

Uniqueness and value distribution for q -shifts of meromorphic functions*

HONG YAN XU^{1,†}, KAI LIU² AND TING BIN CAO²

¹ Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, 333 403 Jiangxi, P. R. China

² Department of Mathematics, Nanchang University, Nanchang, 330 031 Jiangxi, P. R. China

Received June 22, 2011; accepted April 27, 2012

Abstract. In this paper, we deal with value distribution for q -shift polynomials of transcendental meromorphic functions with zero order and obtain some results which improve the previous theorems given by Liu and Qi [18]. In addition, we investigate value sharing for q -shift polynomials of transcendental entire functions with zero order and obtain some results which extend the recent theorem given by Liu, Liu and Cao [17].

AMS subject classifications: 30D35, 39A50

Key words: q -shift, uniqueness, meromorphic function, zero order

1. Introduction and main results

In this paper, meromorphic function f will always mean meromorphic in the complex plane \mathbb{C} . We assume that the reader is familiar with the fundamental results and standard notations of Nevanlinna value distribution theory, such as the proximity function $m(r, f)$, the counting function $N(r, f)$, the characteristic function $T(r, f)$, the first and second main theorems, the lemma on logarithmic derivatives and so on, for details about Nevanlinna theory, see Hayman [10], Yi and Yang [24]. For a meromorphic function f , $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside a possible exceptional set of the finite logarithmic measure, $\mathbb{S}(f)$ denotes the family of all meromorphic functions α such that $T(r, \alpha) = S(r, f) = o(T(r, f))$, where $r \rightarrow \infty$ outside of a possible exceptional set of the finite logarithmic measure. For convenience, we agree that $\mathbb{S}(f)$ includes all constant functions and $\widehat{\mathbb{S}} := \mathbb{S}(f) \cup \{\infty\}$. In addition, by $S_1(r, f)$ we denote any quantity satisfying $S_1(r, f) = o(T(r, f))$ for all r on a set of logarithmic density 1.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. If for some $a \in \mathbb{C} \cup \{\infty\}$, the zeros of $f(z) - a$ and $g(z) - a$ (if $a = \infty$, zeros of $f(z) - a$ and $g(z) - a$ are the poles of $f(z)$ and $g(z)$, respectively) coincide in locations and multiplicities,

*The first author was supported by the NSF of China (11301233, 61202313, 11101201, 11461042, 11301260), the Natural Science Foundation of Jiangxi Province in China (20132BAB211001, 20151BAB201008), and the Foundation of Education Department of Jiangxi (GJJ14644) of China.

[†]Corresponding author. *Email addresses:* xhyhh@126.com (H. Y. Xu), liukai418@126.com (K. Liu), tbc@ncu.edu.cn (T. B. Cao)

we say that $f(z)$ and $g(z)$ share the value a *CM* (counting multiplicities) and if they coincide in locations only, we say that $f(z)$ and $g(z)$ share a *IM* (ignoring multiplicities).

In recent years, there has been an increasing interest in studying difference equations, the difference product and the q -difference in the complex plane \mathbb{C} , a number of papers (including [4, 7, 9, 12, 15, 16, 19, 21, 22, 25]) have focused on the uniqueness of difference analogues of Nevanlinna theory. Halburd and Korhonen [7] established a difference analogue of the Logarithmic Derivative Lemma, and then applied it to prove a number of results on meromorphic solutions of complex difference equations. Afterwards, Barnett, Halburd, Korhonen and Morgan [2] also established an analogue of the Logarithmic Derivative Lemma on q -difference operators.

Liu, Liu and Cao [17], Chen, Huang and Zheng [3], and Luo and Lin [20] studied zeros distributions of difference polynomials of meromorphic functions and obtained the following results:

Theorem 1 (see [17], Theorem 1.2). *Let f be a transcendental meromorphic function of finite order and c a nonzero complex constant. If $n \geq 6$, then the difference polynomial $f(z)^n f(z+c) - \alpha(z)$ has infinitely many zeros, where $\alpha(z) \in \mathbb{S}(f)$.*

Theorem 2 (see [17], Corollary 1.3). *Let $f(z)$ be a transcendental entire function of finite order and c a nonzero complex constant. If $f(z)$ has the Borel exceptional value 0, then $H(z) = f(z)f(z+c)$ takes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Theorem 3 (see [20], Theorem 1). *Let f be a transcendental entire function of finite order σ and c be a nonzero complex constant, Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a nonzero polynomial, where $a_0, a_1, \dots, a_n (\neq 0)$ are complex constants, and let m be the number of the distinct zeros of $P(z)$. Then for $n > m$, $P(f)f(z+c) = a(z)$ has infinitely many solutions, where $a(z) \in \mathbb{S}(f) \setminus \{0\}$.*

For the q -difference of meromorphic functions, Zhang and Korhonen [23] studied value distribution of q -difference polynomials of meromorphic functions and obtained the following result.

Theorem 4 (see [23], Theorem 4.1). *Let f be a transcendental meromorphic (resp. entire) function of zero order and q non-zero complex constant. Then for $n \geq 6$ (resp. $n \geq 2$), $f(z)^n f(qz)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

Recently, Liu and Qi [18] firstly investigated value distributions for a q -shift of the meromorphic function and obtained the following result.

Theorem 5 (see [18], Theorem 3.6). *Let f be a zero-order transcendental meromorphic function, $n \geq 6, q \in \mathbb{C} \setminus \{0\}, \eta \in \mathbb{C}$, and $R(z)$ a rational function. Then the q -shift difference polynomial $f(z)^n f(qz + \eta) - R(z)$ has infinitely many zeros.*

A natural question is what can we get about the zeros of $P(f)f(qz + \eta) = a(z)$ and $P(f)[f(qz + \eta) - f(z)] = a(z)$, where $P(f), a(z)$ are stated as in Theorem 3 and q, η are stated as in Theorem 5? Corresponding to this question, we get the following theorems which are the improvements of Theorems 4 and 5.

Theorem 6. *Let f be a zero-order transcendental meromorphic (resp. entire) function, $q \in \mathbb{C} \setminus \{0\}$, $\eta \in \mathbb{C}$. Then for $n > m+4$ (resp. $n > m$), $P(f)f(qz+\eta) = a(z)$ has infinitely many solutions, where $a(z) \in \mathbb{S}(f) \setminus \{0\}$, $P(f)$ and m are stated in Theorem 3.*

Theorem 7. *Let f be a zero-order transcendental meromorphic (resp. entire) function, $q \in \mathbb{C} \setminus \{0\}$, $\eta \in \mathbb{C}$. Then for $n > m+6$ (resp. $n > m+2$), $P(f)[f(qz+\eta) - f(z)] = a(z)$ has infinitely many solutions, where $a(z) \in \mathbb{S}(f) \setminus \{0\}$, $P(f)$ and m are stated in Theorem 3.*

For the uniqueness of the difference and the q -difference of meromorphic functions, some results were obtained (see [11, 12, 15, 17, 20, 25, 26]). Here, we only state some recent theorems as follows.

Theorem 8 (see [26], Theorem 5.1). *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (resp. entire) functions of zero order. Suppose that q is a non-zero complex constant and n is an integer satisfying $n \geq 8$ (resp. $n \geq 4$). If $f(z)^n f(qz)$ and $g(z)^n g(qz)$ share $1, \infty$ CM, then $f(z) \equiv tg(z)$ for $t^{n+1} = 1$.*

Theorem 9 (see [26], Theorem 5.2). *Let $f(z)$ and $g(z)$ be two transcendental entire functions of zero order. Suppose that q is a non-zero complex constant and $n \geq 6$ is an integer. If $f(z)^n (f(z) - 1) f(qz)$ and $g(z)^n (g(z) - 1) g(qz)$ share 1 CM, then $f(z) \equiv g(z)$.*

Theorem 10 (see [20], Theorem 2). *Let f and g be transcendental entire functions of finite order, c a nonzero complex constant, let $P(z)$ be stated as in Theorem 3, and let $n > 2\Gamma_0 + 1$ be an integer, where $\Gamma_0 = m_1 + 2m_2$, m_1 is the number of the simple zero of $P(z)$ and m_2 is the number of multiple zeros of $P(z)$. If $P(f)f(z+c)$ and $P(g)g(z+c)$ share 1 CM, then one of the following results holds:*

(i) $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = \text{GCD}\{\lambda_0, \lambda_1, \dots, \lambda_n\}$ and

$$\lambda_i = \begin{cases} i+1, & a_i \neq 0, \\ n+1, & a_i = 0, \end{cases} \quad i = 0, 1, 2, \dots, n;$$

(ii) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = P(\omega_1)\omega_1(z+c) - P(\omega_2)\omega_2(z+c)$;

(iii) $f(z) = e^{\alpha(z)}$, $g(z) = e^{\beta(z)}$, where $\alpha(z)$ and $\beta(z)$ are two polynomials, b is a constant satisfying $\alpha + \beta \equiv b$ and $a_n^2 e^{(n+1)b} = 1$.

In this paper, we will investigate the uniqueness problem of q -shifts of entire functions and obtain the following results.

Theorem 11. *Let f and g be transcendental entire functions of zero order, and let $q \in \mathbb{C} \setminus \{0\}$, $\eta \in \mathbb{C}$, $P(f), \Gamma_0, d$ be stated as in Theorem 10. If $P(f)f(qz+\eta)$ and $P(g)g(qz+\eta)$ share 1 CM and $n > 2\Gamma_0 + 1$, then one of the following cases holds:*

(i) $f \equiv tg$ for a constant t such that $t^d = 1$;

- (ii) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = P(\omega_1)\omega_1(qz + \eta) - P(\omega_2)\omega_2(qz + \eta)$;
- (iii) $fg \equiv \mu$, where μ is a complex constant satisfying $a_n^2 \mu^{n+1} \equiv 1$.

To state the other theorem, we can explain some notations and definitions as follows.

Definition 1 (see [13, 14]). Let l be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, by $E_l(a; f)$ we denote the set of all a -points of f where an a -point of multiplicity k is counted k times if $k \leq l$ and $l + 1$ times if $k > l$. If $E_l(a; f) = E_l(a; g)$, we say that f, g share the value a with weight l .

Theorem 12. Under the assumptions of Theorem 11, if $E_l(1; P(f)f(qz + \eta)) = E_l(1; P(g)g(qz + \eta))$ and l, n, m are integers satisfying one of the following conditions:

- (I) $l = 2, n > 2\Gamma_0 + m + 2 - \lambda$;
- (II) $l = 1, n > 2\Gamma_0 + 2m + 3 - 2\lambda$;
- (III) $l = 0, n > 2\Gamma_0 + 3m + 4 - 3\lambda$;
- (IV) $l \geq 3, n > 2\Gamma_0 + 1$.

Then the conclusions of Theorem 11 hold, where $\lambda = \min\{\Theta(0, f), \Theta(0, g)\}$ and m is stated as in Theorem 3.

2. Some lemmas

In what follows, we explain some definitions and notations which are used in this paper. For $a \in \mathbb{C} \cup \infty$, we define

$$\Theta(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r, f)}.$$

For $a \in \mathbb{C} \cup \infty$ and k a positive integer, by $\overline{N}_{(k)}(r, \frac{1}{f-a})$ we denote the counting function of those a -points of f whose multiplicities are not less than k in counting the a -points of f we ignore the multiplicities (see [10]) and $N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2)}(r, \frac{1}{f-a}) + \cdots + \overline{N}_{(k)}(r, \frac{1}{f-a})$.

Definition 2 (see [1]). When f and g share 1 IM, we denote by $\overline{N}_L(r, \frac{1}{f-1})$ the counting function of the 1-points of f whose multiplicities are greater than 1-points of g , where each zero is counted only once; similarly, we have $\overline{N}_L(r, \frac{1}{g-1})$. Let z_0 be a zero of $f - 1$ of multiplicity p and a zero of $g - 1$ of multiplicity q , by $N_{11}(r, \frac{1}{f-1})$ we also denote the counting function of those 1-points of f where $p = q = 1$.

Lemma 1 (see [6]). Let f and g be two meromorphic functions. If f and g share 1 CM, then one of the following three cases holds:

(i)

$$T(r, f) + T(r, g) \leq 2N_2(r, f) + 2N_2(r, g) + 2N_2\left(r, \frac{1}{f}\right) + 2N_2\left(r, \frac{1}{g}\right) \\ + S(r, f) + S(r, g);$$

(ii) $f \equiv g$;(iii) $f \cdot g = 1$.

Lemma 2 (see [5]). *Let f and g be two meromorphic functions, and let l be a positive integer. If $E_l(1; f) = E_l(1; g)$, then one of the following cases must occur:*

(i)

$$T(r, f) + T(r, g) \leq N_2(r, f) + N_2(r, g) + N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{f-1}\right) \\ + \bar{N}\left(r, \frac{1}{g-1}\right) - N_{11}\left(r, \frac{1}{f-1}\right) + \bar{N}^{(l+1)}\left(r, \frac{1}{f-1}\right) \\ + \bar{N}^{(l+1)}\left(r, \frac{1}{g-1}\right) + S(r, f) + S(r, g);$$

(ii) $f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, where $a(\neq 0), b$ are two constants.

Lemma 3 (see [5]). *Let f and g be two meromorphic functions. If f and g share 1 IM, then one of the following cases must occur:*

(i)

$$T(r, f) + T(r, g) \leq 2\left[N_2(r, f) + N_2\left(r, \frac{1}{f}\right) + N_2(r, g) + N_2\left(r, \frac{1}{g}\right)\right] \\ + 3\bar{N}_L\left(r, \frac{1}{f-1}\right) + 3\bar{N}_L\left(r, \frac{1}{g-1}\right) \\ + S(r, f) + S(r, g);$$

(ii) $f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, where $a(\neq 0), b$ are two constants.

Lemma 4 (see [24]). *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1f + a_2f^2 + \cdots + a_nf^n$, where $a_0, a_1, a_2, \dots, a_n$ are constants and $a_n \neq 0$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 5 (see [8], Lemma 2.1). *Let f be a non-constant meromorphic function, $s > 0$, $\alpha < 1$, and let $F \subset \mathbb{R}_+$ be the set of all r such that $T(r, f) \leq \alpha T(r+s, f)$. If the logarithmic measure of F is infinite, that is, $\int_F \frac{dt}{t} = \infty$, then f is of infinite order of growth.*

Lemma 6 (see [4], Theorem 2.1). *Let $f(z)$ be a meromorphic function of finite order ρ and c a non-zero complex constant. Then, for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r).$$

Lemma 7 (see [26], Theorem 1.1 and Theorem 1.3). *Let $f(z)$ be a transcendental meromorphic function of zero order and q a nonzero complex constant. Then*

$$T(r, f(qz)) = (1 + o(1))T(r, f(z))$$

and

$$N(r, f(qz)) = (1 + o(1))N(r, f(z)),$$

on a set of logarithmic density 1.

Remark 1. *Under the assumptions of Lemma 7, from the definition of $S_1(r, f)$ we have*

$$\begin{aligned} T(r, f(qz)) &= T(r, f(z)) + S_1(r, f), \\ N(r, f(qz)) &= N(r, f(z)) + S_1(r, f). \end{aligned}$$

Lemma 8. *Let $f(z)$ be a transcendental meromorphic function of zero order and q, η two nonzero complex constants. Then*

$$\begin{aligned} T(r, f(qz + \eta)) &= T(r, f(z)) + S_1(r, f), \\ N\left(r, \frac{1}{f(qz + \eta)}\right) &\leq N\left(r, \frac{1}{f}\right) + S_1(r, f), \\ N(r, f(qz + \eta)) &\leq N(r, f) + S_1(r, f), \\ \overline{N}\left(r, \frac{1}{f(qz + \eta)}\right) &\leq \overline{N}\left(r, \frac{1}{f}\right) + S_1(r, f), \\ \overline{N}(r, f(qz + \eta)) &\leq \overline{N}(r, f) + S_1(r, f). \end{aligned}$$

Proof. From Lemma 6 and Lemma 7, we easily get the first equality of this lemma.

Next, the idea of the proof of other inequalities of this lemma are from [12, 25]. From Lemma 7, we have

$$\begin{aligned} N\left(r, \frac{1}{f(qz + \eta)}\right) &= N\left(r, \frac{1}{f\left(z + \frac{\eta}{q}\right)}\right) + S_1\left(r, f\left(z + \frac{\eta}{q}\right)\right) \\ &\leq N\left(r + \left|\frac{\eta}{q}\right|, \frac{1}{f}\right) + S_1\left(r, f\left(z + \frac{\eta}{q}\right)\right). \end{aligned} \quad (1)$$

By Lemma 6 and f is meromorphic function of zero order, we have

$$S_1\left(r, f\left(z + \frac{\eta}{q}\right)\right) = S_1(r, f).$$

And by Lemma 5, we have

$$N\left(r + \left|\frac{\eta}{q}\right|, \frac{1}{f}\right) \leq N\left(r, \frac{1}{f}\right) + S(r, f), \quad (2)$$

outside of a possible exceptional set with the finite logarithmic measure.

From (1),(2) and $N\left(r, \frac{1}{f}\right) \leq N\left(r + \left|\frac{\eta}{q}\right|, \frac{1}{f}\right)$, we have

$$N\left(r + \left|\frac{\eta}{q}\right|, \frac{1}{f}\right) = N\left(r, \frac{1}{f}\right) + S(r, f)$$

and

$$N\left(r, \frac{1}{f(qz + \eta)}\right) \leq N\left(r, \frac{1}{f}\right) + S_1(r, f).$$

Similarly, we can get

$$\bar{N}\left(r, \frac{1}{f(qz + \eta)}\right) \leq \bar{N}\left(r, \frac{1}{f}\right) + S_1(r, f).$$

Set $h(z) = \frac{1}{f(z)}$, then $h(qz + \eta) = \frac{1}{f(qz + \eta)}$. From above, we can prove other inequalities. \square

Lemma 9 (see [18], Theorem 2.1). *Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz + \eta)}{f(z)}\right) = S(r, f),$$

on a set of logarithmic density 1.

Lemma 10. *Let f be a transcendental meromorphic function of zero order, $q(\neq 0), \eta$ complex constants, and let $P(z)$ be stated as in Theorem 3. Then we have*

$$(n - 1)T(r, f) + S_1(r, f) \leq T(r, P(f)f(qz + \eta)) \leq (n + 1)T(r, f) + S_1(r, f). \quad (3)$$

If f is a transcendental entire function of zero order, we have

$$T(r, P(f)f(qz + \eta)) = T(r, P(f)f) + S_1(r, f) = (n + 1)T(r, f) + S_1(r, f). \quad (4)$$

Proof. Set $F(z) = P(f)f(qz + \eta)$. If f is a transcendental entire function of zero order, from Lemma 9 and Lemma 4, we have

$$\begin{aligned} T(r, F(z)) &= m(r, F(z)) \leq m(r, P(f)f(z)) + m\left(r, \frac{f(qz + \eta)}{f(z)}\right) \\ &\leq m(r, P(f)f(z)) + S_1(r, f) = T(r, P(f)f(z)) + S_1(r, f) \\ &= (n + 1)T(r, f) + S_1(r, f). \end{aligned}$$

On the other hand, from Lemma 9, we have

$$\begin{aligned} (n + 1)T(r, f) &= T(r, P(f)f(z)) + S(r, f) = m(r, P(f)f(z)) + S(r, f) \\ &\leq m(r, F(z)) + m\left(r, \frac{f(z)}{f(qz + \eta)}\right) \\ &= T(r, F(z)) + S_1(r, f). \end{aligned}$$

Thus, we can get (4).

If f is a meromorphic function of zero order, from Lemma 8 and Lemma 4, we have

$$T(r, P(f)f(qz + \eta)) \leq T(r, P(f)) + T(r, f(qz + \eta)) \leq (n + 1)T(r, f) + S_1(r, f).$$

On the other hand, from Lemma 9 and Lemma 4, we have

$$\begin{aligned} (n + 1)T(r, f) &= T(r, P(f)f) + S(r, f) = m(r, P(f)f) + N(r, P(f)f) + S(r, f) \\ &\leq m\left(r, F(z)\frac{f(z)}{f(qz + \eta)}\right) + N\left(r, F(z)\frac{f(z)}{f(qz + \eta)}\right) + S(r, f) \\ &\leq T(r, F(z)) + 2T(r, f) + S_1(r, f). \end{aligned}$$

Thus, we can get (3). \square

Using the same method as in Lemma 10, we can easily get the following lemma.

Lemma 11. *Let f be a transcendental meromorphic function of zero order, $q(\neq 0), \eta$ complex constants, and let $P(z)$ be stated as in Theorem 3. Then we have*

$$T(r, P(f)[f(qz + \eta) - f(z)]) \geq (n - 1)T(r, f) + S_1(r, f).$$

If f is a transcendental entire function of zero order, we have

$$T(r, P(f)[f(qz + \eta) - f(z)]) \geq nT(r, f) + S_1(r, f).$$

Lemma 12. *Let $f(z)$ and $g(z)$ be transcendental entire functions of zero order, $P(z)$ be stated as in Theorem 3. If $n \geq 2$, and*

$$P(f)f(qz + \eta)P(g)g(qz + \eta) \equiv t, \quad (5)$$

where $q(\neq 0), \eta, t(\neq 0)$ are complex constants, then we have $fg = \mu$, where $a_n^2 \mu^{n+1} = t$.

Proof. Suppose that the roots of $P(z) = 0$ are b_1, b_2, \dots, b_m with multiplicities l_1, l_2, \dots, l_m . Then we have $l_1 + l_2 + \dots + l_m = n$. From (5), we have

$$(f - b_1)^{l_1} (f - b_2)^{l_2} \dots (f - b_m)^{l_m} f(qz + \eta) (g - b_1)^{l_1} (g - b_2)^{l_2} \dots (g - b_m)^{l_m} g(qz + \eta) \equiv t. \quad (6)$$

Since f, g are nonconstant entire functions, from (6), we can deduce that $b_1 = b_2 = \dots = b_m = 0$. In fact, from (6), we can get that b_1, b_2, \dots, b_m are Picard exceptional values. If $m \geq 2$ and $b_j \neq 0 (j = 1, 2, \dots, m)$, by Picard's theorem of the entire function, we can get that Picard's exceptional values of f are at least three. Thus, we can get a contradiction. Hence, $m = 1$ and $l_1 = n$, that is, there exists a complex constant γ satisfying $P(f) = a_n(f - \gamma)^n$ and $P(g) = a_n(g - \gamma)^n$. Then

$$a_n(f - \gamma)^n f(qz + \eta) a_n(g - \gamma)^n g(qz + \eta) \equiv t. \quad (7)$$

Since f, g are transcendental entire functions, by Picard's theorem, we can get that $f - \gamma = 0$ and $g - \gamma = 0$ do not have zeros. Then, we obtain that $f(z) = e^{\alpha(z)} +$

γ , $g(z) = e^{\beta(z)} + \gamma$, where $\alpha(z), \beta(z)$ are two nonconstant functions. From (7), we get that $f(qz + \eta) \neq 0$ and $g(qz + \eta) \neq 0$. Thus, we can get $\gamma = 0$, that is,

$$a_n^2 f(z)^n f(qz + \eta) g(z)^n g(qz + \eta) \equiv t. \quad (8)$$

Set $M(z) = f(z)g(z)$. If $M(z)$ is nonconstant, from (8), we have

$$a_n^2 M(z)^n M(qz + \eta) \equiv t,$$

that is,

$$a_n^2 M(z)^n \equiv \frac{t}{M(qz + \eta)}. \quad (9)$$

Since f, g are transcendental entire functions of zero order, from (9), Lemma 4, Lemma 8 and $n \geq 2$, we can get a contradiction.

Thus, $M(z)$ is a constant. From (9), we can get $f(z)g(z) \equiv \mu$, where μ is a complex constant satisfying $a_n^2 \mu^{n+1} \equiv t$.

Therefore, the proof of Lemma 12 is complete. \square

3. Proofs of Theorems 6 and 7

3.1. The proof of Theorem 6

Proof. Case 1. If f is a transcendental meromorphic function of zero order, we first suppose that $P(f)f(qz + \eta) = a(z)$ has finitely many solutions. From Lemma 10, we have $S(r, P(f)f(qz + \eta)) = S(r, f)$. By the Second Fundamental Theorem, Lemma 8 and the definition of m , we have

$$\begin{aligned} T(r, P(f)f(qz + \eta)) &\leq \bar{N}(r, P(f)f(qz + \eta)) + \bar{N}\left(r, \frac{1}{P(f)f(qz + c)}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{P(f)f(qz + c) - a(z)}\right) + S(r, f) \\ &\leq 2\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{P(f)}\right) + \bar{N}\left(r, \frac{1}{f(qz + c)}\right) + S(r, f) \\ &\leq (m + 3)T(r, f) + S_1(r, f). \end{aligned} \quad (10)$$

From Lemma 10 and (10), we have

$$(n - 1)T(r, f) \leq (m + 3)T(r, f) + S_1(r, f),$$

that is,

$$(n - m - 4)T(r, f) \leq S_1(r, f). \quad (11)$$

Since $n > m + 4$ and f is a transcendental meromorphic function, we can get a contradiction. Thus, $P(f)f(qz + \eta) = a(z)$ has infinitely many solutions when f is a transcendental meromorphic function of zero order.

Case 2. If f is a transcendental entire function, we suppose that $P(f)f(qz + \eta) = a(z)$ has finitely many solutions. By using the same argument as in Case 1 and (4), we have

$$(n + 1)T(r, f) \leq (m + 1)T(r, f) + S_1(r, f).$$

Since $n > m$ and f is transcendental, we can get a contradiction.

Thus, we can get the conclusions of Theorem 6. \square

3.2. The Proof of Theorem 7

Proof. Similarly to the proof of Theorem 6, and using Lemma 12, we can easily prove Theorem 7. \square

4. Proofs of Theorems 11 and 12

In this section, set $F(z) = P(f)f(qz + \eta)$ and $G(z) = P(g)g(qz + \eta)$.

4.1. The proof of Theorem 11

Proof. From the assumptions of Theorem 11, we have that $F(z), G(z)$ share 1 CM. Then, the following three cases will be considered.

Case 1. Suppose that $F(z), G(z)$ satisfy Lemma 1(i). Since $f(z), g(z)$ are entire functions of zero order, from Lemma 10, we have $S(r, F) = S(r, f)$, $S(r, G) = S(r, g)$. Then, from Lemma 1(i) and Lemma 8, we have

$$\begin{aligned}
 T(r, F(z)) + T(r, G(z)) &\leq 2N_2 \left(r, \frac{1}{F} \right) + 2N_2 \left(r, \frac{1}{G} \right) + S(r, f) + S(r, g) \quad (12) \\
 &\leq 2N_2 \left(r, \frac{1}{P(f)} \right) + 2N_2 \left(r, \frac{1}{f(qz + \eta)} \right) + 2N_2 \left(r, \frac{1}{P(g)} \right) \\
 &\quad + 2N_2 \left(r, \frac{1}{g(qz + \eta)} \right) + S(r, f) + S(r, g) \\
 &\leq 2\Gamma_0 T(r, f) + 2\Gamma_0 T(r, g) + 2N \left(r, \frac{1}{f(qz + \eta)} \right) \\
 &\quad + 2N \left(r, \frac{1}{f(qz + \eta)} \right) + S_1(r, f) + S_1(r, g) \\
 &\leq 2(\Gamma_0 + 1)[T(r, f) + T(r, g)] + S_1(r, f) + S_1(r, g).
 \end{aligned}$$

From Lemma 9 and (12), we have

$$(n + 1)[T(r, f) + T(r, g)] \leq 2(\Gamma_0 + 1)[T(r, f) + T(r, g)] + S_1(r, f) + S_1(r, g),$$

that is,

$$(n - 2\Gamma_0 - 1)[T(r, f) + T(r, g)] \leq S_1(r, f) + S_1(r, g). \quad (13)$$

Since $n > 2\Gamma_0 + 1$ and f, g are transcendental functions, we can get a contradiction.

Case 2. If $F(z) \equiv G(z)$, that is,

$$P(f)f(qz + \eta) \equiv P(g)g(qz + \eta). \quad (14)$$

Set $h = \frac{f}{g}$. If h is not a constant, from (14), we can get that f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = P(\omega_1)\omega_1(qz + c) - P(\omega_2)\omega_2(qz + c)$.

If h is a constant. Substituting $f = gh$ into (14), we can get

$$g(qz + \eta)[a_n g^n (h^{n+1} - 1) + a_{n-1} g^{n-1} (h^n - 1) + \cdots + a_0 (h - 1)] \equiv 0, \quad (15)$$

where $a_n (\neq 0), a_{n-1}, \dots, a_0$ are constants.

Since g is a transcendental entire function, we have $g(qz + \eta) \not\equiv 0$. Then, from (15), we have

$$a_n g^n (h^{n+1} - 1) + a_{n-1} g^{n-1} (h^n - 1) + \cdots + a_0 (h - 1) \equiv 0, \quad (16)$$

If $a_n \neq 0$ and $a_{n-1} = a_{n-2} = \cdots = a_0 = 0$, then from (16) and g being a transcendental function, we can get $h^{n+1} = 1$.

$a_n \neq 0$ and there exists $a_i \neq 0 (i \in \{0, 1, 2, \dots, n-1\})$. Suppose that $h^{n+1} \neq 1$, by Lemma 4 and (16), we have $T(r, g) = S(r, g)$ which is a contradiction with a transcendental function g . Then $h^{n+1} = 1$. Similarly to this discussion, we can get that $h^{j+1} = 1$, when $a_j \neq 0$ for some $j = 0, 1, \dots, n$.

Thus, from the definition of d , we can get that $f \equiv tg$, where t is a constant such that $t^d = 1$, $d = GCD\{\lambda_0, \lambda_1, \dots, \lambda_n\}$.

Case 3. If $F(z)G(z) \equiv 1$. From Lemma 12, we can get that $fg = \mu$ for a constant μ such that $a_n^2 \mu^{n+1} \equiv 1$.

Thus, this completes the proof of Theorem 11. \square

4.2. The proof of Theorem 12

From the assumptions of Theorem 12, we have $E_l(1; F(z)) = E_l(1; G(z))$.

Proof. (I) $l = 2$. Since

$$\begin{aligned} \bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{G-1}\right) - N_{11}\left(r, \frac{1}{F-1}\right) \\ + \frac{1}{2}\bar{N}_{(l+1)}\left(r, \frac{1}{F-1}\right) + \frac{1}{2}\bar{N}_{(l+1)}\left(r, \frac{1}{G-1}\right) \\ \leq \frac{1}{2}N\left(r, \frac{1}{F-1}\right) + \frac{1}{2}N\left(r, \frac{1}{G-1}\right) \\ \leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, F) + S(r, G). \\ \bar{N}_{(l+1)}\left(r, \frac{1}{F-1}\right) \leq \frac{1}{2}N\left(r, \frac{F}{F'}\right) = \frac{1}{2}N\left(r, \frac{F'}{F}\right) + S(r, F) \leq \frac{1}{2}\bar{N}\left(r, \frac{1}{F}\right) + S(r, F) \\ \leq \frac{m}{2}T(r, f) + \frac{1}{2}\bar{N}\left(r, \frac{1}{f}\right) + S_1(r, f), \end{aligned} \quad (17)$$

and

$$\bar{N}_{(l+1)}\left(r, \frac{1}{G-1}\right) \leq \frac{m}{2}T(r, g) + \frac{1}{2}\bar{N}\left(r, \frac{1}{g}\right) + S_1(r, g).$$

Case 1. If $F(z), G(z)$ satisfy Lemma 2(i), from transcendental entire function $f(z), g(z)$ and (17), we have

$$\begin{aligned} T(r, F(z)) + T(r, G(z)) &\leq 2N_2\left(r, \frac{1}{F}\right) + 2N_2\left(r, \frac{1}{G}\right) + mT(r, f) + mT(r, g) \\ &\quad + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) + S_1(r, f) + S_1(r, g). \end{aligned}$$

From Lemma 10 and $\lambda = \min\{\Theta(0, f), \Theta(0, g)\}$, for any $\varepsilon(0 < \varepsilon < n - 2\Gamma_0 - m - 2 + \lambda)$, we have

$$(n - 2\Gamma_0 - m - 2 + \lambda - \varepsilon)[T(r, f) + T(r, g)] \leq S_1(r, f) + S_1(r, g). \quad (18)$$

Since $n > 2\Gamma_0 + m + 2 - \lambda$ and f, g are transcendental functions, we can get a contradiction.

Case 2. If $F(z), G(z)$ satisfy Lemma 2(ii), that is,

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, \quad (19)$$

where $a(\neq 0), b$ are two constants.

We consider three cases as follows.

Subcase 2.1. $b \neq 0, -1$. If $a - b - 1 \neq 0$, then by (19) we know

$$\bar{N}\left(r, \frac{1}{G + \frac{a-b-1}{b+1}}\right) = \bar{N}\left(r, \frac{1}{F}\right).$$

Since f, g are entire functions of zero order, by the Second Fundamental Theorem and Lemma 7 and Lemma 8, we have

$$\begin{aligned} T(r, G) &\leq \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{G + \frac{a-b-1}{b+1}}\right) + S(r, g) \\ &\leq \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{F}\right) + S(r, g) \\ &\leq (m+1)T(r, g) + mT(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + S_1(r, f) + S_1(r, g). \end{aligned}$$

Then from Lemma 8, we have

$$(n - m)T(r, g) \leq mT(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + S_1(r, f) + S_1(r, g).$$

Similarly, we have

$$(n - m)T(r, f) \leq mT(r, g) + \bar{N}\left(r, \frac{1}{g}\right) + S_1(r, f) + S_1(r, g).$$

From the above two inequalities, we have

$$(n - 2m - 1 + \lambda - \varepsilon)[T(r, f) + T(r, g)] \leq S_1(r, f) + S_1(r, g). \quad (20)$$

From the definitions of m and Γ_0 , we have $m = m_1 + m_2$. Since $2\Gamma_0 + m + 2 - \lambda - (2m + 1 - \lambda) \geq 0$, that is, $n > 2\Gamma_0 + m + 2 - \lambda \geq 2m + 1 - \lambda$. From (20) and since f, g are transcendental, we can get a contradiction.

If $a - b - 1 = 0$, then by (19) we know $F = ((b + 1)G)/(bG + 1)$. Since f, g are entire functions, we get that $-\frac{1}{b}$ is a Picard's exceptional value of $G(z)$. By the Second Fundamental Theorem, we have

$$T(r, G) \leq \bar{N}\left(r, \frac{1}{G}\right) + S(r, G) \leq (m + 1)T(r, g) + S_1(r, g).$$

Then, from Lemma 10 and $n > 2\Gamma_0 + m + 2 - \lambda$, we know $T(r, g) \leq S_1(r, g)$, a contradiction.

Subcase 2.2. $b = -1$. Then (19) becomes $F = a/(a + 1 - G)$.

If $a + 1 \neq 0$, then $a + 1$ is a Picard's exceptional value of G . Similarly to the discussion in Subcase 2.1, we can deduce a contradiction again.

If $a + 1 = 0$, then $FG \equiv 1$, that is,

$$P(f)f(qz + \eta)P(g)g(qz + \eta) \equiv 1.$$

Since $n > 2\Gamma_0 + m + 2 - \lambda \geq 2$, by Lemma 12, we can get that $fg = \mu$ for a constant μ such that $a_n^2 \mu^{n+1} \equiv 1$.

Subcase 2.3. $b = 0$. Then (19) becomes $F = (G + a - 1)/a$.

If $a - 1 \neq 0$, then $\bar{N}\left(r, \frac{1}{G+a-1}\right) = \bar{N}\left(r, \frac{1}{F}\right)$. Similarly to the discussion in Subcase 2.1, we can deduce a contradiction again.

If $a - 1 = 0$, then $F \equiv G$, that is,

$$P(f)f(qz + \eta) \equiv P(g)g(qz + \eta).$$

Using the same argument as in the proof of Case 2 in Theorem 11, we can get that f, g satisfy Theorem 11(ii).

(II) $l = 1$. Since

$$\begin{aligned} & \bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{G-1}\right) - N_{11}\left(r, \frac{1}{F-1}\right) \\ & \leq \frac{1}{2}N\left(r, \frac{1}{F-1}\right) + \frac{1}{2}N\left(r, \frac{1}{G-1}\right) \\ & \leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, F) + S(r, G). \end{aligned} \quad (21)$$

From Lemma 8, we have

$$\begin{aligned} \bar{N}_{(2)}\left(r, \frac{1}{F}\right) & \leq N\left(r, \frac{F}{F'}\right) = N\left(r, \frac{F'}{F}\right) + S(r, f) \leq \bar{N}\left(r, \frac{1}{F}\right) + S(r, f) \\ & \leq mT(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + S_1(r, f), \end{aligned} \quad (22)$$

and

$$\bar{N}_{(2)}\left(r, \frac{1}{G}\right) \leq mT(r, g) + \bar{N}\left(r, \frac{1}{g}\right) + S_1(r, g). \quad (23)$$

Case 1. If $F(z), G(z)$ satisfy Lemma 2(i), from f, g as entire functions and (21)-(23), we have

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2(\Gamma_0 + m + 1)[T(r, f) + T(r, g)] + 2\bar{N}\left(r, \frac{1}{f}\right) \\ &\quad + 2\bar{N}\left(r, \frac{1}{g}\right) + S_1(r, f) + S_1(r, g). \end{aligned}$$

From Lemma 10 and $\lambda = \min\{\Theta(0, f), \Theta(0, g)\}$, for any $\varepsilon(0 < \varepsilon < n - 2\Gamma_0 - 2m - 3 + 2\lambda)$, we have

$$[n - 2\Gamma_0 - 2m - 3 + 2\lambda - \varepsilon][T(r, f) + T(r, g)] \leq S_1(r, f) + S_1(r, g). \quad (24)$$

Since $n > 2\Gamma_0 + 2m + 3 - 2\lambda$, from (24) and since f, g are transcendental, we can get a contradiction.

Case 2. If $F(z), G(z)$ satisfy Lemma 2(ii). Similarly to the proof of Case 2 in (I), we can get the conclusions of Theorem 12.

(III) $l = 0$, that is, $F(z), G(z)$ share 1 *IM*. From the definitions of $F(z), G(z)$, we have

$$\begin{aligned} \bar{N}_L\left(r, \frac{1}{F-1}\right) &\leq N\left(r, \frac{F}{F'}\right) = N\left(r, \frac{F'}{F}\right) + S(r, F) \leq \bar{N}\left(r, \frac{1}{F}\right) + S(r, F) \quad (25) \\ &\leq mT(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + S_1(r, f); \end{aligned}$$

similarly, we have

$$\bar{N}_L\left(r, \frac{1}{G-1}\right) \leq mT(r, g) + \bar{N}\left(r, \frac{1}{g}\right) + S_1(r, g). \quad (26)$$

Case 1. Suppose that $F(z), G(z)$ satisfy Lemma 3(i). From (25) and (26), we have

$$\begin{aligned} T(r, F(z)) + T(r, G(z)) &\leq 2N_2\left(r, \frac{1}{F}\right) + 2N_2\left(r, \frac{1}{G}\right) + 3mT(r, f) + 3mT(r, g) \\ &\quad + 3\bar{N}\left(r, \frac{1}{f}\right) + 3\bar{N}\left(r, \frac{1}{g}\right) + S_1(r, f) + S_1(r, g). \end{aligned}$$

From Lemma 10, for any $\varepsilon(0 < \varepsilon < n - 2\Gamma_0 - 3m - 4 + 3\lambda)$, we can get

$$(n - 2\Gamma_0 - 3m - 4 + 3\lambda - \varepsilon)[T(r, f) + T(r, g)] \leq S_1(r, f) + S_1(r, g). \quad (27)$$

Since $n > 2\Gamma_0 + 3m + 4 - 3\lambda$, we can get a contradiction.

Case 2. Suppose that $F(z), G(z)$ satisfy Lemma 3(ii). Similarly to the proof of Case 2 in (I), we can easily get the conclusions of Theorem 12.

(IV) $l \geq 3$. Since

$$\begin{aligned} \bar{N}\left(r, \frac{1}{F(z)-1}\right) &+ \bar{N}\left(r, \frac{1}{G(z)-1}\right) + \bar{N}_{(l+1)}\left(r, \frac{1}{F(z)-1}\right) \\ &+ \bar{N}_{(l+1)}\left(r, \frac{1}{G(z)-1}\right) - N_{11}\left(r, \frac{1}{F(z)-1}\right) \\ &\leq \frac{1}{2}N\left(r, \frac{1}{F(z)-1}\right) + \frac{1}{2}N\left(r, \frac{1}{G(z)-1}\right) + S(r, F) + S(r, G) \\ &\leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, F) + S(r, G). \end{aligned}$$

Case 1. Suppose that $F(z), G(z)$ satisfy Lemma 2(i). From Lemmas 8 and 9, we have

$$(n+1)[T(r, f) + T(r, g)] \leq 2(\Gamma_0 + 1)[T(r, f) + T(r, g)] + S_1(r, f) + S_1(r, g),$$

that is,

$$(n - 2\Gamma_0 - 1)[T(r, f) + T(r, g)] \leq +S_1(r, f) + S_1(r, g). \quad (28)$$

Since $n > 2\Gamma_0 + 1$ and f, g are transcendental functions, we can get a contradiction.

Case 2. Suppose that $F(z), G(z)$ satisfy Lemma 2(ii). Similarly to the proof of Case 2 in (I), we can easily get the conclusions of Theorem 12.

Thus, the proof of Theorem 12 is complete. \square

Acknowledgments

The author would like to thank the referee(s) for reading the manuscript very carefully and making a number of valuable and kind comments which improved the presentation.

References

- [1] A. BANERJEE, *Weighted sharing of a small function by a meromorphic function and its derivative*, Comput. Math. Appl. **53**(2007), 1750–1761.
- [2] D. C. BARNETT, R. G. HALBURD, R. J. KORHONEN, W. MORGAN, *Nevanlinna theory for the q -difference operator and meromorphic solutions of q -difference equations*, Proc. Roy. Soc. Edinburgh Sect. A **137**(2007), 457–474.
- [3] Z. X. CHEN, Z. B. HUANG, X. M. ZHENG, *On properties of difference polynomials*, Acta Math. Sci. Ser. B Engl. Ed. **31**(2011), 627–633.
- [4] Y. M. CHIANG, S. J. FENG, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J. **16**(2008), 105–129.
- [5] C. Y. FANG, M. L. FANG, *Uniqueness of meromorphic functions and differential polynomials*, Comput. Math. Appl. **44**(2002), 607–617.
- [6] M. L. FANG, X. H. HUA, *Entire functions that share one value*, J. Nanjing Unvi. Math. Biquarterly **13**(1996), 44–48.

- [7] R. G. HALBURD, R. J. KORHONEN, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. **314**(2006), 477–487.
- [8] R. G. HALBURD, R. J. KORHONEN, *Finite-order meromorphic solutions and the discrete Painlevé equations*, Proc. London Math. Soc. **94**(2007), 443–474.
- [9] R. G. HALBURD, R. J. KORHONEN, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. **31**(2006), 463–478.
- [10] W. K. HAYMAN, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.
- [11] J. HEITTOKANGAS, R. J. KORHONEN, I. LAINE, J. RIEPPO, *Uniqueness of meromorphic functions sharing values with their shifts*, Complex Var. Elliptic Equ. **56**(2011), 81–92.
- [12] J. HEITTOKANGAS, R. J. KORHONEN, I. LAINE, J. RIEPPO, J. L. ZHANG, *Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity*, J. Math. Anal. Appl. **355**(2009), 352–363.
- [13] I. LAHIRI, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J. **161**(2001), 193–206.
- [14] I. LAHIRI, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl. **46**(2001), 241–253.
- [15] I. LAINE, C. C. YANG, *Value distribution of difference polynomials*, Proc. Japan Acad. Ser. A **83**(2007), 148–151.
- [16] K. LIU, *Meromorphic functions sharing a set with applications to difference equations*, J. Math. Anal. Appl. **359**(2009), 384–393.
- [17] K. LIU, X. L. LIU, T. B. CAO, *Value distributions and uniqueness of difference polynomials*, Adv. Differential Equations **2011**(2011), Art. ID 234215, pp.12.
- [18] K. LIU, X. G. QI, *Meromorphic solutions of q -shift difference equations*, Ann. Polon. Math. **101**(2011), 215–225.
- [19] K. LIU, L. Z. YANG, *Value distribution of the difference operator*, Arch. Math. **92**(2009), 270–278.
- [20] X. D. LUO, W. C. LIN, *Value sharing results for shifts of meromorphic functions*, J. Math. Anal. Appl. **377**(2011), 441–449.
- [21] L. N. WANG, H. Y. XU, T. S. ZHAN, *Properties of q -shift difference-differential polynomials of meromorphic functions*, Adv. Difference Equ. **2014**(2014), Art. 249, doi:10.1186/1687-1847-2014-249.
- [22] H. Y. XU, *On the value distribution and uniqueness of difference polynomials of meromorphic functions*, Adv. Difference Equ. **2013**(2013), Art. 290, doi:10.1186/1687-1847-2013-90.
- [23] L. YANG, *Value Distribution Theory*, Springer-Verlag, Berlin, 1993.
- [24] H. X. YI, C. C. YANG, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht, 2003; Chinese original: Science Press, Beijing, 1995.
- [25] J. L. ZHANG, *Value distribution and shared sets of differences of meromorphic functions*, J. Math. Anal. Appl. **367**(2010), 401–408.
- [26] J. L. ZHANG, R. J. KORHONEN, *On the Nevanlinna characteristic of $f(qz)$ and its applications*, J. Math. Anal. Appl. **369**(2010), 537–544.