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ORTHOGONAL PROJECTION OF AN INFINITE ROUND CONE IN REAL HILBERT SPACE

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Abstract. We fully characterize orthogonal projections of infinite right circular (round) cones in real Hilbert spaces. Another interpretation is that, given two vectors in a real Hilbert space, we establish the optimal estimate on the angle between the orthogonal projections of the two vectors. The estimate depends on the angle between the two vectors and the position of only one of the two vectors. Our results also make a contributions to Cauchy-Bunyakovsky-Schwarz type inequalities.

1. Introduction and literature overview

Let us introduce the topic in two simple settings.

Example 1.1. Let $C_{\mathbb{R}^2}(v,\varphi) = \left\{u \in \mathbb{R}^2 \,:\, \langle u, v\rangle \geq \cos\varphi \,\|u\|\,\|v\|\right\}$. We call it a filled angle or a one-sided infinite cone in $\mathbb{R}^2$. Let V be any line trough the origin and P any projection on V. Not just orthogonal, but any projection of $C(v,\varphi)$ on V can be characterized as either a whole line, or a closed half-line (bounded with the origin), or just the origin.

From this point on, we investigate orthogonal projections only.

Problem 1.1. *The one-sided right circular (round) infinite cone in* $\mathbb{R}^3$ *with apex in the origin, half-aperture* $\varphi \in [0,\pi]$*, and axis direction given by vector* $v \in \mathbb{R}^3$ *is defined by*

$$C_{\mathbb{R}^3}(v,\varphi) \stackrel{\text{def}}{=} \left\{u \in \mathbb{R}^3 \,:\, \langle u, v\rangle \geq \cos\varphi \,\|u\|\,\|v\|\right\}. \tag{1.1}$$

Let V *be a two-dimensional subspace in* $\mathbb{R}^3$*. What is the orthogonal projection of* $C_{\mathbb{R}^3}(v,\varphi)$ *onto* V*? If the orthogonal projection is a round infinite cone in* V*, what is its direction and aperture?*

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The aim of this paper is to solve a generalization of Problem 1.1 to real Hilbert spaces. Our main result is Theorem 2.2. When applied to Problem 1.1, Theorem 2.2 distinguishes among three cases, based on φ (the angle between u and v) and the angle between v and V. Interestingly, orthogonal projections of round infinite cones in any real Hilbert space produce only these three cases already present in three dimensional Euclidean space.

1. In the case when $\angle(v,V) > \frac{\pi}{2} - \varphi$, the orthogonal projection of the cone is the whole subspace V.
2. When $\angle(v,V) < \frac{\pi}{2} - \varphi$, the orthogonal projection $P[C_{\mathbb{R}^3}(v,\varphi)]$ is $C_{\mathbb{R}^2}(Pv,\varphi_1)$, which is a cone in V with apex in the origin, the axis direction given by Pv and half-aperture[1]

$$\varphi_1 = \arccos\sqrt{\frac{\cos^2\varphi - \sin^2\angle(v,V)}{1-\sin^2\angle(v,V)}}\,. \tag{1.2}$$

3. The border case, when $\angle(v,V) = \frac{\pi}{2} - \varphi$, further depends on φ.
   (a) When $\varphi = 0$ and $v \perp V$, then $P[C_{\mathbb{R}^3}(v,0)] = \{(0,0,0)\}$.
   (b) When $v \in V$ and $\varphi = \pi/2$, then $P[C_{\mathbb{R}^3}(v,\pi/2)]$ is a cone in V with half-aperture $\pi/2$ and axis given by Pv (a closed half-space in V).
   (c) When $0 < \varphi < \pi/2$ and $\angle(v,V) = \frac{\pi}{2} - \varphi$, then the projection $P[C_{\mathbb{R}^3}(v,\varphi)]$ is the union of the interior of the half-space in V oriented by Pv and the origin:

$$P[C_{\mathbb{R}^3}(v,\varphi)] = \{(0,0,0)\} \cup \{y \in V \,:\, \langle y, Pv\rangle > 0\}\,.$$

Cones are well known objects in Hilbert space [3, p. 86], of which round cones (right circular cones) are a special case. Infinite round cone is a rotational body with a filled angle as a radial cross-section. Therefore, the problem of round cone projection is related to problems of planar angle projection, which have long been studied in three-dimensional (3D) Euclidean space [2,6,9–12]. Papers [9] and [2] studied the relationship between a fixed planar angle and the orthogonal projection of that angle to another plane, which is another planar angle. Their method was to set up the appropriate coordinate system and then analytically calculate the formulas for value and position of the projected angle. These formulas could be applied to the rotational body in order to deduce formula (1.2) in 3D Euclidean space. On the other hand, the projection of an infinite round cone in 3D could be computed by "extending to infinity" the projected area of a finite right circular cone [7]. Note that applying coordinatization to circular cones in [14] provided results valid only in finite dimensions.

Our method is different and suits naturally to general Hilbert space setting. We express geometric intuition from 3D in terms of the standard inner

[1] Formula for φ_1 in Theorem 2.2 is in terms of $\angle\left(v, V^{\perp}\right)$.

product calculus and real analysis. In equation (1.1), a round cone in 3D is defined using a so-called reverse Cauchy-Bunyakovsky-Schwarz (CBS) type inequality, and we will use the same inequality to define a round cone in general Hilbert space (Section 2). In Section 3, we show few applications, among which Example 3.5 is in (infinite dimensionsional) Lebesgue space.

Fifty years ago, paper [4, p. 89] mentioned connection between cones and reverse[2] triangle inequality in Hilbert spaces. Thus, it is not surprising that results on round infinite cone projections are directly related to new CBS-type inequalities (Section 4).

Known reverse CBS inequalities [5] provide ways to estimate $\cos\varphi$ in (1.1) from below, based on some knowledge about the projections of u and v. For example, the Pólya-Szegö inequality in $\mathbb{R}^n$ [8] estimates $\cos\varphi$ in (1.1) based on lower and upper bounds of coordinates $m_u \le u_i \le M_u$ and $m_v \le v_i \le M_v$. Cassels' inequality [13, page 330] and its refinement by Andrica and Badea [1] provide a bound on $\cos\varphi$ in (1.1) based on the bounds of the ratio $m \le {}^{u_i}\!/_{v_i} \le M$. On the other hand, in this paper we estimate the angle (1.2) between orthogonal projections Pu and Pv based on the value of $\cos\varphi$ in (1.1). Our result on a reverse CBS inequality, Theorem 4.3, by contraposition gives a sufficient condition for an estimate that is more strict then the classical CBS: $\langle u, v\rangle \le \alpha\|u\|\|v\|$ (see Example 4.2).

2. Assumptions, Notation and the Main result

Throughout the remainder of the paper we make two assumptions.

1. H is a real Hilbert space, $\|x\| = \sqrt{\langle x, x\rangle}$ denotes vector norm and O denotes zero vector,
2. V is a closed subspace of H, and $V^\perp$ denotes its orthogonal complement.

From classical Hilbert space theory we know that there exists the unique orthogonal projection onto V, which we denote by $P : H \to V$. Given any set $\Omega \subseteq H$, its orthogonal projection onto V is denoted by $P[\Omega]$. We define angles between vectors u and v, and between vector u and subspace S with[3]

$$\angle(u, v) \overset{\text{def}}{=} \sup\{\varphi \in [0, \pi] : \langle u, v\rangle \le \cos\varphi\, \|u\|\, \|v\|\}, \tag{2.1}$$

$$= \begin{cases} \arccos \frac{\langle u,v\rangle}{\|u\|\|v\|}, & \text{if } u \neq O \text{ and } v \neq O, \\ \pi, & \text{if } u = O \text{ or } v = O. \end{cases} \tag{2.2}$$

$$\angle(u, S) \overset{\text{def}}{=} \inf\{\angle(u, v) : v \in S\} \tag{2.3}$$

[2] In [4] the inequality is called "complementary" to CBS or "running the other way".

[3] Angle definition in (2.1) allows the trivial cases $V = H$, $V = \{O\}$, and $v = O$ to be naturally included in Theorem 2.2, without any special considerations. We use sign $\overset{\text{def}}{=}$ throughout the paper in order to indicate that the relation is actually a definition.

Thus $\angle(u,\{O\}) = \pi = \angle(O,V)$. Note that $\angle\left(v,V^{\perp}\right) + \angle\left(v,V\right) = \frac{\pi}{2}$, except when $v = 0$, or $V = H$, or $V = \{O\}$.

Definition 2.1. *The infinite one-sided solid right circular (round) cone in H with apex $a \in H$, axis direction given by $v \in H$, and half-aperture $\varphi \in [0,\pi]$ is defined by*

$$C_H\left(a,v,\varphi\right) \overset{\text{def}}{=} \left\{u \in H \,:\, \langle u-a, v\rangle \geq \cos\varphi \,\|u-a\|\,\|v\|\right\}.$$

The infinite one-sided solid right circular (round) cone with the apex included but with the rest of the boundary excluded is defined by

$$C_H^{\circ}\left(a,v,\varphi\right) \overset{\text{def}}{=} \{a\} \cup \left\{u \in H \,:\, \langle u-a, v\rangle > \cos\varphi \,\|u-a\|\,\|v\|\right\}.$$

When the apex is O, the notation is abbreviated: $C_H\left(v,\varphi\right) \overset{\text{def}}{=} C_H\left(O,v,\varphi\right)$ and $C_H^{\circ}\left(v,\varphi\right) \overset{\text{def}}{=} C_H^{\circ}\left(O,v,\varphi\right)$. By definition, we have $C_H\left(a,O,\varphi\right) = H$ and $C_H^{\circ}\left(a,O,\varphi\right) = \{a\}$. By CBS inequality, we have $C_H\left(a,v,\pi\right) = H$ and $C_H^{\circ}(a,v,0) = \{a\}$. Dilation (Minkowski addition) is denoted by $X + Y = \{x+y \,:\, x \in X \text{ and } y \in Y\}$. Thus, $C_H\left(a,v,\varphi\right) = C_H\left(v,\varphi\right) + \{a\}$.

Given $a \in H$, $v \in H$ and $\varphi \in [0,\pi]$, the aim of this paper is to determine the projection $P\left[C_H\left(a,v,\varphi\right)\right]$. The main result is the following theorem.

Theorem 2.2. *Let $C_H\left(a,v,\varphi\right)$ be an infinite one-sided solid cone in H and*

$$\varphi_1 = \begin{cases} \arccos\sqrt{\dfrac{\cos^2\varphi - \cos^2\angle\left(v,V^{\perp}\right)}{1-\cos^2\angle\left(v,V^{\perp}\right)}}, & \angle\left(v,V^{\perp}\right) \in \left\langle 0, \frac{\pi}{2}\right] \\ \varphi, & \textit{else.} \end{cases} \tag{2.4}$$

Given the values of φ and $\angle\left(v,V^{\perp}\right)$ in the first two columns of the following table, the orthogonal projection P of $C_H\left(a,v,\varphi\right)$ onto a closed subspace V can be determined from the third column of the same table.

<table>
<tr><th>φ</th><th>$\angle\left(v,V^{\perp}\right)$</th><th>$P\left[C_H\left(a,v,\varphi\right)\right]$</th></tr>
<tr><td>$\varphi = 0$</td><td>$\angle\left(v,V^{\perp}\right) = 0$</td><td>$\{P\,a\}$</td></tr>
<tr><td rowspan="2">$\varphi = \angle\left(v,V^{\perp}\right)$</td><td>$\angle\left(v,V^{\perp}\right) \in \langle 0, \pi/2\rangle$</td><td>$C_V^{\circ}(P\,a,\ P\,v,\ \pi/2)$</td></tr>
<tr><td>$\angle\left(v,V^{\perp}\right) \geq \pi/2$</td><td rowspan="2">$C_V(P\,a,\ P\,v,\ \varphi_1)$</td></tr>
<tr><td>$\varphi < \angle\left(v,V^{\perp}\right)$</td><td>$\angle\left(v,V^{\perp}\right) > 0$</td></tr>
<tr><td colspan="2">$\varphi > \angle\left(v,V^{\perp}\right)$</td><td>$V$</td></tr>
</table>

Proof. Suppose that the apex is $a = O$. We prove the theorem on a case by case basis. Technical work is deferred to Section 4, which deals with reverse CBS inequalities that underlay the definition of the cone.

Case $v = O$. This case is trivial as $C_H(O,\varphi) = H$, $P[C_H(O,\varphi)] = V = C_V(O,\varphi_1)$, $\angle(O,V^\perp) = \pi$, and $\varphi_1 = \varphi$.

Case $\dim V = 0$. Projection collapses everything to $V = \{O\}$, $\varphi_1 = \varphi$, and $\angle(v,V^\perp) = 0$. We have either $\varphi = \angle(v,V^\perp) = 0$, or $\varphi > \angle(v,V^\perp)$ and both corresponding rows in the table of the theorem provide the valid answer.

Case $V = H$. Let $C_H(v,\varphi)$ be a cone, such that $v \neq O$. We have $P\,v = v$, $P[C_H(v,\varphi)] = C_V(v,\varphi)$, $\angle(v,V) = 0$, $\angle(v,V^\perp) = \pi$ and $\varphi \leq \angle(v,V^\perp)$. From formula (2.4), $\varphi_1 = \varphi$. Thus, $P[C_H(v,\varphi)] = C_H(P\,v,\,\varphi)$, which is the conclusion of the theorem.

Case $\varphi = 0$. Note that $\cos\varphi = 1$, $C_H(v,0) = \{t\,v : t \geq 0\}$ and $P[C_H(v,0)] = \{t\,P\,v : t \geq 0\} = C_V(P\,v,0)$ unless $P\ v = O$. Note also that formula (2.4) produces $\varphi_1 = 0$ when $\varphi = 0$. In the special case when $P\,v = O$, then $\angle(v,V^\perp) = 0$ and $P[C_H(v,0)] = \{O\}$.

Case $\varphi < \angle(v,V^\perp)$. As $\varphi \geq 0$, we must have $\angle(v,V^\perp) > 0$. Therefore $P\,v \neq O$. The cases $\dim V = 0$, and $V = H$ have already been solved. In this case the main part of the proof is Theorem 4.3. The first part of Theorem 4.3 states that: $u \in C_H(v,\varphi)$ implies $P\,u \in C_V(P\,v,\,\varphi_1)$, i.e. $P[C_H(v,\varphi)] \subseteq C_V(P\,v,\,\varphi_1)$. In the subcase $\dim V \geq 2$, the second part of Theorem 4.3 establishes existence of $u \in C_H(v,\varphi)$ such that $Pu \neq O$, and $\langle P\,u, P\,v\rangle = \cos\varphi_1\,\|P\,u\|\,\|P\,v\|$. By Lemma 4.9, we get $C_V(P\,v,\,\varphi_1) \subseteq P[C_H(v,\varphi)]$, and the subcase $\dim V \geq 2$ is solved.

In the subcase $\dim V = 1$, there are just 2 unit vectors in V, which we denote with 1_V and -1_V. Also, there are just 4 different "cones" with the apex O in V: $\{0\}$, V, $C_V(1_V,0)$ and $C_V(-1_V,0)$ (cf. Example 1.1). Therefore $C_V(P\,v,\pi/2) = C_V(P\,v,\varphi_1) = C_V(P\,v,0)$, as $\varphi_1 \in [0,\pi/2]$. Without the loss of generality, we can assume that 1_V and $P\,v$ are pointing in the same direction (we are free to swap the names between 1_V and -1_V). Thus, we get $C_V(1_V,0) = \{t\,Pv : t \in \mathbb{R}\} \subseteq P[C_H(v,\varphi)]$ and $C_V(P\,v,0) = C_V(1_V,0)$. Together, we have $C_V(P\,v,\varphi_1) \subseteq P[C_H(v,\varphi)]$. As noted before, from Theorem 4.3 we get $P[C_H(v,\varphi)] \subseteq C_V(P\,v,\,\varphi_1)$. Therefore, $P[C_H(v,\varphi)] = C_V(P\,v,\,\varphi_1)$.

Case $\varphi > \angle(v,V^\perp)$. This implies $V \neq H$, because of the angle definition in (2.1). The main part of the proof in this case is moved to Proposition 4.4 and Lemma 4.9. Proposition 4.4 shows that there is $u \in C_H(v,\varphi)$ such that $Pu \neq O$, and $\langle P\,u, P\,v\rangle = \cos\pi\,\|P\,u\|\,\|P\,v\|$. From there, Lemma 4.9 concludes that $C_V(P\,v,\pi) \subseteq P[C_H(v,\varphi)]$. By the definition of cone and by the CBS inequality we know that $C_V(Pv,\pi) = V$. Thus we get $P[C_H(v,\varphi)] = V$, which is just what the theorem states in this case.

Case $\varphi = \angle(v,V^\perp)$. Because of the other cases that have been discussed already, we can safely assume that $\varphi = \angle(v,V^\perp) \in \langle 0,\pi/2]$ and $\dim V \geq 1$. Note that $Pv \neq O$ and $\varphi_1 = \pi/2$.

First, suppose $\dim V = 1$. As we noted earlier, there are just four cones to distinguish in V. Note that $C_V^\circ(P\,v, {}^{\pi}/_2) = C_V(P\,v, {}^{\pi}/_2) = C_V\,(P\,v, 0)$. From $Pv \neq O$ we get $C_V\,(P\,v, 0) \subseteq P\,[C_H\,(v,\varphi)]$. From Lemma 4.6 we get $P\,[C_H\,(v,\varphi)] \subseteq C_V(P\,v, {}^{\pi}/_2)$. Thus, we conclude $P\,[C_H\,(v,\varphi)] = C_V(P\,v, {}^{\pi}/_2) = C_V^\circ(P\,v, {}^{\pi}/_2)$.

Next, suppose $\dim V \geq 2$ and $V \neq H$. We discuss two subcases: either $\varphi = \angle\left(v, V^\perp\right) \in \langle 0, {}^{\pi}/_2\rangle$ or $\varphi = \angle\left(v, V^\perp\right) = {}^{\pi}/_2$. Suppose $\varphi = \angle\left(v, V^\perp\right) \in \langle 0, {}^{\pi}/_2\rangle$. By Proposition 4.5 and Lemma 4.9: $C_V\,(Pv, {}^{\pi}/_2 - \varepsilon) \subseteq P\,[C_H\,(v,\varphi)]$, for each $\varepsilon \in \langle 0, {}^{\pi}/_2]$. Furthermore, Proposition 4.8 yields $C_V\,(-Pv, {}^{\pi}/_2) \cap P\,[C_H\,(v,\varphi)] = \{O\}$, and therefore $P\,[C_H\,(v,\varphi)] = C_V^\circ\,(Pv, {}^{\pi}/_2)$.

Finally, we prove the subcase $\varphi = \angle\left(v, V^\perp\right) = {}^{\pi}/_2$ and $\dim V \geq 2$. Then, $Pv = v$ and $C_V\,(v, {}^{\pi}/_2) = C_H\,(v, {}^{\pi}/_2) \cap V \subseteq P\,[C_H\,(v, {}^{\pi}/_2)]$. Lemma 4.6 shows that $\langle u, v\rangle \geq 0$ implies $\langle Pu, Pv\rangle \geq 0$. Thus $P\,[C_H\,(v, {}^{\pi}/_2)] \subseteq C_V\,(v, {}^{\pi}/_2)$. From $\varphi = \varphi_1 = {}^{\pi}/_2$ we conclude $P\,[C_H\,(v,\varphi)] = C_V\,(Pv, \varphi_1)$.

We have proved the theorem for apex $a = O$. The general case follows from the properties of dilation

$$P\,[C_H\,(a,v,\varphi)] = P\,[C_H\,(v,\varphi) + \{a\}] = P\,[C_H\,(v,\varphi)] + \{P\,a\}\,.$$

∎

3. A Few Applications

Remark 3.1. Let d be a vector orthogonal to V. Then $\Pi \stackrel{\text{def}}{=} V + \{d\}$ is an affine subspace in H. Orthogonal projection of u onto Π is defined by $P_\Pi u = P\,u + d$. Orthogonal projection of cone $C_H\,(a,v,\varphi)$ onto Π yields $P_\Pi\,[C_H\,(a,v,\varphi)] = P\,[C_H\,(a,v,\varphi)] + \{d\} \subseteq \Pi$, where $P\,[C_H\,(a,v,\varphi)]$ is given in Theorem 2.2.

Remark 3.2. We can use Theorem 2.2 to describe projections of solid cones with the apex included but without the rest of the boundary. It is easy to see that

$$P\,[C_H^\circ\,(a,v,\varphi)] = P\,[\,\bigcup_{\varepsilon>0} C_H\,(a,v,\varphi-\varepsilon)\,] = \bigcup_{\varepsilon>0} P\,[C_H\,(a,v,\varphi-\varepsilon)]\,. \tag{3.1}$$

Therefore, if we extend the formula for φ_1 in (2.4) by setting $\varphi_1 = 0$ when $\varphi = \angle\left(v, V^\perp\right) = 0$ we get

$$P\,[C_H^\circ\,(a,v,\varphi)] = \begin{cases} C_V^\circ\,(Pa, Pv, \varphi_1)\,, & \varphi \leq \angle\left(v, V^\perp\right), \\ V, & \varphi > \angle\left(v, V^\perp\right). \end{cases} \tag{3.2}$$

Furthermore, a relation equivalent to (3.2) is also valid for "open" cones.

Example 3.3. Let $v \in H$ such that $Pv \neq O$. Which is the widest half aperture φ of an infinite solid cone with the apex $a \in H$, axis and direction given by v such that the half aperture of a projected cone is at most $\varphi_1 < {}^{\pi}/_2$?

Solving formula (1.2) for φ yields

$$\varphi = \arccos \sqrt{\cos^2 \angle (v, V^\perp) + \cos^2 \varphi_1 - \cos^2 \angle (v, V^\perp) \cos^2 \varphi_1}. \tag{3.3}$$

Theorem 2.2 establishes $P\left[C_H\left(a, v, \varphi\right)\right] = C_V\left(Pa,\, Pv,\, \varphi_1\right)$ and any larger φ would yield aperture of projected cone larger then φ_1.

FACT 3.4. *The widest half aperture of an one-sided infinite cone that fits inside an orthant (hyperoctant) of* $\mathbb{R}^n$ *is* $\varphi = \arccos \sqrt{\frac{n-1}{n}}$. *As dimension* $n \to \infty$, *the aperture* $\varphi \to 0$.

PROOF. All the projections onto coordinate 2D planes V_{ij} of such a infinite one-sided cone need to fit into a quadrant, which is a directed cone with half aperture $\varphi_1 = \pi/4$ around directed axis $P\,v = (1, 1)$. Therefore, the directed cone with the widest aperture needs to have the axis $v = (1, 1, \ldots, 1)$. By formula (4.1) $\angle (v, V_{ij}) = \arccos \frac{\sqrt{2}}{\sqrt{n}}$ and $\cos^2 \angle \left(v, V_{ij}^\perp\right) = \frac{n-2}{n}$. Formula (1.2) yields $\varphi = \arccos \sqrt{\frac{n-1}{n}}$ and Theorem 2.2 establishes the fact. ∎

EXAMPLE 3.5. Let $\alpha \in (0, 1)$ and $H = L^2\left(0, 1\right)$ Lebesgue space. What is the smallest $t > 0$ such that for all $u \in H$,

$$\underbrace{\int_0^1 u \geq \alpha \sqrt{\int_0^1 u^2}}_{\star} \quad \Longrightarrow \quad \int_0^t u \geq 0 \quad ? \tag{3.4}$$

Let $v = \mathbf{1}_{(0,1)} \in H$, where we denote with $\mathbf{1}_X$ the characteristic function of X. Then $u \in C_H(v, \arccos \alpha)$ if and only if u satisfies $\star$ in (3.4). Let $V_t = \{f \in L^2\left(0, 1\right) : f(x) = 0 \text{ for almost all } x \in (t, 1)\}$. V_t is a closed subspace of H and isometrically isomorphic with $L^2\left(0, t\right)$. Let P_{V_t} be the orthogonal projection onto V_t. Then $P_{V_t}(v) = \mathbf{1}_{(0,t)}$. We calculate the angles $\angle(v, V_t) = \angle(v, \mathbf{1}_{(0,t)}) = \arccos \sqrt{t}$ and $\angle(v, V_t^\perp) = \angle(v, \mathbf{1}_{(t,1)}) = \arccos \sqrt{1-t}$.

From Theorem 2.2, if $\arccos \alpha > \arccos \sqrt{1-t}$, then $P_{V_t}[C_H(v, \arccos \alpha)] = V_t$. In other words, when $t < 1 - \alpha^2$, then for appropriate u that satisfies $\star$ in (3.4) we can get $\int_0^t u < 0$. On the other hand, for $\arccos \alpha \leq \arccos \sqrt{1-t}$, $P_{V_t}[C_H(v, \arccos \alpha)]$ is either $C_{V_t}(\mathbf{1}_{(0,t)}, \varphi_1)$ or $C^\circ_{V_t}(\mathbf{1}_{(0,t)}, \pi/2)$, where $\varphi_1 < \pi/2$ can be calculated from (2.4) with $\varphi = \arccos \alpha$. In any case, if $t \geq 1 - \alpha^2$, then for any u that satisfies $\star$ in (3.4) we have $\int_0^t u \geq \sqrt{t - 1 + \alpha^2} \sqrt{\int_0^t u^2}$. Therefore we have shown that $t = 1 - \alpha^2$ is the smallest number that satisfies (3.4).

4. Reverse CBS Inequalities

In this section we provide technical results used in the proof of Theorem 2.2 on cone projections. Cones have been defined in terms of reverse CBS inequality. Given a reverse CBS inequality in Hilbert space H, we need to establish an "optimal" reverse CBS inequality for orthogonal projections onto a closed subspace V. This is the converse from reverse CBS inequalities in survey [5, Section 5]. The contraposition of our results provides sufficient conditions for an estimate that is more strict then the classical CBS inequality: $\langle u,v\rangle < \alpha\,\|u\|\,\|v\|$. This is described in Example 4.2.

Remark 4.1. Throughout this section we use subscripts to denote $u_1 \stackrel{\text{def}}{=} P\,u$ and $u_2 \stackrel{\text{def}}{=} u - P\,u$, for any $u \in H$. For the unique decomposition of $u \in H$ as the sum of two orthogonal vectors, one from V and other from $V^\perp$ we write $u = u_1 \dotplus u_2$. For $u \neq O$ we use formulas

$$\angle\left(u, V^\perp\right) = \arccos\frac{\|u_2\|}{\|u\|} = \begin{cases} \arctan\frac{\|u_1\|}{\|u_2\|}, & u_2 \neq O, \\ \frac{\pi}{2}, & u_2 = O, \end{cases} \tag{4.1}$$

Example 4.2. Suppose we have set $v \in H$, V closed subspace of H, and $\alpha \in (0,1)$. We are looking for a sufficient condition on the component $u_1 = P\,u$, that can establish an inequality stronger then the CBS inequality: $\langle u,v\rangle < \alpha\,\|u\|\,\|v\|$.

We use contraposition of Theorem 4.3. Condition $\alpha = \cos\varphi > \cos\angle\left(v, V^\perp\right) = \frac{\|v_2\|}{\|v\|}$ gives $\|v_2\| < \frac{\alpha}{\sqrt{1-\alpha^2}}\|v_1\|$. If that condition is met, then $\langle u_1, v_1\rangle < \cos\varphi_1\,\|u_1\|\,\|v_1\|$ is the sufficient condition for $\langle u,v\rangle < \alpha\,\|u\|\,\|v\|$, where φ_1 is the same as in (2.4). Therefore,

$$\left(\|v_2\| < \frac{\alpha\|v_1\|}{\sqrt{1-\alpha^2}} \text{ and } \langle u_1, v_1\rangle < \sqrt{\frac{\alpha^2\|v\|^2 - \|v_2\|^2}{\|v\|^2 - \|v_2\|^2}}\,\|u_1\|\,\|v_1\|\right) \implies \langle u,v\rangle < \alpha\,\|u\|\,\|v\|\,. \tag{4.2}$$

The following result is a cornerstone in the proof of Theorem 2.2.

Theorem 4.3. *Let $v \in H$, V closed subspace of H, φ such that $0 \le \varphi < \angle\left(v, V^\perp\right)$ and φ_1 as in (2.4). Let $\alpha = \cos\varphi$ and $\alpha_1 = \cos\varphi_1$. Then for arbitrary $u \in H$,*

$$\langle u,v\rangle \geq \alpha\,\|u\|\,\|v\| \qquad \implies \qquad \langle P\,u,\, P\,v\rangle \geq \alpha_1\,\|Pu\|\,\|Pv\|\,. \tag{4.3}$$

Moreover, when $\dim V \geq 2$ and $v \neq O$ then our α_1 is the largest possible in (4.3). In other words, when $\dim V \geq 2$, then there exists $u \in H$ such that $Pu \neq O$, $\langle u,v\rangle \geq \alpha\,\|u\|\,\|v\|$ and $\langle P\,u,\, P\,v\rangle = \alpha_1\,\|Pu\|\,\|Pv\|$.

Proof. We use notation $v = v_1 \dotplus v_2$ and $u = u_1 \dotplus u_2$ as in Remark 4.1. If $v_1 = O$ or $V = H$, (4.3) is trivial. Thus, without the loss of generality we can suppose $v_1 \neq O$ and $0 \leq \varphi < \angle\left(v, V^{\perp}\right) \leq \pi/2$. Then[4],

$$0 \leq \frac{\|v_2\|}{\|v\|} \overset{(4.1)}{=} \cos\angle\left(v, V^{\perp}\right) < \cos\varphi \leq 1. \tag{4.4}$$

$$\frac{\|v_2\|}{\|v_1\|} = \frac{\cos\angle\left(v, V^{\perp}\right)}{\sqrt{1-\cos^2\angle\left(v, V^{\perp}\right)}} \tag{4.5}$$

When $u_1 = O$ then (4.3) is trivial. The main part of the proof investigates

$$\begin{aligned} \min \quad \frac{\langle u_1, v_1\rangle}{\|u_1\|\,\|v_1\|} &= \min \quad \frac{\cos\theta\ \|u\|\,\|v\| - \langle u_2, v_2\rangle}{\|u_1\|\,\|v_1\|} = \\ &= \min \quad \underbrace{\cos\theta\sqrt{1+\frac{\|u_2\|^2}{\|u_1\|^2}}\sqrt{1+\frac{\|v_2\|^2}{\|v_1\|^2}} - \left\langle \frac{u_2}{\|u_1\|}, \frac{v_2}{\|v_1\|}\right\rangle}_{(\blacksquare)} \end{aligned} \tag{4.6}$$

under the conditions that $u_1 \neq O$ and $\langle u, v\rangle = \cos\theta\,\|u\|\,\|v\| \geq \cos\varphi\ \|u\|\,\|v\|$, for some $\theta \in [0, \varphi]$ that depends on u and v. Under these conditions

$$(\blacksquare) \geq \cos\varphi\sqrt{1+\frac{\|u_2\|^2}{\|u_1\|^2}}\sqrt{1+\frac{\|v_2\|^2}{\|v_1\|^2}} - \frac{\|u_2\|\,\|v_2\|}{\|u_1\|\,\|v_1\|} = f\left(\frac{\|u_2\|}{\|u_1\|}, \frac{\|v_2\|}{\|v_1\|}\right)$$

where $f(a,b) = \cos\varphi\sqrt{1+a^2}\sqrt{1+b^2} - ab$.

As $\|u_2\|/\|u_1\| \geq 0$ and v has been fixed from the start, together with condition (4.4), it is sufficient to examine the function $a \overset{g}{\longmapsto} f(a,b)$ for all $a \geq 0$ and a fixed b, taking into account that $\cos\varphi > b/\sqrt{1+b^2}$. The continuity of g, the first, and the second derivative of g, together show that g is convex with the only minimizer $a = b/\sqrt{\cos^2\varphi(1+b^2)-b^2}$ and the minimum $\sqrt{\cos^2\varphi\ (1+b^2)-b^2}$. Therefore

$$(\blacksquare) \geq \sqrt{\cos^2\varphi\left(1+\frac{\|v_2\|^2}{\|v_1\|^2}\right) - \frac{\|v_2\|^2}{\|v_1\|^2}} \overset{(4.5)}{=} \sqrt{\frac{\cos^2\varphi - \cos^2\angle\left(v, V^{\perp}\right)}{1-\cos^2\angle\left(v, V^{\perp}\right)}} > 0.$$

Thus $\langle u_1, v_1\rangle \geq \sqrt{\frac{\cos^2\varphi - \cos^2\angle\left(v, V^{\perp}\right)}{1-\cos^2\angle\left(v, V^{\perp}\right)}}\ \|u_1\|\,\|v_1\|$ whenever $\|u\| \neq O$. The first part of the theorem has been proved without the assumption $\dim V \geq 2$.

The assumptions for the second part of the theorem include $v \neq O$ and $0 < \angle\left(v, V^{\perp}\right)$. Therefore, $v_1 \neq O$. Another assumption of the second part of the theorem is $\dim V \geq 2$, and so $z \in V$ can be chosen such that $\|z\| = 1$ and

[4]Formula numbers above and under (in)equality sign establish a cross reference that can help to understand the relationship. This notation is used throughout the article.

$z \perp v_1$. It is straightforward to check that for $u = \cos\varphi_1\, v_1 + \|v_1\| \sin\varphi_1\, z + \frac{1}{\cos\varphi_1}\, v_2$:

$$\|u_1\| = \sqrt{\cos\varphi_1^2\, \|v_1\|^2 + \|v_1\|^2 \sin^2\varphi_1\, \|z\|^2} = \|v_1\| \neq 0 \tag{4.7}$$

$$\langle u_1, v_1\rangle = \cos\varphi_1\ \|v_1\|^2 = \cos\varphi_1\ \|u_1\|\,\|v_1\|$$

$$\|u\| = \sqrt{\|u_1\|^2 + \frac{\|v_2\|^2}{\cos^2\varphi_1}} \overset{(4.7)}{=} \|v_1\| \sqrt{1 + \frac{\|v_2\|^2}{\|v_1\|^2 \cos^2\varphi_1}}$$

$$\underset{(4.5)}{\overset{(2.4)}{=}} \|v_1\| \frac{\cos\varphi}{\sqrt{\cos^2\varphi - \cos^2\angle(v, V^\perp)}} \tag{4.8}$$

$$\|v\| = \|v_1\| \sqrt{1 + \frac{\|v_2\|^2}{\|v_1\|^2}} \overset{(4.5)}{=} \frac{\|v_1\|}{\sqrt{1 - \cos^2\angle(v, V^\perp)}} \tag{4.9}$$

Then,

$$\langle u, v\rangle = \cos\varphi_1\ \|v_1\|^2 + \frac{\|v_2\|^2}{\cos\varphi_1} = \|v_1\|^2 \left(\cos\varphi_1 + \frac{\|v_2\|^2}{\cos\varphi_1\ \|v_1\|^2}\right)$$

$$\underset{(4.5)}{\overset{(2.4)}{=}} \|v_1\|^2 \frac{\cos^2\varphi}{\sqrt{\cos^2\varphi - \cos^2\angle(v, V^\perp)}\sqrt{1 - \cos^2\angle(v, V^\perp)}} \underset{(4.9)}{\overset{(4.8)}{=}} \cos\varphi\, \|u\|\,\|v\|$$

Thus, when $\dim V \geq 2$, formula (2.4) gives the smallest possible $\varphi_1 \in [0, \pi]$ and $\alpha_1 = \cos\varphi_1$ is the largest possible in (4.3). ∎

From Theorem 4.3 we were able to get a useful estimate on the angle between projections when $\angle(u, v) < \angle\left(v, V^\perp\right)$. On the other hand, the following result shows that when the angle between vectors $\angle(u, v) > \angle\left(v, V^\perp\right)$, then no useful estimate on the angle between projections can be provided. In other words, the worst case scenario $\angle(Pu, Pv) = \pi$ is possible.

Proposition 4.4. *Let V be a closed subspace of Hilbert space H, with $\dim V \geq 1$ and $V \neq H$. Let P be the orthogonal projection onto V. Let $v \in H$, $v \neq O$, $\varphi \in \left\langle \angle\left(v, V^\perp\right), \pi\right]$, and $\alpha = \cos\varphi$. Then there exists $u \in H$ such that*

$$Pu \neq O, \quad \langle u, v\rangle \geq \alpha\ \|u\|\,\|v\| \quad \text{and} \quad \langle Pu,\, Pv\rangle = (-1)\ \|Pu\|\,\|Pv\|\,. \tag{4.10}$$

Proof. We use notation from Remark 4.1. Assumption of the proposition is that either $v_1 \neq O$ or $v_2 \neq O$. We will prove the proposition on a case by case basis. We will choose a different continuous parametrization $u(t)$ for each case, and then investigate the continuous function

$$f(t) \overset{\text{def}}{=} \langle u(t), v\rangle - \cos\varphi\ \|u(t)\|\,\|v\|\,. \tag{4.11}$$

Consider the case $v_1 \neq O$ and $v_2 \neq O$. Then $\pi/2 > \angle\big(v, V^\perp\big) > 0$. Let $u(t) = tv_1 \dotplus v_2$ and f as in (4.11). Then

$$\begin{aligned}(4.12)\quad f(t) &= \|v_1\|^2 t + \|v_2\|^2 - \cos\varphi \sqrt{t^2 \|v_1\|^2 + \|v_2\|^2}\sqrt{\|v_1\|^2 + \|v_2\|^2} \\ &\overset{(4.1)}{=} \|v_1\|^2 t + \|v_2\|^2 - \cos\varphi \sqrt{t^2 \|v_1\|^2 + \|v_2\|^2}\frac{\|v_2\|}{\cos\angle\left(v, V^\perp\right)}.\end{aligned}$$

Plugging $t = 0$ in (4.12) gives $f(0) = \|v_2\|^2 - \dfrac{\cos\varphi}{\cos\angle\left(v, V^\perp\right)}\|v_2\|^2 > 0$. By the continuity of f, there exists some $t_0 < 0$ such that $f(t_0) > 0$. Now $u = u(t_0)$ satisfies (4.10): the first part as $Pu = t_0 v_1 \neq O$, the second part because $0 < f(t_0) = \langle u(t_0), v\rangle - \cos\varphi \, \|u(t_0)\| \, \|v\|$ and the third as $\langle u_1, v_1\rangle = \langle t_0 v_1, v_1\rangle = -\|t_0 v_1\| \, \|v_1\| = -\|u_1\| \, \|v_1\|$.

Next, consider the case $v_1 = O$ and $v_2 \neq O$. As $\dim V \geq 1$ we can choose $z \in V$ such that $\|z\| = 1$. Let $u(t) = tz \dotplus v_2$ and f as in (4.11). Then

$$(4.13)\quad f(t) = \|v_2\|^2 - \cos\varphi \sqrt{t^2 + \|v_2\|^2}\;\|v_2\| \;\text{ and }\; f(0) = \|v_2\|^2 (1 - \cos\varphi).$$

As $\varphi > \angle\big(v, V^\perp\big) = 0$ thus $f(0) > 0$. By the continuity of f, there exists some $t_0 < 0$ such that $f(t_0) > 0$. Vector $u = u(t_0)$ satisfies (4.10) because $Pu = t_0 z \neq O$, $0 < f(t_0) = \langle u, v\rangle - \cos\varphi \, \|u\| \, \|v\|$ and $\langle u_1, v_1\rangle = 0 = -\|u_1\| \, \|v_1\|$.

Consider the final case $v_1 \neq O$ and $v_2 = O$. Then $\pi \geq \varphi > \angle\big(v, V^\perp\big) = \pi/2$. As $V \neq H$ there exists $z \in V^\perp$ such that $\|z\| = 1$. Let $u(t) = tv_1 \dotplus z$ and f as in (4.11). Then

$$f(t) = \|v_1\|^2 t - \cos\varphi \sqrt{t^2 \|v_1\|^2 + 1}\;\|v_1\|.$$

As $\cos\varphi < 0$ therefore $f(0) = -\cos\varphi \|v_1\| > 0$. By the continuity of f there exists some $t_0 < 0$ such that $f(t_0) > 0$. Now $u = u(t_0)$ satisfies (4.10): the first part as $u_1 = t_0 v_1 \neq O$, the second part because $0 < f(t_0) = \langle u, v\rangle - \cos\varphi \, \|u\| \, \|v\|$ and the third as $\langle u_1, v_1\rangle = \langle t_0 v_1, v_1\rangle = -\|t_0 v_1\| \, \|v_1\| = -\|u_1\| \, \|v_1\|$. ∎

The previous two results discussed the cases $\angle(u, v) < \angle\big(v, V^\perp\big)$ and $\angle(u, v) > \angle\big(v, V^\perp\big)$. The border case $\angle(u, v) = \angle\big(v, V^\perp\big)$ is different, as:

- $\inf\angle(Pu, Pv) = \pi/2$ (combine Proposition 4.5 and Lemma 4.6),
- the infimum is achieved in the case $v \in V$ (see Proposition 4.8),
- the infimum is not achieved in the case $v \notin V$ (see Proposition 4.8).

Proposition 4.5. *Let P be an orthogonal projection onto a closed subspace V of Hilbert space H, with $\dim V \geq 2$. Let $v \in H$, $v \neq O$, $\angle\big(v, V^\perp\big) > 0$, and $\varepsilon \in \langle 0, 1]$. Then there exists $u \in H$ such that $Pu \neq O$,*

$$\langle u, v\rangle \geq \cos\angle\big(v, V^\perp\big)\;\|u\|\,\|v\| \qquad \textit{and} \qquad \langle Pu, Pv\rangle = \varepsilon\,\|Pu\|\,\|Pv\|.$$

Proof. We use notation from Remark 4.1. As $\angle\left(v,V^{\perp}\right)>0$, we have $\|v_1\|>0$. Without the loss of generality we assume that $\|v_1\|=1$ (if the statement of the proposition is true for vector $v/\|v_1\|$ then it is also true for v).

As $\dim V\geq 2$, there exists $z\in V$ such that $z\perp v_1$ and $\|z\|=\sqrt{1-\varepsilon^2}$. Let $u(t)\overset{\text{def}}{=}\varepsilon v_1+tv_2+z$ and $u_1(t)\overset{\text{def}}{=}P\ u(t)$. Then $u_1(t)=\varepsilon v_1+z\neq O$, $\|u_1(t)\|=\sqrt{\varepsilon^2\|v_1\|^2+\|z\|^2}=1$, and $\langle u_1(t),\, v_1\rangle=\varepsilon=\varepsilon\,\|u_1(t)\|\,\|v_1\|$. So we only need to find t such that $\langle u(t),v\rangle\geq\cos\angle\left(v,V^{\perp}\right)\ \|u(t)\|\,\|v\|$.

We investigate the real function

$$f(t)\overset{\text{def}}{=}\langle u(t),v\rangle-\cos\angle\left(v,V^{\perp}\right)\ \|u(t)\|\,\|v\|=\varepsilon+t\,\|v_2\|^2-\|v_2\|\sqrt{1+t^2\,\|v_2\|^2}\,.$$

Note that f is differentiable and strictly increasing, with $\lim\limits_{t\to+\infty}f(t)=\varepsilon>0$. Therefore, f assumes positive value for some $t_0\in\mathbb{R}$. Vector $u(t_0)$ satisfies the conclusion of the proposition. ∎

Lemma 4.6. *Let P be an orthogonal projection onto a closed subspace V of Hilbert space H, with $V\neq H$. Let $v\in H$, $v\neq O$ such that $\angle\left(v,V^{\perp}\right)>0$. Then*

$$\forall u\in H,\qquad \langle u,v\rangle\geq\cos\angle\left(v,V^{\perp}\right)\ \|u\|\,\|v\|\quad\Longrightarrow\quad\langle Pu,Pv\rangle\geq 0.$$

Proof. If $V=\{O\}$, $v=O$, $Pv=O$ ($\angle(v,V^{\perp})=0$) or $Pu=O$ the conclusion of the lemma is trivial. Otherwise, apply Lemma 4.7. ∎

Lemma 4.7. *Let P be an orthogonal projection onto a closed subspace V of Hilbert space H, with $V\neq H$ and $V\neq\{O\}$. Let $v\in H$, such that $0<\angle\left(v,V^{\perp}\right)<\pi/2$. Then for all $u\in H$,*

$$\left(\langle u,v\rangle\geq\cos\angle\left(v,V^{\perp}\right)\ \|u\|\,\|v\|\ \text{ and } Pu\neq O\right)\quad\Longrightarrow\quad\langle Pu,Pv\rangle>0.$$

Proof. We use notation from Remark 4.1. As $0<\angle\left(v,V^{\perp}\right)<\pi/2$ so $v\neq O$, $v_1\neq O$, and $v_2\neq O$.

To prove the conclusion of the lemma, we assume $u_1\neq O$ and

$$\langle u_1,v_1\rangle+\langle u_2,v_2\rangle=\langle u,v\rangle\geq\cos\angle\left(v,V^{\perp}\right)\ \|u\|\,\|v\|\overset{(4.1)}{=}\|u\|\,\|v_2\|\tag{4.14}$$

From (4.14), as $\|u_2\|<\|u\|$ we get

$$\|u\|\,\|v_2\|-\langle u_1,v_1\rangle\overset{(4.14)}{\leq}\langle u_2,v_2\rangle\leq\|u_2\|\,\|v_2\|<\|u\|\,\|v_2\|\,.$$

Thus, $\langle u_1,v_1\rangle>0$ follows by subtraction of $\|u\|\,\|v_2\|$ from both sides of the previous inequality. ∎

Proposition 4.8. *Let P be an orthogonal projection onto a closed subspace V of Hilbert space H, such that $V\neq H$ and $\dim V\geq 2$. Let $v\in H$,* *such that* $v \neq O$ *and* $\angle(v, V^{\perp}) > 0$. *Then* $\angle(v, V^{\perp}) < \pi/2$ *if and only if for all* $u \in H$,

$$\big(\langle u, v\rangle \geq \cos\angle(v, V^{\perp})\ \|u\|\,\|v\| \ \text{ and } Pu \neq O\big) \implies \langle Pu, Pv\rangle > 0. \tag{4.15}$$

Proof. One implication is proved in lemma 4.7.

Instead of proving that (4.15) implies $\angle(v, V^{\perp}) < \pi/2$, we prove the contraposition: $\angle(v, V^{\perp}) \geq \pi/2$ implies that there exists an $u \neq O$ such that $\langle u, v\rangle \geq \cos\angle\left(v, V^{\perp}\right) \|u\|\,\|v\|$, $Pu \neq O$ and $\langle Pu, Pv\rangle \leq 0$. From (4.1), $\angle\left(v, V^{\perp}\right) \geq \pi/2$, the proposition assumptions $V \neq H$ and $v \neq O$, we can conclude that $\angle\left(v, V^{\perp}\right) = \pi/2$. Therefore $\cos\angle\left(v, V^{\perp}\right) = 0$ and $v \in V$. As $\dim V \geq 2$ there exists $z \in V$ such that $z \perp v$ and $\|z\| = 1$. For $u = z$ we get $\langle u, v\rangle = 0 \geq \cos\angle(v, V^{\perp})\ \|u\|\,\|v\|$ and $\langle Pu, Pv\rangle = \langle z, v\rangle = 0$. ◻

The following lemma allows us to go from just one $u \in C_H(v, \varphi)$ such that Pu is on the boundary of $C_V(Pv, \theta)$ and conclude that all of $C_V(Pv, \theta)$ is inside of the projection of $C_H(v, \varphi)$.

Lemma 4.9. *Let* $\theta \in [0, \pi]$, $u \in C_H(v, \varphi)$, $Pu \neq O$ *and* $\langle Pu, Pv\rangle = \cos\theta\,\|Pu\|\,\|Pv\|$. *Then* $C_V(Pv, \theta) \subseteq P\left[C_H(v, \varphi)\right]$.

Proof. We use notation from Remark 4.1. Note that $u_1 \neq O$, and so if $\varphi = 0$, then the case $v_1 = 0$ is excluded from the conclusion of the lemma. On the other hand, when $\varphi \neq 0$ then the case $v_1 = 0$ is included in the lemma. Notice that $u \in C_H(v, \varphi)$ corresponds to $\langle u, v\rangle \geq \cos\varphi\,\|u\|\,\|v\|$.

We will prove that for each $w \in V$ such that $\langle w, v_1\rangle \geq \cos\theta\,\|w\|\,\|v_1\|$, there exists $z \in V^{\perp}$ such that $\tilde{u} \overset{\text{def}}{=} w \dotplus z$, that satisfies $\langle \tilde{u}, v\rangle \geq \cos\varphi\,\|\tilde{u}\|\,\|v\|$. Note that by the definition of $\tilde{u}$ we get $P\tilde{u} = w$. Take $z = u_2 \dfrac{\|w\|}{\|u_1\|}$, then $\|\tilde{u}\| = \dfrac{\|w\|}{\|u_1\|}\ \|u\|$ and

$$\begin{aligned}\langle \tilde{u}, v\rangle = \langle w, v_1\rangle + \frac{\|w\|}{\|u_1\|}\langle u_2, v_2\rangle \geq \cos\theta\,\|w\|\,\|v_1\| + \frac{\|w\|}{\|u_1\|}\langle u_2, v_2\rangle = \\ = \frac{\|w\|}{\|u_1\|}\left(\cos\theta\,\|u_1\|\,\|v_1\| + \langle u_2, v_2\rangle\right) = \frac{\|w\|}{\|u_1\|}\left(\langle u_1, v_1\rangle + \langle u_2, v_2\rangle\right) = \\ = \frac{\|w\|}{\|u_1\|}\langle u, v\rangle \geq \frac{\|w\|}{\|u_1\|}\cos\varphi\ \|u\|\,\|v\| = \cos\varphi\ \|\tilde{u}\|\,\|v\|\,.\end{aligned}$$

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Ortogonalna projekcija beskonačnog kružnog konusa u realnom Hibertovom prostoru

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Sažetak. Dajemo potpuni opis ortogonalnih projekcija beskonačnog kružnog stošca u realnim Hilbertovim prostorima. Druga interpretacija je da smo za dva vektora dobili optimalnu ocjenu kuta između ortogonalnih projekcija tih vektora. Ta ocjena ovisi o kutu između polazna dva vektora i položaju samo jednog od njih. Među rezultatima je također doprinos nejednakostima tipa Cauchy-Bunyakovsky-Schwarz.

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