a convolution on the space $\mathcal{H}(G)$. It is often called the Duhamel product [10]. The operator $\mathcal{J}$ is a multiplier of the Duhamel product [2] and $f \circledast g = D(f*g)$.

Let us describe all derivation operators with respect to the Duhamel product. Assume that an operator $T \in \mathcal{L}(\mathcal{H}(G))$ is a derivation operator with respect to the Duhamel pruduct. Then the following equality holds:

$$T(f \circledast g) = (Tf) \circledast g + f \circledast (Tg) \tag{11}$$

for any $f, g \in \mathcal{H}(G)$. Let $T1 = \psi$. Since $f \circledast 1 = 1$ for any $f \in \mathcal{H}(G)$, setting $f = g = 1$ in (11) we obtain that $\psi \equiv 0$ in G.

Replacing f and g by $\mathcal{J}f$ and $\mathcal{J}g$ in (11), respectively, and using the fact that the operator $\mathcal{J}$ is a multiplier of the Duhamel product $\circledast$ we get

$$(T\mathcal{J})(f*g) = ((T\mathcal{J})f)*g + f*((T\mathcal{J})g) \tag{12}$$

for any $f, g \in \mathcal{H}(G)$. Hence, $T\mathcal{J}$ is a derivation operator with respect to the convolution $*$. Then by Theorem 1 this operator can be represented in the form $(T\mathcal{J})f = D^2((Uf)*\varphi)$, where φ is a function of $\mathcal{H}(G)$. Let $\delta_0(f) = f(0)$, $f \in \mathcal{H}(G)$. Then $f = (\mathcal{J}D)f + \delta_0(f)$, $f \in \mathcal{H}(G)$. Hence, for an arbitrary function f of $\mathcal{H}(G)$ we have

$$Tf = (T\mathcal{J})(Df) + \delta_0(f)\psi = D^2((UDf) * \varphi) = D((UDf)\circledast\varphi).$$

Thus, we have proved the necessary part of the following theorem.

Theorem 2. *Let G be a starlike domain of the complex plane with respect to the origin. In order to have an operator T of the class $\mathcal{L}(\mathcal{H}(G))$ as a derivation operator with respect to the Duhamel product $\circledast$ it is necessary and sufficient that this operator is of the form*

$$Tf = D((UDf)\circledast\varphi), \tag{13}$$

where φ is a function of the space $\mathcal{H}(G)$.

Proof. Sufficiency. Suppose $\varphi \in \mathcal{H}(G)$. Then formula (13) defines an operator T of the class $\mathcal{L}(\mathcal{H}(G))$. Let $Tf = T_1 Df$, where $T_1 = D^2((Uf) * \varphi)$. Since according to Theorem 1 the operator T_1 is a derivation operator with respect to the Duhamel convolution, we get

$$\begin{aligned}
T(f\circledast g) =& T_1\left(D^2(f*g)\right) = T_1\left(D^2((\mathcal{J}Df + \delta_0(f)) * (\mathcal{J}Dg + \delta_0(g)))\right)\\
=& T_1\left(D^2(\mathcal{J}Df) * (\mathcal{J}Dg)\right) + T_1 D^2\left(\delta_0(f) * \delta_0(g)\right) + T_1 D^2\left((\mathcal{J}Df) * \delta_0(g)\right)\\
&+ T_1 D^2\left(\delta_0(f) * (\mathcal{J}Dg)\right) = T_1(Df * Dg) + \delta_0(g)T_1(Df) + \delta_0(f)T_1(Dg)\\
=& (T_1 Df) * (Dg) + (Df) * (T_1 Dg) + \delta_0(g)T_1(Df) + \delta_0(f)T_1(Dg),
\end{aligned}$$

for any $f, g \in \mathcal{H}(G)$. On the other hand, we have

$$\begin{aligned}
(Tf)\circledast g + f\circledast(Tg) =& D(Tf * g) + D(f * Tg)\\
=& D((T_1 Df) * (\mathcal{J}Dg + \delta_0(g))) + +D((\mathcal{J}Df + \delta_0(f)) * (T_1 Dg))\\
=& D((T_1 Df) * (\mathcal{J}Dg)) + D((\mathcal{J}Df) * (T_1 Dg))\\
&+ \delta_0(g)D(T_1 Df * 1) + \delta_0(f)(1 * (T_1 Dg))\\
=& (T_1 Df) * (Dg) + (Df) * (T_1 Dg) + \delta_0(g)T_1(Df) + \delta_0(f)T_1(Dg).
\end{aligned}$$

Thus, equality (11) holds, i.e., T is a derivation operator with respect to the Duhamel product. □

In the light of the above-proved theorems, there naturally arises an interesting problem of the description of all pairs of linear continuous operators A and B on the space $\mathcal{H}(G)$ such that

$$A(f*g) = (Af) * (Bg) + (Ag) * (Bf) \tag{14}$$

for any $f, g \in \mathcal{H}(G)$. When the convolution $*$ coincides with the product, the corresponding equation (14) is called the Rubel's equation. All solutions of the Rubel's equation in the class of linear continuous operators that act in spaces of analytic functions in arbitrary simply connected domains were described in [11]. In [12], Rubel's equation was solved in the class of linear operators that act in spaces of analytic functions in domains.

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