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ROBUST STABILITY OF SINGULARLY IMPULSIVE DYNAMICAL SYSTEMS

Summary

In this paper, we present results of the robust stability analysis for the class of nonlinear uncertain singularly impulsive dynamical systems. We present sufficient conditions for the robust stability of a class of nonlinear uncertain singularly impulsive dynamical systems. The problem of evaluating performance bounds for a nonlinear-nonquadratic hybrid cost functional depending upon a class of nonlinear uncertain singularly impulsive dynamical systems is considered. It turns out that the cost bound can be evaluated in closed form as long as the hybrid cost functional is related in a specific way to an underlying Lyapunov function that guarantees robust stability over a prescribed uncertainty set. Then, results for the case of uncertain singularly impulsive dynamical systems are presented. The results obtained for the nonlinear case are further specialized to linear singularly impulsive dynamical systems.

Key words: *singular systems, singularly impulsive dynamical systems, mathematical model, robust control*

1. Introduction

Finally, we use the following standard notation. Let $\mathbf{R}$ denote the set of real numbers, let $\mathbf{N}$ denote the set of nonnegative integers, let $\mathbf{R}^n$ denote the set of $(n \times 1)$ real column vectors, let $\mathbf{R}^{n \times m}$ denote the set of $(n \times m)$ real matrices, let $\mathbf{S}^n$ denote the set of $(n \times n)$ symmetric matrices, let $\mathbf{N}^n$ (resp. $\mathbf{P}^n$) denote the set of $(n \times n)$ nonnegative (resp., positive) definite matrices, and let $\mathbf{I}^n$ or $\mathbf{I}$ denote the $(n \times n)$ identity matrix. Furthermore, $A \geq 0$ (resp., $A > 0$) denotes the fact that the Hermitian matrix is nonnegative (resp., positive) definite and $A \geq B$ (resp., $A > B$) denotes the fact that $A - B \geq 0$ (resp., $A - B > 0$). In addition, we write $V'(x)$ for the Fréchet derivative of $V(\cdot)$ at x, and $\partial\mathbf{S}, \overset{\circ}{\mathbf{S}}, \overline{\mathbf{S}}$ denote the boundary, the interior, and the closure of the subset $\mathbf{S} \subset \mathbf{R}^n$, respectively. Finally, let C^0 denote the set of continuous functions and C^r the set of functions with r continuous derivatives.

In this paper, we give the robust stability analysis results for the class of nonlinear uncertain singularly impulsive dynamical systems presented in [5]. For that purpose, we generalize the robust stability results developed in [3]. First, given a hybrid performance functional, we develop sufficient conditions for the robust stability of the class of nonlinear uncertain singularly impulsive dynamical systems. Next, for the dynamics of the system written in the form of nominal dynamics plus perturbation, we present robust stability results. Then, the results are specialized to the linear singularly impulsive case.

2. Robust Stability Analysis of Nonlinear, Uncertain, Singularly Impulsive, Dynamical Systems

In this section we present sufficient conditions for the robust stability of a class of *nonlinear uncertain singularly impulsive dynamical systems*. We consider the problem of evaluating the performance bound of a nonlinear-nonquadratic hybrid cost functional depending upon a class of nonlinear uncertain singularly impulsive dynamical systems. It turns out that the cost bound can be evaluated in closed form as long as the hybrid cost functional is related in a specific way to an underlying Lyapunov function that guarantees robust stability over a prescribed uncertainty set. Here, we confine our attention to the nonlinear *state-dependent singularly impulsive dynamical system* [5] G given by

$$E_c\dot{x}(t) = f_c(x(t)), \quad x(0) = 0, \quad x(t) \notin \mathbf{Z}, \tag{2.1}$$

$$E_d \Delta x(t) = f_d(x(t)), \quad x(t) \in \mathbf{Z}, \tag{2.2}$$

where $t \geq 0, \mathbf{x}(t) \in \mathbf{D} \subseteq \mathbf{R}^n$, $\mathbf{D}$ is an open set, t_k denotes the k^{th} instant of time at which $x(t)$ intersects $\mathbf{Z}$, $0 \in \mathbf{D}$, $fc : \mathsf{D} \to \mathbf{R}^n$ is the continuous in the sense of Lipschitz and satisfies $\mathbf{f}_c(0) = \mathbf{0}$, $\mathbf{f}_d : \mathbf{D} \to \mathbf{R}^n$ is the continuous and satisfies $\mathbf{f}_d(0) = 0$, and $\mathbf{Z} \subset \mathbf{R}^n$ is the resetting set. Matrices E_c, E_d may be singular matrices. We refer to the differential equation (2.1) as the continuous time dynamics, and we refer to the difference equation (2.2) as the resetting law. Furthermore, F_c and F_d denote the class of nonlinear uncertain singularly impulsive dynamical systems with $f_{c0}(\cdot) \in F_c$ and $f_{d0}(\cdot) \in F_d$ defining the nominal nonlinear singularly impulsive dynamical system for the continuous time and the resetting dynamics, respectively. Note that since the resetting set $\mathbf{Z}$ is a subset of the state space $\mathbf{D}$ and is independent of time, state-dependent singularly impulsive dynamical systems are time-invariant. In this paper we assume that existence and uniqueness properties of a given state-dependent singularly impulsive dynamical system are satisfied in forward time. For details see [8]. For the following result, let $L_c, L_d : \mathbf{D} \to \mathbf{R}$, and let $F_c \subset \left\{ fc : \mathbf{D} \to \mathbf{R}^n, f_c(0) = 0 \right\}$ and let $F_d \subset \left\{ f_d : \mathbf{D} \to \mathbf{R}^n, f_d(0) = 0 \right\}$ denote the class of nonlinear uncertain singularly impulsive dynamical systems with $f_{c0}(\cdot) \in F_c$ and $f_{d0}(\cdot) \in F_d$ defining the nominal nonlinear singularly impulsive dynamical system on continuous time and discrete time dynamics, respectively. For the following result and the remainder of the chapter we denote the resetting times $\tau_k(x_0)$ by t_k, and for simplicity, we also define $(f_c(\cdot), f_d(\cdot)) \in F_c \times F_d = \mathbf{F}$ and $\mathbf{N}_{[0,t)} = \{k : 0 \leq t_k < t\}$. Within the context of robustness analysis, it is assumed that the zero solution $x(t) \equiv 0$ to the nominal nonlinear singularly impulsive dynamical system given by (2.1) and (2.2) is asymptotically stable. Furthermore, we assume that an infinite number of resetting occurs.

Theorem 2.1. *Consider the nonlinear uncertain singularly impulsive dynamical system* G *given by (2.1) and (2.2), where* $(f_c(\cdot), f_d(\cdot)) \in \mathbf{F}$, *with performance functional*

$$J(E_c x_0) = \int_0^\infty L_c(E_c(x(t))dt + \sum_{k \in N[0,\infty)}^{\infty} L_d(E_d x(t_k)). \tag{2.3}$$

Furthermore, assume there exist functions $\Gamma_c, \Gamma_d : \mathbf{D} \to \mathbf{R}$ *and* $V : \mathbf{D} \to \mathbf{R}$, *where* $V(\cdot)$ *is* C^1 *function, such that*

$$V(0) = 0, \tag{2.4}$$

$$V(E_{c/d}\, x) \geq 0, \quad x \in \mathbf{D}, x \neq 0, \tag{2.5}$$

$$V'(E_c x) f_c(x) \leq V'(E_c x) f_{c0}(x) + \Gamma_c(x), \quad x \notin \mathbf{Z}, f_c(\cdot) \in F_c, \tag{2.6}$$

$$V'(E_c x) f_{c0}(x) + \Gamma_c(x) < 0, \quad x \notin \mathbf{Z}, x \neq 0, \tag{2.7}$$

$$L_c(E_c x) + V'(E_c x) f_{c0}(x) + \Gamma_c(x) = 0, \quad x \notin \mathbf{Z}, \tag{2.8}$$

$$V(E_d x + f_d(x)) - V(E_d x) \leq V(E_d x + f_{d0}(x)) - V(E_d x) + \Gamma_d(x), \quad x \in \mathbf{Z}, f_d(\cdot) \in F_d, \tag{2.9}$$

$$V'(E_d x + f_{d0}(x)) - V(E_d x) + \Gamma_d(x) \leq 0, \quad x \in \mathbf{Z}, \tag{2.10}$$

$$L_d(E_d x) + V(E_d x + f_{d0}(x)) - V(E_d x) + \Gamma_d(x) = 0, \quad x \in \mathbf{Z}, \tag{2.11}$$

where $(f_{c0}(\cdot), f_{d0}(\cdot)) \in \mathbf{F}$ *defines the nominal nonlinear singularly impulsive dynamical system. Then, there exists a neighbourhood* $\mathbf{D}_0 \subset \mathbf{D}$ *of the origin such that if* $x_0 \subset \mathbf{D}_0$, *then the zero solution* $x(t) \equiv 0$ *to (2.1) and (2.2) is locally asymptotically stable for all* $(f_{c0}(\cdot), f_{d0}(\cdot)) \in \mathbf{F}$, *and*

$$\sup_{(f_c(\cdot), f_d(\cdot)) \in F} J(E_c x_0) \leq \mathcal{J}(E_c x_0) = V(E_c x_0), \tag{2.12}$$

where

$$\mathcal{J}(E_c x_0) = \int_0^\infty \left[L_c(E_c x(t)) + \Gamma_c(x(t))dt + \sum_{k \in N[0,\infty)} \left[L_d(E_d x(t_k)) + \Gamma_d(x(t_k)) \right] \right], \tag{2.13}$$

and where $x(t)$, $t \geq 0$, *is a solution to (2.1) and (2.2) with* $(f_c(x(t)), f_d(x(t_k))) = (f_{c0}(x(t)), f_{d0}(x(t_k)))$. *Finally, if* $\mathbf{D} = \mathbf{R}^n$ *and*

$$V\left(E_{c/d}x\right)\to\infty \quad as \quad \|x\|\to\infty, \tag{2.14}$$

then the zero solution $x(t)\equiv 0$ *to (2.1) and (2.2) is globally asymptotically stable for all* $(f_c(\cdot), f_d(\cdot))\in\mathbf{F}$, [3], [7].

Proof. Let $(f_c(\cdot), f_d(\cdot))\in\mathbf{F}$ and $x(t)$ satisfy (2.1) and (2.2). Then,

$$V\left(E_c x(t)\right)=\frac{d}{dt}V\left(E_c x(t)\right)=V'\left(E_c x(t)\right)f_c\left(x(t)\right), \quad x(t)\notin\mathbf{Z},\ t_k<t\le t_{k+1}. \tag{2.15}$$

Hence, it follows from (2.6) and (2.7) that

$$V\left(E_c x(t)\right)<0, \quad x(t)\notin\mathbf{Z}, x(t)\neq 0, \quad t_k<t\le t_{k+1}. \tag{2.16}$$

Furthermore,

$$\Delta V\left(E_d x(t_k)\right)=V\left(E_d x(t_k)+f_d\left(x(t_k)\right)\right)-V\left(E_d x(t_k)\right), \quad x(t_k)\in\mathbf{Z}. \tag{2.17}$$

Hence, from (2.9) and (2.10) it follows that

$$\Delta V\left(E_d x(t_k)\right)\le 0, \quad x(t_k)\in\mathbf{Z}, \tag{2.18}$$

Thus, from (2.4), (2.5), (2.16), and (2.18) and from Theorem 3.2 of [4] it follows that $V()$ is the Lyapunov function for (2.1) and (2.2), which proves the local asymptotic stability of the zero solution $x(t)\equiv 0$ for all $(f_c(\cdot), f_d(\cdot))\in\mathbf{F}$. Consequently, $x(t)\to 0$ as $t\to\infty$ for all initial conditions $x_0\in\mathbf{D}_0$ for some neighbourhood $\mathbf{D}_0\subset\mathbf{D}$ of the origin. Now, (2.15) and (2.17) imply that

$$0=-\dot V\left(E_c x(t)\right)+V'\left(E_c x(t)\right)f_c\left(x(t)\right), \quad x(t)\notin\mathbf{Z},\ t_k<t<t_{k+1}, \tag{2.19}$$

$$0=-\Delta V\left(E_d x(t_k)\right)+V\left(E_d x(t_k)+f_d\left(x(t_k)\right)\right)-V\left(E_d x(t_k)\right), \quad x(t_k)\in\mathbf{Z}. \tag{2.20}$$

From (2.19), using (2.6) and (2.8),

$$\begin{aligned}Lc\left(x(t)\right)&=-\dot V\left(E_c x(t)\right)+Lc\left(E_c x(t)\right)+\dot V\left(E_c x(t)\right)f_c\left(x(t)\right)\\ &\le-\dot V\left(E_c x(t)\right)+L_c\left(E_c x(t)\right)+V\left(E_c x(t)\right)f_{c0}x(t)+\Gamma_c\left(x(t)\right)\\ &=-\dot V\left(E_c x(t)\right), \quad x(t)\in Z,\ t_k<t\le t_{k+1}.\end{aligned} \tag{2.21}$$

From (2.20), using (2.9) and (2.11),

$$\begin{aligned}L_d\left(E_d x(t_k)\right)&=-\Delta V\left(E_d x(t_k)\right)+L_d\left(E_d x(t_k)\right)+V\left(E_d x(t_k)\right)+f_d\left(x(t_k)\right)-V\left(E_d x(t_k)\right)\\ &\le-V\left(E_d x(t_k)\right)+L_d\left(E_d x(t_k)\right)+V\left(E_d x(t_k)+f_{d0}x(t_k)\right)-V\left(E_d x(t_k)\right)+\Gamma_d\left(x(t_k)\right)\\ &=-\Delta V\left(E_d x(t_k)\right), \quad x(t_k)\in\mathbf{Z}.\end{aligned} \tag{2.22}$$

Now, integrating over the interval $[0,t)$ with $\mathbf{N}_{[0,t)}=\{1,2,\ldots i,\ldots j\}$, (2.21) and (2.22) yield

$$\begin{aligned}
\int_0^t Lc\big(E_c x(s)\big)ds+\sum_{k\in N[0,t)} L_d\left(E_d x(t_i)\right)&=\int_0^{t_i} Lc\big(E_c x(s)\big)ds+L_d\left(E_d x(t_i)\right)\\
&+\int_{t_i}^{t_{i+1}} Lc\big(E_c x(s)\big)ds+L_d\left(E_d\left(x(t_{i+1})\right)\right)\\
&+\ldots+\int_{t_j}^{t_{j+1}} Lc\big(E_c x(s)\big)ds+L_d\left(E_d x(t_{j+1})\right)\\
&+.\int_{t_{j+1}}^{t} Lc\big(E_c x(s)\big)ds\\
&\le V\left(E_c x(t_i)\right)-V\left(E_c x_0\right)+V(E_d x((t_i))-V\left(E_d x(t_i)+f_d x(t_i)\right)\\
&+V\left(E_c x(t_{i+1})\right)-V\left(E_c x(t_i)\right)+V(E_d x((t_{i+1}))-V\left(E_d x(t_{i+1})+f_d x(t_{i+1})\right)\\
&+\ \ldots\ +V\left(E_c x(t_{j+1})\right)-V\left(E_c x(t_j)\right)+V(E_d x((t_{j+1}))-V\left(E_d x(t_{j+1})+f_d x(t_{j+1})\right)\\
&+V\left(E_c x(t)\right)-V\left(E_c x(t_{j+1})\right)\\
&\le V\left(E_c\mathbf{x}(t)\right)-V\left(E_c\mathbf{x}_0\right).
\end{aligned}\tag{2.23}$$

Letting $t\to\infty$ and noting that $V\big(E_{c/d}x(t)\big)\to 0$ for all $x_0\in\mathbf{D}_0$ yields $J(f_c,f_d)x_0\le V(E_c x_0)$. Next, let $\mathbf{x}(t), t\ge 0$ satisfy (2.1) and (2.2) with $\left(f_c(\cdot),f_d(\cdot)\right)=\left(f_{c0}(\cdot),f_{d0}(\cdot)\right)$. Now, from (2.19), using (2.8),

$$\begin{aligned}
L_c(E_c x(t))+\Gamma_c(x(t))&=-\dot{V}(E_c x(t))+L_c(E_c x(t))+V'(E_c x(t))f_{c0}(x(t))+\Gamma_c x(t)\\
&=-\dot{V}(E_c x(t)),\quad x(t)\notin\mathbf{Z},\ t_k<t\le t_{k+1}.
\end{aligned}\tag{2.24}$$

From (2.20), using (2.11),

$$\begin{aligned}
L_d\left(E_d x(t_k)\right)+\Gamma_d\left(x(t_k)\right)&=-\Delta V\left(E_d x(t_k)\right)+L_d\left(E_d x(t_k)\right)+V\left(E_d x(t_k)+f_{d0}\left(x(t_k)\right)\right)\\
&-V\left(E_d x(t_k)\right)+\Gamma_d\left(x(t_k)\right)=-\Delta V\left(E_d x(t_k)\right),\quad x(t_k)\in\mathbf{Z},
\end{aligned}\tag{2.25}$$

Now, integrating over the interval $[0,t)$ with $\mathbf{N}_{[0,t)}=\{1,2,\ldots i,\ldots j\}$, (2.24) and (2.25) yield

$$\begin{aligned}
\int_0^t\left[L_c(E_c x(t))+\Gamma_c(x(t))\right]dt+\sum_{k\in N[0,\infty)}\left[L_d\left(E_d x(k)\right)+\Gamma_d(x(k))\right]=\\
-V(E_{c/d}x(t))+V(E_{c/d}x_0).
\end{aligned}\tag{2.26}$$

Letting $t\to\infty$ and noting that $V\left(E_{c/d}\,x(t)\right)\to 0$ for all $x_0\in\mathbf{D}_0$ yields $J\left(E_c f_{c0}\right)=V\left(E_c x_0\right)$. Finally, for $\mathbf{D}=\mathbf{R}^n$ and for all $\left(f_c(\cdot),f_d(\cdot)\right)$, the global asymptotic stability of the zero solution $x(t)\equiv 0$ to (2.1), (2.2) is a direct consequence of Theorem 3.2 of [4] using the radially unbounded condition (2.14) on $V\left(E_{c/d}\,x\right)$, $x_0=\mathbf{R}^n$.

Remark 2.1. *Theorem 2.1 provides sufficient conditions for the robust stability of a class of nonlinear uncertain singularly impulsive dynamical systems* $\left(f_c(\cdot),f_d(\cdot)\right)\in\mathbf{F}$ *. Specifically,* *(2.4) and (2.5) assume that* $V(x)$ *is a Lyapunov function candidate for the nonlinear uncertain singularly impulsive dynamical system given by (2.1) and (2.2). Conditions (2.6), (2.7), (2.9), and (2.10) imply* $\dot{V}(E_c x(t)) < 0$, $x(t) \notin \mathbf{Z}, t > 0$, *and* $\dot{V}(E_d x(t_k)) \leq 0, x(t_k) \in \mathbf{Z}, k \in \mathbf{N}$, *for* $x(\cdot)$ *satisfying (2.1)and (2.2) for all* $(f_c(\cdot), f_d(\cdot)) \in \mathbf{F}$, *and hence* $V(\cdot)$ *is the Lyapunov function guaranteeing the robust stability of the nonlinear uncertain singularly impulsive dynamical system given by (2.1) and (2.2). It is important to note that Conditions (2.7) and (2.10) are verifiable conditions since they are independent of the uncertain system parameters* $(f_c(\cdot), f_d(\cdot)) \in \mathbf{F}$. *To apply Theorem 2.,1 we specify the bounding functions* $\Gamma_c(\cdot), \Gamma_d(\cdot)$ *for the uncertain set* $F_c \times F_d$ *such that* $\Gamma_c(\cdot), \Gamma_d(\cdot)$ *bound* $F_c \times F_d$. *If* $\mathbf{F}$ *consists only of the nominal nonlinear singularly impulsive dynamical system* $(f_{c0}(\cdot), f_{d0}(\cdot)) \in \mathbf{F}$, *then* $\Gamma_c(x) = 0, \Gamma_d(x) = 0$ *for all* $x \in \mathbf{D}$ *satisfy (2.6) and (2.9), respectively, and hence* $J(f_{c0}, f_{d0}) = \mathrm{J}(E_c x_0)$. *Finally, the worst-case upper bound to the nonlinear-nonquadratic hybrid performance functional is given in terms of the Lyapunov function which can be interpreted in terms of an auxiliary cost defined for the nominal singularly impulsive dynamical system.*

Next, we specialize Theorem 2.1 to nonlinear uncertain singularly impulsive dynamical systems of the form

$$E_c \dot{x}(t) = f_{c0}(x(t)) + \Delta f_c(x(t)), \quad x(t) \notin Z, x(0) = x_0, \tag{2.27}$$

$$E_d \Delta x(k) = f_{d0}(x(k)) + \Delta f_d(x(k)), \quad x(t) \in Z, \tag{2.28}$$

where $t \geq 0$, $f_{c0}, f_{d0} : \mathbf{D} \to \mathbf{R}^n$ and satisfies $f_{c0}(0) = 0, f_{d0}(0) = 0$ and $(\Delta f_c, \Delta f_d) \in f_c \times f_d = \mathbf{F}$, where

$$F_c \subset \left\{ \Delta f_c : \mathbf{D} \to R^n, \Delta f_c(0) = 0 \right\}, \tag{2.29}$$

$$F_d \subset \left\{ \Delta f_d : \mathbf{D} \to R^n, \Delta_d(0) = 0 \right\}. \tag{2.30}$$

Corollary 2.1. *Consider the nonlinear uncertain singularly impulsive dynamical systems given by (2.27) and (2.28) with performance functional (2.3). Furthermore, assume there exist functions* $\Gamma_c(\cdot), \Gamma_d(\cdot) : \mathbf{D} \to \mathbf{R}, P_{1f_d} : \mathbf{D} \to \mathbf{R}^{1\times n}, P_{2f_d} : \mathbf{D} \to \mathbf{N}^{1\times n}$, *and* $V : \mathbf{D} \to \mathbf{R}$, *where* $V(\cdot)$ *is a* C^1 *function, such that (2.4), (2.5), (2.10), and (2.11) hold and*

$$V'(E_c x) \Delta f_c(x) \leq \Gamma_c(x), \qquad x \notin \mathbf{Z}, \Delta f_c(\cdot) \in F_c, \tag{2.31}$$

$$V'(E_c x) f_{c0}(x) + \Gamma_c(x) < 0, \quad x \notin \mathbf{Z}, x \neq 0, \tag{2.32}$$

$$L_c(E_c x) + V'(E_c x) f_{c0}(x) + \Gamma_c(x) = 0, \quad x \notin \mathbf{Z} \tag{2.33}$$

$$P_{1fd}(0) = 0 \tag{2.34}$$

$$\Delta f_d^T(x) P_{1f_d}(x) + P_{1f_d}(x)\Delta f_d(x) + \Delta f_d^T(x) P_{2f_d}(x)\Delta f_d(x) \le \Gamma_d(x), \;\; x \in \mathbf{Z}, \Delta f_d(\cdot) \in \mathrm{F}_d, \tag{2.35}$$

$$\begin{aligned} V\left(E_d x + f_{d0}(x) + \Delta f_d(x)\right) - V\left(E_d x\right) \le V\left(E_d x + f_{d0}(x)\right) - V\left(E_d x\right) + \Delta f_d^T(x) P_{1f_d(x)}^T \\ + P_{1f_d(x)}^T \Delta f_d(x) + \Delta f_d{}^T(x) P_{2f_d(x)} \Delta f_d(x), \qquad x \in \mathbf{Z}, \Delta f_d(\cdot) \in F_d. \end{aligned} \tag{2.36}$$

Then, there exists a neighbourhood $V : \mathbf{D}_0 \subset \mathbf{D}$ *of the origin such that if* $x_0 \in \mathbf{D}_0$*, then the zero solution* $x(t) \equiv 0$ *to (2.27) and (2.28) is locally asymptotically stable for all* $(f_c(\cdot), f_d(\cdot)) \in \mathbf{F}$*; and the hybrid performance functional (2.3) satisfies*

$$\sup_{(\Delta f_c(\cdot),\, \Delta f_d(\cdot)) \in \mathbf{F}} J(E_c x_0) \le J(E_c x_0) = V(E_c x_0), \tag{2.37}$$

where

$$J(E_c x_0) = \int_0^t L_c(E_c x(t)) + \Gamma(x(t)) dt + \sum_{k \in N[0,\infty)} \left[L_d(E_d x(t_k)) + \Gamma_d x(t_k)\right] \tag{2.38}$$

and where $x(t), t \ge 0,$ *is the solution to (2.27) and (2.28) with* $(\Delta f_c(\cdot), \Delta f_d(\cdot)) = (0, 0)$*. Finally, if* $\mathbf{D} = \mathbf{R}^n$ *and* $V(E_{c/d}\, x), x \in \mathbf{R}^n$ *satisfy (2.14), then the zero solution* $x(t) \equiv 0$ *to (2.27) and (2.28) is globally asymptotically stable for all* $(\Delta f_c(\cdot), \Delta f_d(\cdot)) \in \mathbf{F}$, [3], [7].

Proof. The result is a direct consequence of Theorem 2.1 with $f_c(x) = f_{c0}(x) + \Delta f_c(x)$, for $x \notin \mathbf{Z}$ and ϕd(ξ) = ϕd₀(ξ) + Δϕd(ξ), for ξ ∈ Z. Specifically, in this case, it follows from (2.31) and (2.32) that ς ′(Ecξ)ϕc(ξ) ≤ ς ′(Ecξ)ϕc₀(ξ)+ ¡c(ξ) < 0 for all $x \ne 0, x \notin \mathbf{Z}$, and $\Delta f_c(\cdot) \in F_c$. Further, it follows from (2.35) and (2.36) that $V(E_d x + f_d(x)) - V(E_d x) \le V(E_d x + f_{d0}(x)) - V(E_d x) + \Gamma_d(x)$, for all $x \in \mathbf{Z}$ and $\Delta f_d(\cdot) \in F_d$. Hence, all the conditions of Theorem 2.1 are satisfied.

The following corollary specializes Theorem 2.1 to a class of linear uncertain singularly impulsive dynamical systems. Specifically, we take $\mathbf{F} = F_c \times F_d$ to be a set of linear uncertain singularly impulsive dynamical systems, with

$$F_c = \left\{ (A_c + \Delta A_c)x : x \in \mathbf{R}^n, A_c \in \mathbf{R}^{n\times n}, \Delta A_c \in \Delta A_c \right\},$$

$$F_d = \left\{ (A_d + \Delta A_d)x : x \in \mathbf{R}^n, A_d \in \mathbf{R}^{n\times n}, \Delta A_d \in \Delta A_d \right\},$$

where $\Delta A_c, \Delta A_d \subset \mathbf{R}^{n\times n}$ are the given bounded uncertainty sets of uncertain perturbations $\Delta A_c, \Delta A_d$ of the nominal asymptotically stable system matrices A_c, A_d such that $0 \in \Delta A_c$, $0 \in \Delta A_d$. For simplicity, we also define $(\Delta A_c, \Delta A_d) \in \Delta_{A_c} \times \Delta_{A_d} = \mathbf{\Delta}$.

Corollary 2.2. *Let* $R_c \in \mathbf{P}^n$ *and* $R_d \in \mathbf{N}^n$*. Consider the linear state-dependent uncertain singularly impulsive dynamical system*

$$E_c \dot{x}(t) = (A_c + \Delta A_c)x(t), \quad x(0) = x_0, \quad t \ge 0, \quad x(t) \notin \mathbf{Z}, \tag{2.39}$$

$$E_d \Delta x(k) = (A_d + \Delta A_d - E_d)x(k), \qquad x(t_k) \notin \mathbf{Z}, \tag{2.40}$$

with performance functional

$$J(\Delta A_c, \Delta A_d)(E_c x_0) = \int_0^\infty x^T(t) E_c^T R_c E_c x(t)dt + \sum_{k \in N[0,\infty)} x^T(t_k) E_d^T R_d E_d x(t_k), \tag{2.41}$$

Where $(\Delta A_c, \Delta A_d) \in \mathbf{\Delta}$. *Let* $c, d : N_P \subseteq \mathbf{S}^n \to \mathbf{N}^n$, $P \in \mathbf{P}^n$ *be such that*

$$x^T\left(\Delta A_c^T P E_c + E_c^T P \Delta A_c\right)x \le x^T E_c^T \Omega_c(P) E_c x, \quad x \notin \mathbf{Z}, \Delta A_c \in \Delta A_c, \tag{2.42}$$

$$x^T\left(A_d^T P \Delta A_d + \Delta A_d^T P A_d + A_d^T P \Delta A_d\right)x \le x^T E_d^T \Omega_d(P)x, \quad x \in \mathbf{Z}, \Delta A_d \in \Delta A_d. \tag{2.43}$$

Furthermore, suppose there exists $P \in \mathbf{P}^n$ *satisfying*

$$0 = x^T\left(A_c^T P E_c + E_c^T P A_c + \Omega_c(P) + E_c^T R_c E_c\right)x, \quad x \notin \mathbf{Z}, \tag{2.44}$$

$$0 = x^T\left(A_d^T P A_d - E_d^T P E_d + \Omega_d(P) + E_d^T R_d E_d\right)x, \quad x \in \mathbf{Z}, \tag{2.45}$$

Then the zero solution $x(t) \equiv 0$ *to (2.39) and (2.40) is globally asymptotically stable for all* $(\Delta A_c, \Delta A_d) \in \mathbf{\Delta}$, *and the hybrid performance functional (2.41) satisfies*

$$\sup_{(\Delta A_c, \Delta A_d) \in \Delta J(\Delta A_c, \Delta A_d)} J_{\Delta A_c}(E_c x_0) \le \mathrm{J}(E_c x_0) = x_0^T E_c^T P E_c x_0, \quad x_0 \in \mathbf{R}^n, \tag{2.46}$$

where

$$\mathrm{J}(E_c x_0) = \int_0^\infty x^T(t)\left(\Omega c(P) + E_c^T R_c E_c\right)x(t)dt + \sum_{k \in N[0,\infty)} x^T(t_k)\left(\Omega_d(P) + E_d^T R_d E_d\right)x(t_k), \tag{2.47}$$

and where $x(t), t \ge 0$, *is the solution to (2.39) and (2.40) with* $(\Delta A_c, \Delta A_d) = (0, 0)$, [3], [7].

Proof. The result is a direct consequence of Theorem 2.1 with $f_c(x) =$, $(A_c + \Delta A_c)x$, $f_{c0}(x) = A_c x$, $L_c(E_c x) = x^T E_c^T R_c E_c x, \Gamma_c(E_c x) = x^T \Omega_c(P)x$, for $x \notin \mathbf{Z}$, $f_d(x) = (A_d + \Delta A_d - E_d)x$, $f_{d0}(x) = A_c x$, $L_d(E_c x) = x^T E_d^T R_d E_d x, \Gamma_d(E_d x) = x^T \Omega_d(P)x$ for $x \in Z$, $V(E_{c/d}x) = x^T E_{c/d}^T P E_{c/d} x$, with arguments $E_c x, E_d x$, and $\mathbf{D} = \mathbf{R}^n$. Specifically, conditions (2.4) and (2.5) are trivially satisfied. Now, for the argument $E_c x$, $V(E_c x) f_c(x) =$ $x^T\left(A_c^T P E_c + E_c^T P A_c\right)x + x^T\left(\Delta A_c^T P E_c + E_c^T P D A_c\right)x$, for all $x \ne 0, x \notin \mathbf{Z}$ and $\Delta A_c \in \Delta \mathbf{A}_c$, it follows from (2.42) that $V'(E_c x) f_c(x) \le$ $V'(E_c x) f_{c0}(x) + \Gamma_c(x) = x^T\left(A_c^T P E_c + E_c^T P A_c + \Omega_c(P)\right)x$, for all $x(t) \ne 0$, $x(t) \notin \mathbf{Z}$. Similarly, for the argument $E_d x$, $V(E_d x + f_d(x)) - V(E_d x) = x^T\left(A_d^T P A_d + E_d^T P E_d\right)x +$

$x^T\left(A_d^T P\Delta A_d + \Delta A_d^T PA_d + \Delta A_d^T P\Delta A_d\right)x, x\in\mathbf{Z}$ and $\Delta A_d \in \Delta\mathbf{A}_d$, it follows from (2.43) that $V(E_d x + f_d(x)) - V(E_d x) =$
$V(E_d x + f_{d0}(x)) - V(E_d x) + \Gamma_d(x) = x^T\left(A_d^T PA_d + E_d^T PE_d + \Omega_d(P)\right)x, \quad x\in\mathbf{Z}$. Furthermore, it follows from (2.44) that $L_c(E_c x) + V'(E_c x)f_{c0}(x) + \Gamma_c(x) = 0, x\notin Z$ and hence $V'(E_c x)f_{c0}(x) + \Gamma_c(x) < 0$, for all $x(t)\neq 0$, $x\notin\mathbf{Z}$. Similarly, it follows from (2.45) that $L_d(E_d x) + V(E_d x + f_{d0}(x)) - V(E_d x) + \Gamma_d(x) = 0, x\in\mathbf{Z}$ and hence $V(E_d x + f_{d0}(x)) - V(E_d x) + \Gamma_d(x) \leq 0, x\in\mathbf{Z}$, so that all the conditions of Theorem 2.1 are satisfied. Finally, since $V(E_{c/d}x), x\in\mathbf{R}^n$, is radially unbounded, the system given by (2.39) and (2.40) is globally asymptotically stable for all $(\Delta A_c, \Delta A_d)\in\mathbf{\Delta}$.

3. Conclusion

In this paper we presented the results of the robust stability analysis for the class of nonlinear uncertain singularly impulsive dynamical systems. We presented sufficient conditions for the robust stability of a class of nonlinear uncertain singularly impulsive dynamical systems. Results are then presented for the case of uncertain singularly impulsive dynamical systems. The results obtained for the nonlinear case are further specialized to linear singularly impulsive dynamical systems.

4. Future Work

In further work we will develop the results presented in this paper to nonnegative, compartmental and large scale dynamical systems. Results will extend to time-delay systems. Sensitivity and robustness are very important qualitative properties to be conserved in biological systems and we will focus on obtaining results applicable to particular examples of biological systems.

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