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FORMULAS FOR QUADRATIC SUMS THAT INVOLVE GENERALIZED FIBONACCI AND LUCAS NUMBERS

Zvonko Čerin

Abstract. We improve on Melham's formulas in [10, Section 4] for certain classes of finite sums that involve generalized Fibonacci and Lucas numbers. Here we study the quadratic sums where products of two of these numbers appear. Our results show that most of his formulas are the initial terms of a series of formulas, that the analogous and somewhat simpler identities hold for associated dual numbers and that besides the alternation according to the numbers $(-1)^{\frac{n(n+1)}{2}}$ it is possible to get similar formulas for the alternation according to the numbers $(-1)^{\frac{n(n-1)}{2}}$. We also consider twelve quadratic sums with binomial coefficients that are products.

1. Introduction

The main goal of this paper is to improve several results by R. S. Melham in [10, Section 4]. His idea was to consider identities for those finite sums of products of two generalized Fibonacci and Lucas numbers where the right hand side has a pleasing form. In order to achieve a balance between elegance and generality, he choose to employ the following four sequences that we now define.

Let a, b and p be arbitrary complex numbers such that $p^2 \neq -2, -4$ and $p \neq 0$. The roots $\alpha = \frac{p+\sqrt{\Delta}}{2}$ and $\beta = \frac{p-\sqrt{\Delta}}{2}$ of the equation $z^2 - p\,z - 1 = 0$ are distinct, where $\Delta = p^2 + 4$. Now the first two sequences are given by their Binet forms as

$$W_n = W(a,b,p,n) = \frac{A\,\alpha^n - B\,\beta^n}{\alpha - \beta}, \quad X_n = X(a,b,p,n) = A\,\alpha^n + B\,\beta^n$$

for any integer n, where $A = b - a\,\beta$ and $B = b - a\,\alpha$.

For $(a,b) = (0,1)$, we write $W_n = U_n$ and $X_n = V_n$. Then $\{U_n\}_{n\in\mathbb{Z}}$ and $\{V_n\}_{n\in\mathbb{Z}}$ are the third and the fourth sequence, respectively. Notice that W_n and X_n generalize U_n and V_n, respectively, which in turn generalize F_n and

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L_n (Fibonacci and Lucas numbers), respectively. Aspects of W_n and X_n have been treated, for example, in [1], [7], and [15], and more recently in [9].

We shall employ also another four similar sequences that we call dual and denote by the corresponding small letters which come from the equation $z^2 - p\,z + 1 = 0$ under the assumption that $p \neq 0$ and $p^2 \neq 2, 4$. Let $\delta = p^2 - 4$.

Let i, k and s denote arbitrary integers. Let $\ell = k + i$ and $m = k - i$.

As in [10], in each of our sums the lower limit is allowed to vary. Accordingly, we always assume the upper limit to be greater than the lower limit, and that either limit may be negative.

In Sections 2-5 we present our results that are collected into four sets of sums similar to the quadratic sums in Section 4 of [10] and then conclude by giving in Section 6 a sample proof.

2. The sums of products

Let a, b, c, d and p be arbitrary complex numbers. In this section besides the numbers $W_n = W(a, b, p, n)$, etc. we shall also use the numbers $W_n^* = W(c, d, p, n)$, etc. Hence, the presence of the star only indicates different initial values.

The formulas (4.1) and (4.2) in [10] consider the sums $\sum_{j=i}^{k} W_j\, U_j$ and $\sum_{j=i}^{k} W_j\, V_j$ for the products $W_j\, U_j = W(a, b, p, j)\, W(0, 1, p, j)$ and $W_j\, V_j = W(a, b, p, j)\, X(0, 1, p, j)$ when i and k are assumed to have different parities. Our extension replaces the products $W_j\, U_j$ and $W_j\, V_j$ with the products $W_{sj}\, W_{sj}^*$, $W_{sj}\, X_{sj}^*$ and $X_{sj}\, X_{sj}^*$, where s is any odd integer. Hence, our second factor is more general and instead of a single formula we obtain a sequence of formulas (one for each value of s).

Let $\underline{s} = 2\,s$ and $\overline{s} = 2\,s + 1$. Let Z and Z_j be $W_{\overline{s}\,\overline{i}}\, W^*_{\overline{s}\,\overline{k}} + W_{\overline{s}\,\underline{i}}\, W^*_{\overline{s}\,\underline{k}}$ and $W_{\overline{s}\,j}\, W^*_{\overline{s}\,j}$, respectively. Recall that $m = k - i$. For all integers i, k and s, the following identities hold:

$$U_{2\,\overline{s}} \sum_{j=e}^{f} Z_j = \begin{cases} U_{\overline{s}(\underline{m}+2)}\, Z, & \text{if } e = \underline{i} \text{ and } f = \overline{k}; \\ U_{\overline{s}\,\underline{m}}\, Z, & \text{if } e = \overline{i} \text{ and } f = \underline{k}. \end{cases} \tag{2.1}$$

We note that these sums for the products $W_{\overline{s}\,j}\, X^*_{\overline{s}\,j}$ and $X_{\overline{s}\,j}\, X^*_{\overline{s}\,j}$ are similar. One simply replaces Z with $W_{\overline{s}\,\overline{i}}\, X^*_{\overline{s}\,\overline{k}} + W_{\overline{s}\,\underline{i}}\, X^*_{\overline{s}\,\underline{k}}$ and $X_{\overline{s}\,\overline{i}}\, X^*_{\overline{s}\,\overline{k}} + X_{\overline{s}\,\underline{i}}\, X^*_{\overline{s}\,\underline{k}}$, respectively.

The analogous alternating sums of the numbers w_n are also products. Let $z_u = w_{\overline{s}\,\overline{i}+1}\, w^*_{\overline{s}\,\underline{k}} - w_{\overline{s}\,\overline{i}}\, w^*_{\overline{s}\,\underline{k}+1}$, $z_v = w_{\overline{s}\,\overline{i}}\, w^*_{\overline{s}\,\underline{k}+1} - w_{\overline{s}\,\overline{i}-1}\, w^*_{\overline{s}\,\underline{k}}$ and $z_j = w_{\overline{s}\,j}\, w^*_{\overline{s}\,j}$. Then it holds

$$v_{\overline{s}} \sum_{j=e}^{f} (-1)^j\, z_j = \begin{cases} u_{\overline{s}(\underline{m}+2)}\, z_u, & \text{if } e = \underline{i} \text{ and } f = \overline{k}; \\ u_{\overline{s}\,\underline{m}}\, z_v, & \text{if } e = \overline{i} \text{ and } f = \underline{k}. \end{cases} \tag{2.2}$$

For the numbers w_n the even multiplies of indices could be used. Let z_u, z_v and z_j denote $w_{\underline{s}\,\underline{i}+i}\, w^*_{\underline{s}\,\overline{k}-1} - w_{\underline{s}\,\underline{i}-1}\, w^*_{\underline{s}\,\overline{k}}$, $w_{\underline{s}\,\underline{i}+1}\, w^*_{\underline{s}\,\overline{k}} - w_{\underline{s}\,\underline{i}}\, w^*_{\underline{s}\,\overline{k}-1}$ and $w_{\underline{s}\,j}\, w^*_{\underline{s}\,j}$, respectively. Then

$$(2.3)\qquad v_{\underline{s}} \sum_{j=e}^{f} (-1)^j\, z_j = \begin{cases} u_{\underline{s}\,(\underline{m}+2)}\, z_u, & \text{if } e = \underline{i} \text{ and } f = \overline{k}; \\ u_{\underline{s}\,\underline{m}}\, z_v, & \text{if } e = \overline{i} \text{ and } f = \underline{k}. \end{cases}$$

The alternating sums of the products $W_j W_j^*$ need not be equal to products. However, for the following three products $Z_\alpha = W_{\underline{j}} W^*_{\underline{j}}$, $Z_\beta = W_{\underline{j}} W^*_{\overline{j}}$ and $Z_\gamma = W_{\underline{j}} W^*_{\underline{j+1}}$ these sums are products. Let S_α, S_β and S_γ denote the following sums $W_{4i+1} W^*_{4k+2} + W_{4i} W^*_{4k+1}$, $W_{4i+2} W^*_{4k+2} + W_{4i+1} W^*_{4k+1}$ and $W_{4i+2} W^*_{4k+3} + W_{4i+1} W^*_{4k+2}$. For $t = \alpha,\, \beta,\, \gamma$ the following identities are true:

$$(2.4)\qquad V_2 \sum_{j=e}^{f} (-1)^j\, Z_t = \begin{cases} -U_{4(m+1)}\, S_t, & \text{if } e = \underline{i} \text{ and } f = \overline{k}; \\ U_{4\,m}\, S_t, & \text{if } e = \overline{i} \text{ and } f = \underline{k}. \end{cases}$$

When for $t = \beta,\, \gamma$ in (2.4) we increase each index of the terms in S_t by one, we shall get formulas for alternating sums of the products $W_{\overline{j}} W^*_{\overline{j}+1}$ and $W_{\overline{j}} W^*_{\overline{j}+2}$, respectively.

Let $K = W_{\overline{\ell}} X^*_{\overline{\ell}} - (-1)^s (a\,d - b\,c)$. Let Z_g denote either $W_{\overline{j}s} W^*_{(\underline{j}-1)s}$ or $W_{(\underline{j}-1)s} W^*_{\overline{j}s}$. The following result also provides infinitely many alternating sums that are products:

$$(2.5)\qquad V_{\underline{s}} \sum_{j=e}^{f} (-1)^j\, Z_g = \begin{cases} -U_{4\,s(m+1)}\, K, & \text{if } e = \underline{i} \text{ and } f = \overline{k}; \\ U_{4\,s\,m}\, K, & \text{if } e = \overline{i} \text{ and } f = \underline{k}. \end{cases}$$

When $c = a$ and $d = b$, the above formula reduces to the following identities. Let $Z_h = W_{\overline{j}s} W_{(\underline{j}-1)s}$ and $Z_n = W_{\overline{\ell}\,s} X_{\overline{\ell}\,s}$. Then

$$(2.6)\qquad V_{\underline{s}} \sum_{j=e}^{f} (-1)^j\, Z_h = \begin{cases} -U_{4\,s(m+1)}\, Z_n, & \text{if } e = \underline{i} \text{ and } f = \overline{k}; \\ U_{4\,s\,m}\, Z_n, & \text{if } e = \overline{i} \text{ and } f = \underline{k}. \end{cases}$$

Similar sums that alternate according to the numbers $(-1)^{\frac{j(j+1)}{2}}$ are also products and are given by the following identities. Let μ_3, μ_4, μ_5 and μ_6 be pairs $(4i, 4k+1)$, $(4i, 4k+3)$, $(4i+2, 4k+1)$ and $(4i+2, 4k+3)$ and let $\omega = (e,\, f)$ be either μ_3, μ_4, μ_5 or μ_6. Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ denote the sum $W_{8i+x} W^*_{8k+x} + W_{8i+y} W^*_{8k+y}$ for $(x, y) = (2,1), (4,3), (6,5)$, respectively. Then

$$(2.7)\qquad \frac{V_4}{U_2} \sum_{j=e}^{f} (-1)^{\frac{j(j+1)}{2}}\, W_{\underline{j}} W^*_{\overline{j}} = \begin{cases} -V_{4\,\overline{m}}\, \mathfrak{A}, & \text{if } \omega = \mu_3; \\ V_{4\,\underline{m}}\, \mathfrak{B}, & \text{if } \omega = \mu_4; \\ -U_{4\,\underline{m}}\, \mathfrak{B}, & \text{if } \omega = \mu_5; \\ V_{4\,\overline{m}}\, \mathfrak{C}, & \text{if } \omega = \mu_6. \end{cases}$$

Let ν_3 and ν_4 be pairs of integers $(4i+1, 4k)$ and $(4i+3, 4k+2)$ and let $\omega = (e,\, f)$ be either ν_3 or ν_4. Then

$$\frac{V_4}{V_2\, U_{4\underline{k-i}}} \sum_{j=e}^{f} (-1)^{\frac{j(j+1)}{2}}\, W_{\underline{j}} W^*_{\overline{j}} = \begin{cases} \mathfrak{A}, & \text{if } \omega = \nu_3; \\ -\mathfrak{C}, & \text{if } \omega = \nu_4. \end{cases} \tag{2.8}$$

The products appear also in analogous sums that alternate according to the numbers $(-1)^{\frac{j(j-1)}{2}}$. Let μ_3, μ_4, μ_5 and μ_6 be pairs $(4i+1, 4k)$, $(4i+1, 4k+2)$, $(4i+3, 4k)$ and $(4i+3, 4k+2)$ and let $\omega = (e,\, f)$ be either μ_3, μ_4, μ_5 or μ_6. Then

$$\frac{V_4}{U_2} \sum_{j=e}^{f} (-1)^{\frac{j(j-1)}{2}}\, W_{\underline{j}} W^*_{\overline{j}} = \begin{cases} U_{4\,\underline{m}}\, \mathfrak{A}, & \text{if } \omega = \mu_3; \\ -V_{4\,\overline{m}}\, \mathfrak{B}, & \text{if } \omega = \mu_4; \\ V_{4\,\underline{m}-4}\, \mathfrak{B}, & \text{if } \omega = \mu_5; \\ -U_{4\,\underline{m}}\, \mathfrak{C}, & \text{if } \omega = \mu_6. \end{cases} \tag{2.9}$$

Let ν_3 and ν_4 be pairs of integers $(4i, 4k+3)$ and $(4i+2, 4k+1)$ and let $\omega = (e,\, f)$ be either ν_3 or ν_4. Then

$$\frac{V_4}{V_2\, \mathfrak{B}} \sum_{j=e}^{f} (-1)^{\frac{j(j-1)}{2}}\, W_{\underline{j}} W^*_{\overline{j}} = \begin{cases} -U_{4\,\underline{m}+8}, & \text{if } \omega = \nu_3; \\ U_{4\,\underline{m}}, & \text{if } \omega = \nu_4. \end{cases} \tag{2.10}$$

3. Sums of squares

Recall that the first formula from the following pair

$$\sum_{j=1}^{k} F_j^2 = F_k\, F_{k+1}, \qquad \sum_{j=1}^{k} L_j^2 = L_k\, L_{k+1} - 2 \tag{3.1}$$

was the motivation in [10] to undertake a task to explore those sums of (generalized) Fibonacci and Lucas numbers where the right side has a pleasing form. The identity [10, (4.18)] is the only other sum for squares of Fibonacci numbers that appears in [10]. In this section a more extensive study of sums of squares of (generalized) Fibonacci and Lucas numbers is given. In this exposition from less to more general formulas, it is possible that some of those identities are already known.

Let H denote either F or L. The following generalizations

$$\sum_{j=i}^{\ell} H_j^2 = H_\ell\, H_{\ell+1} - H_i\, H_{i-1} \tag{3.2}$$

suggests that we might vary lower and upper limits of summation as much as possible (preferably keep them arbitrary as in (3.2)) provided the right side is still relatively simple.

When the part k in the upper limit $\ell = k + i$ is odd, then these sums factor as follows. Let $M_3 = F_\ell\, L_{\ell+1} + (-1)^\ell = F_\ell^2 + F_{\ell+1}^2$. Then

$$\sum_{j=i}^{\overline{k}+i} F_j^2 = \tfrac{1}{5} \sum_{j=i}^{\overline{k}+i} L_j^2 = F_{\overline{k}+1}\, M_3. \tag{3.3}$$

The sums that can be expressed as a product are particularly pleasing. For example, the factor $F_{\overline{k}+1}$ in (3.3) depends only on the number of terms in the sum and therefore has a greater importance.

When this k is even, then the sums almost factor as follows. Let M_4 denote $F_i^2 + F_{\underline{k}}\, F_{\underline{\ell}+1}$ while $N_4 = L_i^2 + 5\, F_{\underline{k}}\, F_{\underline{\ell}+1} = 2(-1)^i - 5\, F_{\underline{i}} + L_{\underline{k}}\, L_{\underline{\ell}+1}$.

$$\sum_{j=i}^{\underline{k}+i} H_j^2 = \begin{cases} M_4, & \text{if } H = F; \\ N_4, & \text{if } H = L. \end{cases} \tag{3.4}$$

The word "almost" refers that the right hand side is a product plus or minus a number that is small in respect to the sum itself. In the following versions of these identities this distance from a product is even smaller. Let $M_5 = L_{\overline{k}}\, L_{\underline{\ell}}$. Then

$$\sum_{j=i}^{\underline{k}+i} F_j^2 = \tfrac{1}{5} \left[\, M_5 - 2(-1)^i \,\right], \qquad \sum_{j=i}^{\underline{k}+i} L_j^2 = M_5 + 2(-1)^i. \tag{3.5}$$

Let Y be either W or X. Let $\widetilde{Y}$ be X if $Y = W$ and W if $Y = X$. The dual numbers are y and $\widetilde{y}$. For the numbers Y_n, the analogue of (3.2) is as follows:

$$p \sum_{j=i}^{\ell} Y_j^2 = Y_\ell\, Y_{\ell+1} - Y_i\, Y_{i-1}. \tag{3.6}$$

Let $\lambda = a^2 - b^2 + a\, b\, p$, $\Theta_W = 1$ and $\Theta_X = \Delta$. The relations (3.3)–(3.5) extend similarly. The upper or lower sign applies if $Y = W$ or $Y = X$. Let M_6 denote $\Theta_Y \left[Y_\ell\, \widetilde{Y}_{\ell+1} \mp (-1)^\ell\, \lambda \right]$. Note that $M_6 = Y_\ell^2 + Y_{\ell+1}^2$. Then we have

$$p \sum_{j=i}^{\overline{k}+i} Y_j^2 = U_{\overline{k}+1}\, M_6. \tag{3.7}$$

Let $M_7 = W_{\underline{i}+1} + 2\, W_{\underline{i}-1} + W_{\underline{i}-3}$. Then

$$p\, \Theta_{\widetilde{Y}} \sum_{j=i}^{\underline{k}+i} Y_j^2 = X_{\underline{k}+1}\, X_{\underline{\ell}} - a\, M_7 \pm 2(-1)^i\, p\lambda. \tag{3.8}$$

Let $M_8 = a\,\Delta\,[b + U_{i-1}\,Q_+]$ and $N_8 = a\,\Delta\left[b + \frac{V_{i-1}\,Q_-}{p}\right]$, where $Q_\pm = W_{\underline{i}+1}$ $\pm W_{\underline{i}-1}$. Then

$$p\,\Theta_{\widetilde{Y}} \sum_{j=i}^{\underline{k}+i} Y_j^2 = \begin{cases} X_{\underline{k}+1}\,X_{\underline{\ell}} - M_8 \mp 2\,p\lambda, & \text{if } i \text{ is odd;} \\ X_{\underline{k}+1}\,X_{\underline{\ell}} - N_8 \pm 2\,p\lambda, & \text{if } i \text{ is even.} \end{cases} \tag{3.9}$$

Another variation is the following:

$$p \sum_{j=i+1}^{\underline{k}+i} Y_j^2 = Y_{\underline{k}}\,Y_{\underline{\ell}+1} - Y_0\,Y_{\underline{i}+1}. \tag{3.10}$$

The factorization occurs also in the following sums. Let $t = 3\,\ell$. Then

$$\frac{V_3}{U_{6(k+1)}} \sum_{j=i}^{\overline{k}+i} Y_{3j}^2 = Y_{t+2}^2 + Y_{t+1}^2 = Y_t\,\widetilde{Y}_{t+3} \mp (-1)^\ell\,\lambda\,U_3. \tag{3.11}$$

This is the second formula (for $s = 1$) of the the following family of formulas beginning with the above formula (3.7) (for $s = 0$). Let M_9 be $Y_{s+\overline{s}\,\ell+1}^2 + Y_{s+\overline{s}\,\ell}^2$. Note that

$$M_9 = \Theta_Y \left[Y_{\overline{s}\,\ell}\widetilde{Y}_{\overline{s}(\ell+1)} \mp (-1)^\ell\,\lambda\,U_{\overline{s}}\right].$$

It holds

$$V_{\overline{s}} \sum_{j=i}^{\overline{k}+i} Y_{\overline{s}j}^2 = \Theta_Y\,U_{(\overline{k}+1)\overline{s}}\,M_9. \tag{3.12}$$

The sums of consecutive squares $Y_{\underline{s}j}^2$ are not products. However, they retain some overall similarities with the above sums. Let M_{10} and N_{10} be $X_{\underline{s}(\underline{\ell}+2)} + X_{\underline{s}\underline{\ell}}$ and $W_{\underline{s}\,\underline{i}-\underline{s}+1} + W_{\underline{s}\,\underline{i}-\underline{s}-1}$, respectively. Then

$$\Theta_{\widetilde{Y}}\,U_{4s} \sum_{j=i}^{\overline{k}+i} Y_{\underline{s}j}^2 = W_{\underline{s}(\overline{k}+1)}\,M_{10} - V_{\underline{s}}\left[a\,N_{10} \mp 4\,\lambda(k+1)\,U_{\underline{s}}\right]. \tag{3.13}$$

When the limits of summation are i and $\underline{k} + i$, then we have the following formulas. Let $\xi_3 = \frac{V_{\overline{s}}}{U_{\overline{s}\underline{k}}}$ and $\zeta_3 = \xi\,Y_{\overline{s}\,i}^2$. Then

$$\xi_3 \sum_{j=i}^{\underline{k}+i} Y_{\overline{s}j}^2 = \zeta_3 + M_9. \tag{3.14}$$

The alternating sum $\sum_{j=i}^{\overline{k}+i} (-1)^j\,Y_j^2$ is not a product. However, for the numbers y_n, we have the following result. Let $\mu = a^2 + b^2 - abp$, $\theta_w = 1$,

$\theta_x = \delta$ and $\xi_4 = (-1)^i \, \frac{p}{u_{\underline{k+1}}}$. Then

$$\xi_4 \sum_{j=i}^{\overline{k}+i} (-1)^j \, y_j^2 = \mp \, \theta_y \, [\mu \pm y_\ell \, \widetilde{y}_{\ell+1}] = y_\ell^2 - y_{\ell+1}^2. \tag{3.15}$$

These formulas are the initial ones of the following series of formulas. Let $\xi_5 = (-1)^i \, \frac{v_{\overline{s}}}{u_{\overline{s}(\overline{k}+1)}}$ and

$$m_9 = y^2_{s+\overline{s}\,\ell} - y^2_{s+\overline{s}\,\ell+1} = \mp \, \theta_y \left[\mu \, u_{\overline{s}} \pm y_{\overline{s}\,\ell} \, \widetilde{y}_{\overline{s}(\ell+1)}\right].$$

Then we have

$$\xi_5 \sum_{j=i}^{\overline{k}+i} (-1)^j \, y^2_{\overline{s}j} = m_9. \tag{3.16}$$

The following formula gives the factorization for the alternating sum of an even number of consecutive squares of the numbers $Y_{s\,j}$ for an even number s:

$$V_{\underline{s}} \sum_{j=i}^{\overline{k}+i} (-1)^j \, Y^2_{\underline{s}\,j} = (-1)^{i+1} \, \Theta_Y \, U_{\underline{s}(\overline{k}+1)} \, Y_{s+\underline{s}\,\ell} \, \widetilde{Y}_{s+\underline{s}\,\ell}. \tag{3.17}$$

For the numbers y_n the similar formulae hold. One has to replace the letters V, Y, Θ and U by the corresponding small letters.

4. Alternation according to $(-1)^{\frac{j(j+1)}{2}}$ and $(-1)^{\frac{j(j-1)}{2}}$

The most identities in [10, Section 4] (more precisely (4.4)-(4.15)) consider sums of products $W_n \, U_n$, $X_n \, U_n$, $W_n \, V_n$ and $X_n \, V_n$ that alternate according to the numbers $(-1)^{\frac{n(n+1)}{2}}$. In this section we treat analogous sums for squares of the numbers W_n and its duals numbers w_n and as above include also alternation according to the numbers $(-1)^{\frac{n(n-1)}{2}}$.

For an integer f, let $f_0 = s + 4f\,\overline{s} + 1$, $f_1 = s + (4f+1)\overline{s} + 1$ and $f_2 = s + (4f+2)\overline{s} + 1$. Let $S_\uparrow(f,g) = V_{2\,\overline{s}} \sum_{j=f}^{g} (-1)^{\frac{j(j+1)}{2}} \, W^2_{\overline{s}\,j}$. If $q_a = \frac{S_\uparrow(4i,\,4k+3)}{V_{\overline{s}}\,U_{4(m+1)\overline{s}}}$, $q_b = \frac{S_\uparrow(4i+1,\,4k+2)}{U_{\overline{s}}\,V_{2(2\,m+1)\overline{s}}}$, $q_c = \frac{S_\uparrow(4i+2,\,4k+1)}{V_{\overline{s}}\,U_{4\,m\overline{s}}}$, $q_d = \frac{S_\uparrow(4i+3,\,4k)}{U_{\overline{s}}\,V_{2(2\,m-1)\overline{s}}}$ and $\mathfrak{q} = W_{i_1} \, W_{k_1} + W_{i_1-1} \, W_{k_1-1}$, then

$$q_a = -q_b = -q_c = q_d = \mathfrak{q}. \tag{4.1}$$

Let $\mathfrak{P}$ and $\mathfrak{R}$ be $W_{i_0} \, X_{k_0} + W_{i_0-1} \, X_{k_0-1}$ and $W_{i_2} \, X_{k_2} + W_{i_2-1} \, X_{k_2-1}$. Two additional such sums are the following:

$$\frac{S_\uparrow(4i+1,\,4k)}{U_{\overline{s}}\,U_{4\,m\overline{s}}} = \mathfrak{P}, \qquad \frac{S_\uparrow(4i+3,\,4k+2)}{U_{\overline{s}}\,U_{4\,m\overline{s}}} = -\mathfrak{R}. \tag{4.2}$$

The versions of the above formulas for the alternation according to the numbers $(-1)^{\frac{j(j-1)}{2}}$ are the following. Let $\mathfrak{p} = W_{i_0} \, W_{k_0} + W_{i_0-1} \, W_{k_0-1}$, $\mathfrak{r} =$

$W_{i_2}\, W_{k_2} + W_{i_2-1}\, W_{k_2-1}$ and $\mathfrak{Q} = W_{i_1}\, X_{k_1} + W_{i_1-1}\, X_{k_1-1}$. Let $S_{\downarrow}(f,g)$ denote the sum $V_{2\,\overline{s}} \sum_{j=f}^{g} (-1)^{\frac{j(j-1)}{2}}\, W_{\overline{s}\,j}^2$. Then it holds

$$\frac{S_{\downarrow}(4i,\, 4k+1)}{U_{\overline{s}}\, V_{(4\,m+2)\overline{s}}} = \frac{S_{\downarrow}(4i+1,\, 4k)}{V_{\overline{s}}\, U_{4\,m\,\overline{s}}} = \mathfrak{p}, \tag{4.3}$$

$$-\frac{S_{\downarrow}(4i,\, 4k+3)}{U_{\overline{s}}\, U_{(4\,m+1)\overline{s}}} = \frac{S_{\downarrow}(4i+2,\, 4k+1)}{U_{\overline{s}}\, U_{4\,m\,\overline{s}}} = \mathfrak{Q}, \tag{4.4}$$

$$\frac{S_{\downarrow}(4i+2,\, 4k+3)}{U_{\overline{s}}\, V_{(4\,m+2)\overline{s}}} = \frac{S_{\downarrow}(4i+3,\, 4k+2)}{V_{\overline{s}}\, U_{4\,m\,\overline{s}}} = -\mathfrak{r}. \tag{4.5}$$

Let $\sigma_{\uparrow}^{s}(f,g) = \sum_{j=f}^{g} (-1)^{\frac{j(j+1)}{2}}\, w_{s\,j}^2$ and $\sigma_{\downarrow}^{s}(f,g) = \sum_{j=f}^{g} (-1)^{\frac{j(j-1)}{2}}\, w_{s\,j}^2$. These sums are also products for all multiples of the index j. Let q_e and q_f denote the quotients $\frac{v_{\underline{s}}\, \sigma_{\uparrow}^{s}(4i,\, 4k+1)}{v_{4\,m\,s+2s}}$ and $\frac{u_s\, v_{\underline{s}}\, \sigma_{\uparrow}^{s}(4i+1,\, 4k)}{v_s\, u_{4\,m\,s}}$, respectively. Then

$$-q_e = q_f = w_{2\,\ell\,s+s}^2 - w_{2\,\ell\,s}^2. \tag{4.6}$$

Let $q_g = \frac{v_{\underline{s}}\, \sigma_{\uparrow}^{s}(4i+2,\, 4k+3)}{v_{4\,m\,s+2s}}$ and $q_h = \frac{u_s\, v_{\underline{s}}\, \sigma_{\uparrow}^{s}(4i+3,\, 4k+2)}{v_s\, u_{4\,m\,s}}$. Then it holds

$$q_g = -q_h = w_{2\,\ell\,s+3s}^2 - w_{2\,\ell\,s+2s}^2, \tag{4.7}$$

$$v_{2\,\underline{s}}\, \sigma_{\uparrow}^{\underline{s}}(4i+2,\, 4k+1) = u_{\underline{s}}\, v_{4\,m\,\underline{s}} \left[\lambda v_{8\,m\,s} - x_{(8i+3)s}\, x_{(8k+3)s}\,\right], \tag{4.8}$$

$$v_{2\,\overline{s}}\, \sigma_{\uparrow}^{\overline{s}}(4i+2,\, 4k+1) = u_{\overline{s}}\, u_{4\,m\,\overline{s}} \left[w_{i_1-1}\, x_{k_1-1} - w_{i_1}\, x_{k_1}\,\right], \tag{4.9}$$

$$v_{2\,\underline{s}}\, \sigma_{\uparrow}^{\underline{s}}(4i,\, 4k+3) = u_{\underline{s}}\, v_{4\,m\,\underline{s}+8s} \left[x_{4\,m\,\underline{s}+5s}^2 - 2\,\lambda\right], \tag{4.10}$$

$$v_{2\,\overline{s}}\, \sigma_{\uparrow}^{\overline{s}}(4i,\, 4k+3) = u_{\overline{s}}\, u_{4(m+1)\overline{s}} \left[w_{\ell_1}\, x_{\ell_1} - w_{\ell_1-1}\, x_{\ell_1-1}\,\right]. \tag{4.11}$$

The following formulas show the analogous results for the alternation according to $(-1)^{\frac{j(j-1)}{2}}$. Let $\sigma_{\downarrow}(f,g) = v_{\underline{s}}\, \sigma_{\downarrow}^{s}(f,g)$, $q_L = \frac{u_s\, \sigma_{\downarrow}(4i,\, 4k+3)}{v_s\, u_{4(m+1)s}}$,

$$q_M = \frac{\sigma_{\downarrow}(4i+1,\, 4k+2)}{v_{(\underline{m}+1)\underline{s}}},\; q_N = \frac{\sigma_{\downarrow}(4i+2,\, 4k+1)}{v_{(\underline{m}-1)\underline{s}}},\; q_O = \frac{u_s\, \sigma_{\downarrow}(4i+3,\, 4k)}{v_s\, u_{4\,m\,s}}.$$

If $\mathfrak{s}$ denotes the difference $w_{2s(\ell+1)}^2 - w_{s(2\,\ell+1)}^2$, then

$$-q_L = -q_M = q_N = q_O = \mathfrak{s}, \tag{4.12}$$

$$v_{2\,\underline{s}}\, \sigma_{\downarrow}^{\underline{s}}(4i+3,\, 4k+2) = u_{\underline{s}}\, u_{4\,m\,\underline{s}} \left[2\,\lambda - x_{(4\,\ell+5)s}^2\right], \tag{4.13}$$

$$v_{2\,\overline{s}}\, \sigma_{\downarrow}^{\overline{s}}(4i+3,\, 4k+2) = u_{\overline{s}}\, u_{4\,m\,\overline{s}} \left[w_{i_2-1}\, x_{k_2-1} - w_{i_2}\, x_{k_2}\right], \tag{4.14}$$

$$v_{2\,\underline{s}}\, \sigma_{\downarrow}^{\underline{s}}(4i+1,\, 4k) = u_{\underline{s}}\, u_{4\,m\,\underline{s}} \left[w_{4\ell s+s+1}\, x_{4\ell+s} - w_{4\ell+s}\, x_{4\ell+s-1}\right], \tag{4.15}$$

$$v_{2\,\overline{s}}\,\sigma_{\downarrow}^{\overline{s}}(4i+1,\,4k) = u_{\overline{s}}\,u_{4\,m\,\overline{s}}\left[w_{\ell_0}\,x_{\ell_0} - w_{\ell_0-1}\,x_{\ell_0-1}\right]. \tag{4.16}$$

5. Sums with binomial coefficients

In this section we consider certain finite sums of squares with binomial coefficients that are not present in [10]. Of course, we selected those that have pleasing right hand sides.

Recall that $\ell = k+i$. Let $t = \ell + g\,s$. Then

$$\frac{1}{\Theta_Y\,\Delta^{k-1}\,U_{\overline{s}}^{g}}\sum_{j=0}^{g}\binom{g}{j}Y_{i+\overline{s}\,j}^2 = \begin{cases} X_t^2 + 2\,(-1)^{\ell}\,\lambda, & \text{if } g = \underline{k};\\ \Delta\left[W_{t+1}^2 + W_t^2\right], & \text{if } g = \overline{k}.\end{cases} \tag{5.1}$$

Recall that $\delta = p^2 - 4$ and $\mu = a^2 + b^2 - a\,b\,p$. We have

$$\frac{1}{\theta_y\,\delta^{k-1}\,u_{\underline{s}}^{g}}\sum_{j=0}^{g}(-1)^j\binom{g}{j}y_{i+\underline{s}\,j}^2 = \begin{cases} x_{i+g\,s}^2 - 2\,\mu, & \text{if } g = \underline{k};\\ -\delta\,w_{i+g\,s}\,x_{i+g\,s}, & \text{if } g = \overline{k}.\end{cases} \tag{5.2}$$

Let $m_{10} = w_{\ell+g\,s}^2 - w_{\ell+g\,s+1}^2$. Then we have

$$\frac{1}{\theta_y\,\delta^{k-1}\,u_{\overline{s}}^{g}}\sum_{j=0}^{g}(-1)^j\binom{g}{j}y_{i+\overline{s}\,j}^2 = \begin{cases} x_{\ell+g\,s}^2 - 2\,\mu, & \text{if } g = \underline{k};\\ \delta\,m_{10}, & \text{if } g = \overline{k}.\end{cases} \tag{5.3}$$

Let $t = \ell + (g+1)s$ and $m_{11} = X_{(k+1)\overline{s}+i}^2 - 2(-1)^{\ell}\,\lambda$. Then

$$\frac{1}{g\,\Theta_Y\,\Delta^{k-1}\,U_{\overline{s}}^{g-1}}\sum_{j=0}^{g} j\binom{g}{j}Y_{i+\overline{s}\,j}^2 = \begin{cases} W_{t+1}^2 + W_t^2, & \text{if } g = \underline{k};\\ m_{11}, & \text{if } g = \overline{k}.\end{cases} \tag{5.4}$$

Let $m_{12} = w_{(g+1)s+i}\,x_{(g+1)s+i}$ and $m_{13} = 2\,\mu - x_{(g+1)s+i}^2$. Then

$$\frac{1}{g\,\theta_y\,\delta^{k-1}\,u_{\underline{s}}^{g-1}}\sum_{j=0}^{g} j(-1)^j\binom{g}{j}y_{i+\underline{s}\,j}^2 = \begin{cases} m_{12}, & \text{if } g = \underline{k};\\ m_{13}, & \text{if } g = \overline{k}.\end{cases} \tag{5.5}$$

Let $m_{14} = w_{(g+1)s+\ell+1}^2 - w_{(g+1)s+\ell}^2$ and $m_{15} = 2\,\mu - x_{(g+1)s+\ell+1}^2$. Then

$$\frac{1}{g\,\theta_y\,\delta^{k-1}\,u_{\overline{s}}^{g-1}}\sum_{j=0}^{g} j(-1)^j\binom{g}{j}y_{i+\overline{s}\,j}^2 = \begin{cases} m_{14}, & \text{if } g = \underline{k};\\ m_{15}, & \text{if } g = \overline{k}.\end{cases} \tag{5.6}$$

6. A sample proof

Each of our formulas can be discovered and proved with the help of a computer. Here we describe how to do this for the first formula in (3.12) (for $Y = W$).

In Maple V, the following code performs search for those sums of consecutive terms of the sequences $(W_{tj}^2)_{j\in\mathbb{Z}}$ which are products. In order to figure out some pattern, we consider the values i, n and t in some small ranges (for example, $1 \le i \le 4$, $1 \le n \le 10$ and $1 \le t \le 10$). We use the fact that the products have considerably smaller number of parts than the high order polynomials that do not factor.

```
for i from 1 to 4 do for n from 1 to 10 do for t from 1 to 10 do
if nops(factor(add(W(a,b,p,t*j)^2,j=i..i+n)))<8 then
print([i,n,t]);fi;od;od;od;
```

Of course, the function `W` is the Binet form of the numbers W_n from the introduction.

As a result of this search, we conclude that products show up when the integers t and n are both odd.

Next we repeat the above search for $n = \overline{k}$ and $t = \overline{s}$ for a small range of values k and s and take a closer look into the factors. One factor contains a, b and p, while all other factors are polynomials only in p and are (factors of the) polynomials U_n. The trick here is to multiply the sum $\sum_{j=i}^{\overline{k}+i} W_{\overline{s}j}^2$ with $V_{\overline{s}}$ in order to conclude that these factors of the second kind (that do not contain a and b) are in fact $\Theta_Y\, U_{(\overline{k}+1)\overline{s}}$.

Finally, the first factor (the one containing a, b and p) does resemble numbers W_n^2 but not quite. The idea here is to select the index n simply to eliminate the leading term of the first factor (considered as a polynomial in p). It is useful here to remember that polynomials (in p) on both sides of (3.12) have equal degrees. After we subtract this $W^2_{s+\ell\,\overline{s}+1}$, what remains is $W^2_{s+\ell\,\overline{s}}$. The first factor is therefore $W^2_{s+\ell\,\overline{s}+1} + W^2_{s+\ell\,\overline{s}}$.

Hence, so far we know that the identity

$$(*) \qquad V_{\overline{s}} \sum_{j=i}^{\overline{k}+i} W_{\overline{s}j}^2 = U_{(\overline{k}+1)\overline{s}} \left[W^2_{s+\ell\,\overline{s}+1} + W^2_{s+\ell\,\overline{s}} \right]$$

holds for some very small ranges of values i, k and s. In order to prove that it is true for all values of i, k and s, note that $W_n = a\,U_n + b\,U_{n-1}$ holds for any integer n. Since

$$\Delta\, U_n^2 = V_{2n} - 2\,(-1)^n \qquad \text{and} \qquad \Delta\, U_n\, U_{n-1} = V_{2n-1} + p\,(-1)^n,$$

the product $\Delta\, W^2_{\overline{s}\,j}$ can be written as

$$(V_{2\,\overline{s}\,j-2} + 2\,(-1)^j)a^2 + 2(V_{2\,\overline{s}\,j-1} + p\,(-1)^j)a\,b + (V_{2\,\overline{s}\,j} - 2\,(-1)^j)b^2.$$

In this way, the evaluation of the left hand side of $(*)$ is reduced to the sums $\omega_g = \sum_{j=i}^{\overline{k}+i} V_{2\,\overline{s}\,j-g} = \frac{\alpha^{2\,\overline{s}(\ell+k+2)-g} - \alpha^{2\,\overline{s}\,i-g}}{\alpha^{2\,\overline{s}}-1} + \frac{\beta^{\sigma(\ell+k+2)-g} - \beta^{2\,\overline{s}\,i-g}}{\beta^{2\,\overline{s}}-1}$ for $g = 0, 1, 2$. The parts with $(-1)^j$ do not contribute anything because the number of terms in the sum is even.

The left hand side of $(*)$ is $LHS = M\,a^2 + 2\,N\,a\,b + P\,b^2$ while the right hand side of $(*)$ is $RHS = M^*\,a^2 + 2\,N^*\,a\,b + P^*\,b^2$, where $M = V_{\overline{s}}\,\omega_2$, $N = V_{\overline{s}}\,\omega_1$, $P = V_{\overline{s}}\,\omega_0$, $M^* = U_{(\overline{k}+1)\overline{s}}\,(U^2_{s+\ell\,\overline{s}-1} + U^2_{s+\ell\,\overline{s}})$,

$$N^* = U_{(\overline{k}+1)\overline{s}}\,U_{s+\ell\,\overline{s}}(U_{s+\ell\,\overline{s}-1} + U_{s+\ell\,\overline{s}+1})$$

and $P^* = U_{(\overline{k}+1)\overline{s}}\,(U^2_{s+\ell\,\overline{s}+1} + U^2_{s+\ell\,\overline{s}})$.

If we replace each β in the difference $M - M^*$ with $-\frac{1}{\alpha}$ we get zero. Similarly, we conclude that $N = N^*$ and $P = P^*$ so that $LHS = RHS$. Hence, the first formula in (3.12) (for $Y = W$) is true for all integer values of i, k and s.

7. Concluding comments

The concluding comments in [10] explain the role of Russell's papers [11] and [12] in this area and how the contribution [10] and therefore also the paper [4] (only for the linear sums) and the present paper (for quadratic sums) want to explore those cases when these formulas are particularly simple. Of course, this is possible only for more specialized Horadam numbers like Y_n and y_n.

We hope that the paper [4] (for linear sums) and this paper on quadratic sums together constitute a partial realization of Melham's prediction "we expect that there is scope for further research along the lines that we set forth" on the first page of [10].

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Formule za kvadratične sume poopćenih Fibonaccijevih i Lucasovih brojeva

Zvonko Čerin

Sažetak. Ovaj članak poboljšava Melhamove formule u [10, Odjeljak 4] za neke klase konačnih suma poopćenih Fibonaccijevih i Lucasovih brojeva. Ovdje se promatraju kvadratne sume gdje se koriste produkti dva takva broja. Dobiveni rezultati pokazuju da su njegove formule početni članovi nizova formula, da slični i nešto jednostavniji identiteti vrijede za pridružene dualne brojeve i da se pored alternacije po brojevima $(-1)^{\frac{n(n+1)}{2}}$ mogu dobiti slične formule za alternacije po brojevima $(-1)^{\frac{n(n-1)}{2}}$. Pored toga, promatra se dvanaest kvadratnih suma s binomnim koeficijentima koje su produkti.

Zvonko Čerin
Kopernikova 7, 10020 Zagreb, Croatia
E-mail: `cerin@math.hr`

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