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ASYMPTOTIC BEHAVIOUR OF THE ITERATIVE PYTHAGOREAN MEANS

Tomislav Burić

Abstract. Asymptotic expansion and behaviour of the iterative combinations of the Pythagorean means (arithmetic, geometric and harmonic mean) is obtained and analyzed. Results are used for asymptotic comparison of means.

1. Introduction

The well-known Pythagorean means are three classical means: arithmetic, geometric and harmonic mean defined by

$$A(s,t)=\frac{s+t}{2},\qquad G(s,t)=\sqrt{st},\qquad H(s,t)=\frac{2}{\frac{1}{s}+\frac{1}{t}}, \tag{1.1}$$

where s and t are real positive numbers.

These means have following ordering

$$\min(s,t)\,\le\,H(s,t)\,\le\,G(s,t)\,\le\,A(s,t)\,\le\,\max(s,t), \tag{1.2}$$

and there are many results and papers concerning their properties, inequalities and comparison to other classical means.

In recent papers [5–7] authors studied asymptotic expansions of means and developed new technique of deriving and calculating coefficients in this expansions. They succesfully used new method to establish various relations between classical means, see cited papers for details.

The asymptotic expansion of the mean $M(s,t)$ is representation of this mean in the form

$$M(x+s,x+t)=x\sum_{n=0}^{\infty}c_n(s,t)x^{-n},\quad x\to\infty, \tag{1.3}$$

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where $c_n(s,t)$ are homogeneous polynomials of the degree n in variables s and t. It is shown that simpler form of these coefficients is obtained under substitution

$$t = \alpha + \beta, \qquad s = \alpha - \beta, \tag{1.4}$$

so variables α and β will be used in the rest of the paper.

Obviously, expansion of the arithmetic mean is

$$A(x+s, x+t) = x + \alpha \tag{1.5}$$

and it has only these two terms. Asymptotic expansion of the geometric and harmonic mean is derived in [6] and it reads as

$$G(x+s, x+t) \sim x + \alpha - \tfrac{\beta^2}{2}x^{-1} + \tfrac{\alpha\beta^2}{2}x^{-2} - \tfrac{\beta^2}{8}(4\alpha^2 + \beta^2)x^{-3} + \dots \tag{1.6}$$

$$H(x+s, x+t) \sim x + \alpha - \beta^2 x^{-1} + \alpha\beta^2 x^{-2} - \alpha^2\beta^2 x^{-3} + \dots \tag{1.7}$$

Recall that arithmetic-geometric mean is an example of an interesting mean obtained by the iterative combination of arithmetic and geometric mean in the following way. Define $a_0 = s$, $g_0 = t$ and

$$a_{k+1} = \frac{a_k + b_k}{2}, \qquad g_{k+1} = \sqrt{a_k b_k}, \qquad k \geq 0. \tag{1.8}$$

Then both of this sequences converge to the same limit $AG(s,t)$ which is called arithmetic-geometric mean.

Same idea can be used to the other combinations of Pythagorean means as well. For an example, iteration of geometric and harmonic mean

$$g_{k+1} = \sqrt{g_k h_k}, \qquad h_{k+1} = \frac{2}{\frac{1}{g_k} + \frac{1}{h_k}}, \tag{1.9}$$

leads to same limit geometric-harmonic mean and iteration of arithmetic and harmonic mean

$$a_{k+1} = \frac{a_k + h_k}{2}, \qquad h_{k+1} = \frac{2}{\frac{1}{a_k} + \frac{1}{h_k}}, \tag{1.10}$$

defines arithmetic-harmonic mean. It is not hard to see that this mean is exactly equal to the geometric mean.

In a recent paper [4], authors studied arithmetic-geometric mean and derived its asymptotic expansion:

$$AG(x+t, x+s) \sim x + \alpha - \tfrac{\beta^2}{4}x^{-1} + \tfrac{\alpha\beta^2}{4}x^{-2} - \tfrac{\beta^2}{64}(16\alpha^2 + 5\beta^2)x^{-3} + \dots \tag{1.11}$$

By comparing coefficients in this expansion with expansion of A and G mean, one can see that AG clearly lies somewhere in the middle, but is a little bit closer to the geometric mean. This iterative process also showed interesting convergence and stationary properties of the coefficients in the asymptotic expansion of AG mean, for details see [4].

The main aim of this paper is to derive coefficients in asymptotic expansion of the arithmetic-harmonic and harmonic-geometric mean by analyzing their iterative proccesses in a similar way as it has been done for AG mean. These coefficients will be used in the third section where we give relations and inequalities between this means.

We shall need the following fundamental lemma for the transformation of the asymptotic series. The proof is easy, see e.g. [6].

Lemma 1.1. *Let function $f(x)$ have asymptotic expansion ($a_0 = 1$)*

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \quad x \to \infty.$$

Then for all real p it holds

$$[f(x)]^p \sim \sum_{n=0}^{\infty} c_n(p) x^{-n},$$

where $c_0 = 1$ and

$$c_n = \frac{1}{n} \sum_{k=1}^{n} [k(1+p) - n] a_k c_{n-k}. \tag{1.12}$$

We shall also need the following standard result:

Lemma 1.2. *Let $f(x)$ and $g(x)$ have asymptotic expansions ($a_0 = b_0 = 1$):*

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \qquad g(x) \sim \sum_{n=0}^{\infty} b_n x^{-n}.$$

Then their product $f(x)g(x)$ and quotient $f(x)/g(x)$ have asymptotic expansions

$$f(x)g(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \qquad \frac{f(x)}{g(x)} \sim \sum_{n=0}^{\infty} d_n x^{-n},$$

where $c_0 = d_0 = 1$ and

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \qquad d_n = a_n - \sum_{k=1}^{n} b_k d_{n-k}. \tag{1.13}$$

2. Asymptotic expansions of iterative Pythagorean means

To derive asymptotic expansion of the combination of two Pythagorean means, we shall follow its iterative process.

Let us start with arithmetic-harmonic mean. Let sequence (a_n) be limiting sequence which converges above to the limit $AH(s,t)$ and let $A_n(s,t)$ be the value of n-th iteration. Similarly, let $H_n(s,t)$ be defined through the sequence (h_n). A_n and H_n are also means because they are obtained as the composition of arithmetic and harmonic mean to the previous members of these sequences.

Then, let

$$A_n(s,t) = x\sum_{k=0}^{\infty} a_k^{(n)}(s,t)x^{-k}, \tag{2.1}$$

and

$$H_n(s,t) = x\sum_{k=0}^{\infty} h_k^{(n)}(s,t)x^{-k}, \tag{2.2}$$

be asymptotic expansions of the n-th iteration.

We shall show that functions $a_k^{(n)}$ and $h_k^{(n)}$ converge to the same limit $c_k(t,s)$ when $n \to \infty$, and it holds

$$AH(s,t) = x\sum_{k=0}^{\infty} c_k(s,t)x^{-k}. \tag{2.3}$$

Theorem 2.1. *Let n be arbitrary natural number. Then we have*

$$a_k^{(n)} = h_k^{(n)} \tag{2.4}$$

for all $k \le 2n$.

In other words, for fixed k, the sequence $a_k^{(n)}$ is stationary sequence which defines the limiting value c_k. Proof of the theorem follows from the next lemma by mathematical induction.

Lemma 2.2. *Suppose that coefficients $a_k^{(n)}$ and $h_k^{(n)}$, for arbitary $n \ge 1$, satisfy*

$$a_k^{(n)} = h_k^{(n)}, \qquad k = 0, 1, \dots, K.$$

Then it holds

$$a_k^{(j)} = h_k^{(j)} = a_k^{(n)}, \qquad j \ge n+1,\ k = 0, \dots, K, \tag{2.5}$$

and

$$a_{K+1}^{(n+1)} = h_{K+1}^{(n+1)}, \qquad a_{K+2}^{(n+1)} = h_{K+2}^{(n+1)}. \tag{2.6}$$

Therefore, in each following step, at least two new coefficients coincide and remain equal in the future.

PROOF. For the arithmetic series the statement (2.5) is obvious. Let us examine (h_n).

Denote

$$A_{n,k}=\sum_{j=0}^{k} a_j^{(n)}x^{-j+1}, \qquad AR_{n,k}=\sum_{j=k+1}^{\infty} a_j^{(n)}x^{-j+1},$$

and similarly for $H_{n,k}$, $HR_{n,k}$.

Then

$$H_{n+1}=\frac{2}{A_n^{-1}+H_n^{-1}}=\frac{2A_nH_n}{A_n+H_n}=\frac{2(A_{n,k}+AR_{n,k})(H_{n,k}+HR_{n,k})}{A_{n,k}+AR_{n,k}+H_{n,k}+HR_{n,k}}.$$

Since $A_{n,k}=H_{n,k}$, this can be written as

$$H_{n+1}=A_{n,k}\frac{\left(1+\dfrac{AR_{n,k}}{A_{n,k}}\right)\left(1+\dfrac{HR_{n,k}}{A_{n,k}}\right)}{1+\dfrac{AR_{n,k}+HR_{n,k}}{2A_{n,k}}} \tag{2.7}$$

and clearly (2.5) follows.

Now, to prove (2.6), let us find the first two coefficients in the expansion of the quotient $AR_{n,k}/A_{n,k}$. Using Lemma 1.2 (recall that $a_0=g_0=1$), we have

$$\frac{AR_{n,k}}{A_{n,k}}=\frac{a_{k+1}^{(n)}}{x^{k+1}}+\frac{a_{k+2}^{(n)}-a_1^{(n)}a_{k+1}^{(n)}}{x^{k+2}}+\ldots$$

In a same way,

$$\frac{HR_{n,k}}{A_{n,k}}=\frac{h_{k+1}^{(n)}}{x^{k+1}}+\frac{h_{k+2}^{(n)}-a_1^{(n)}h_{k+1}^{(n)}}{x^{k+2}}+\ldots$$

Now, from (2.7) it follows

$$H_{n+1}=x\left(1+\frac{a_1^{(n)}}{x}+\ldots\right)\left(1+\frac{a_{k+1}^{(n)}}{x^{k+1}}+\frac{a_{k+2}^{(n)}-a_1^{(n)}a_{k+1}^{(n)}}{x^{k+2}}+\ldots\right)\cdot$$

$$\cdot\left(1+\frac{h_{k+1}^{(n)}}{x^{k+1}}+\frac{h_{k+2}^{(n)}-a_1^{(n)}h_{k+1}^{(n)}}{x^{k+2}}+\ldots\right)\left(1+\frac{u_{k+1}^{(n)}}{x^{k+1}}+\frac{u_{k+2}^{(n)}-a_1^{(n)}u_{k+1}^{(n)}}{x^{k+2}}+\ldots\right)^{-1},$$

where

$$u_{k+i}^{(n)}=\frac{a_{k+i}^{(n)}+h_{k+i}^{(n)}}{2},$$

and therefore

$$h_{k+1}^{(n+1)}=a_{k+1}^{(n)}+h_{k+1}^{(n)}-u_{k+1}^{(n)}=a_{k+1}^{(n+1)},$$

$$h_{k+2}^{(n+1)}=a_1^{(n)}a_{k+1}^{(n)}+a_1^{(n)}h_{k+1}^{(n)}-a_1^{(n)}u_{k+1}^{(n)}+\left(a_{k+2}^{(n)}-a_1^{(n)}a_{k+1}^{(n)}\right)+$$
$$+\left(h_{k+2}^{(n)}-a_1^{(n)}h_{k+1}^{(n)}\right)-\left(u_{k+2}^{(n)}-a_1^{(n)}u_{k+1}^{(n)}\right)=a_{k+2}^{(n+1)}.$$

□

Now we can calculate coefficients c_k of the asymptotic expansion (2.3). In each iteration, coefficients $a_k^{(n)}$ are easily obtained and for calculating sequence

$$h^{(n)} = \frac{2a^{(n-1)}h^{(n-1)}}{a^{(n-1)} + h^{(n-1)}},$$

we use Lemma 1.2 for multiplication and quotient of asymptotic series.

Here are the first few coefficients in terms of variables α and β:

$$\begin{aligned}
c_0 &= 1, \\
c_1 &= \alpha, \\
c_2 &= -\frac{\beta^2}{2}, \\
c_3 &= \frac{\alpha\beta^2}{2}, \\
c_4 &= -\frac{\beta^2}{8}(4\alpha^2 + \beta^2), \\
c_5 &= \frac{\alpha\beta^2}{8}(4\alpha^2 + 3\beta^2), \\
c_6 &= -\frac{\beta^2}{16}(8\alpha^4 + 12\alpha^2\beta^2 + \beta^4), \\
c_7 &= \frac{\alpha\beta^2}{16}(8\alpha^4 + 20\alpha^2\beta^2 + 5\beta^4). \\
c_8 &= -\frac{\beta^2}{128}(64\alpha^6 + 240\alpha^4\beta^2 + 120\alpha^2\beta^4 + 5\beta^6).
\end{aligned}$$

As expected, since arithmetic-harmonic mean is equal to the geometric mean, we have obtained the same coefficients as in expansion (1.6), but this way we have shown that iterative process for arithmetic-harmonic means has same stationary and convergence properties as arithmetic-geometric mean analyzed in paper [4].

We will now show that this properties are also valid for the geometric-harmonic mean.

As before, let sequence (g_n) be limiting sequence which converges above to the limit $GH(s,t)$ and let $G_n(s,t)$ be the value of n-th iteration. $H_n(s,t)$ is defined as before.

Then,

$$G_n(s,t) = x\sum_{k=0}^{\infty} g_k^{(n)}(s,t)x^{-k}, \tag{2.8}$$

and

$$H_n(s,t) = x\sum_{k=0}^{\infty} h_k^{(n)}(s,t)x^{-k}, \tag{2.9}$$

are asymptotic expansions of the n-th iteration.

Functions $g_k^{(n)}$ and $h_k^{(n)}$ converge to the same limit $d_k(t,s)$ when $n \to \infty$, and it holds

$$GH(s,t) = x\sum_{k=0}^{\infty} d_k(s,t)x^{-k}. \tag{2.10}$$

THEOREM 2.3. *Let n be arbitrary natural number. Then we have*

$$g_k^{(n)} = h_k^{(n)} \tag{2.11}$$

for all $k \le 2n$.

Again, for fixed k, the sequence $g_k^{(n)}$ is stationary sequence which defines the limiting value d_k. Proof follows from the next lemma.

LEMMA 2.4. *Suppose that coefficients $g_k^{(n)}$ and $h_k^{(n)}$, for arbitary $n \ge 1$ satisfy*

$$g_k^{(n)} = h_k^{(n)}, \qquad k = 0, 1, \dots, K.$$

Then it holds

$$g_k^{(j)} = h_k^{(j)} = g_k^{(n)}, \qquad j \ge n+1,\ k = 0, \dots, K, \tag{2.12}$$

and

$$g_{K+1}^{(n+1)} = h_{K+1}^{(n+1)}, \qquad g_{K+2}^{(n+1)} = h_{K+2}^{(n+1)}. \tag{2.13}$$

PROOF. Let us first prove (2.5).

Denote

$$G_{n,k} = \sum_{j=0}^{k} g_j^{(n)}x^{-j+1}, \qquad GR_{n,k} = \sum_{j=k+1}^{\infty} g_j^{(n)}x^{-j+1},$$

and same for $H_{n,k}$, $HR_{n,k}$.

Then

$$H_{n+1} = \frac{2}{G_n^{-1} + H_n^{-1}} = \frac{2G_nH_n}{G_n + H_n} = \frac{2(G_{n,k} + GR_{n,k})(H_{n,k} + HR_{n,k})}{G_{n,k} + GR_{n,k} + H_{n,k} + HR_{n,k}}.$$

Since $G_{n,k} = H_{n,k}$, this can be written as

$$H_{n+1} = G_{n,k}\frac{\left(1 + \dfrac{GR_{n,k}}{G_{n,k}}\right)\left(1 + \dfrac{HR_{n,k}}{G_{n,k}}\right)}{1 + \dfrac{GR_{n,k} + HR_{n,k}}{2G_{n,k}}} \tag{2.14}$$

and clearly (2.12) is valid for the harmonic series.

Similary,

$$G_{n+1} = \sqrt{G_n H_n} = \sqrt{(G_{n,k} + GR_{n,k})(H_{n,k} + HR_{n,k})},$$

which can be written as

$$G_{n+1} = G_{n,k} \sqrt{\left(1 + \frac{GR_{n,k}}{G_{n,k}}\right)\left(1 + \frac{HR_{n,k}}{G_{n,k}}\right)} \tag{2.15}$$

and (2.12) follows.

To prove (2.13), we again start with the first two coefficients in the expansion of the quotients

$$\frac{GR_{n,k}}{G_{n,k}} = \frac{g_{k+1}^{(n)}}{x^{k+1}} + \frac{g_{k+2}^{(n)} - g_1^{(n)} g_{k+1}^{(n)}}{x^{k+2}} + \dots,$$

and

$$\frac{HR_{n,k}}{G_{n,k}} = \frac{h_{k+1}^{(n)}}{x^{k+1}} + \frac{h_{k+2}^{(n)} - g_1^{(n)} h_{k+1}^{(n)}}{x^{k+2}} + \dots.$$

Now, applying binomial formula to (2.15) it follows

$$G_{n+1} = x\left(1 + \frac{g_1^{(n)}}{x} + \dots\right)\left(1 + \binom{\frac{1}{2}}{1}\frac{g_{k+1}^{(n)}}{x^{k+1}} + \binom{\frac{1}{2}}{1}\frac{g_{k+2}^{(n)} - g_1^{(n)} g_{k+1}^{(n)}}{x^{k+2}} + \dots\right) \cdot$$
$$\cdot \left(1 + \binom{\frac{1}{2}}{1}\frac{h_{k+1}^{(n)}}{x^{k+1}} + \binom{\frac{1}{2}}{1}\frac{h_{k+2}^{(n)} - g_1^{(n)} h_{k+1}^{(n)}}{x^{k+2}} + \dots\right),$$

and from (2.14) we have

$$H_{n+1} = x\left(1 + \frac{g_1^{(n)}}{x} + \dots\right)\left(1 + \frac{g_{k+1}^{(n)}}{x^{k+1}} + \frac{g_{k+2}^{(n)} - g_1^{(n)} g_{k+1}^{(n)}}{x^{k+2}} + \dots\right) \cdot$$
$$\cdot \left(1 + \frac{h_{k+1}^{(n)}}{x^{k+1}} + \frac{h_{k+2}^{(n)} - g_1^{(n)} h_{k+1}^{(n)}}{x^{k+2}} + \dots\right)\left(1 + \frac{v_{k+1}^{(n)}}{x^{k+1}} + \frac{v_{k+2}^{(n)} - g_1^{(n)} v_{k+1}^{(n)}}{x^{k+2}} + \dots\right)^{-1},$$

where

$$v_{k+i}^{(n)} = \frac{g_{k+i}^{(n)} + h_{k+i}^{(n)}}{2}.$$

Therefore,

$$h_{k+1}^{(n+1)} = g_{k+1}^{(n)} + h_{k+1}^{(n)} - v_{k+1}^{(n)} = \frac{g_{k+1}^{(n)} + h_{k+1}^{(n)}}{2} = g_{k+1}^{(n+1)},$$

$$h_{k+2}^{(n+1)} = g_1^{(n)} g_{k+1}^{(n)} + g_1^{(n)} h_{k+1}^{(n)} - g_1^{(n)} v_{k+1}^{(n)} + \left(g(n)_{k+2} - g_1^{(n)} g_{k+1}^{(n)}\right) +$$
$$+ \left(h_{k+2}^{(n)} - g_1^{(n)} h_{k+1}^{(n)}\right) - \left(v_{k+2}^{(n)} - g_1^{(n)} v_{k+1}^{(n)}\right)$$

$$
\begin{aligned}
&= \frac{g_1^{(n)} g_{k+1}^{(n)}}{2} + \frac{g_1^{(n)} h_{k+1}^{(n)}}{2} + \frac{g_{k+2}^{(n)} - g_1^{(n)} g_{k+1}^{(n)}}{2} + \frac{h_{k+2}^{(n)} - g_1^{(n)} h_{k+1}^{(n)}}{2} \\
&= g_{k+2}^{(n+1)},
\end{aligned}
$$

and the proof is complete. ∎

Finally, we shall derive coefficients d_k of the asymptotic expansion (2.10). In each iteration, for calculating $h^{(n)}$ we use Lemma 1.2 for multiplication and quotient of asymptotic series, and for $g^{(n)}$ we use Lemma 1.1 with $p = \frac{1}{2}$.

So, the first few coefficients in the expansion of the geometric-harmonic mean, in terms of variables α and β, are:

$$
\begin{aligned}
d_0 &= 1, \\
d_1 &= \alpha, \\
d_2 &= -\frac{3\beta^2}{4}, \\
d_3 &= \frac{3\alpha\beta^2}{4}, \\
d_4 &= -\frac{\beta^2}{64}(48\alpha^2 + 7\beta^2), \\
d_5 &= \frac{3\alpha\beta^2}{64}(16\alpha^2 + 7\beta^2), \\
d_6 &= -\frac{\beta^2}{256}(192\alpha^4 + 168\alpha^2\beta^2 + 11\beta^4), \\
d_7 &= \frac{\alpha\beta^2}{256}(192\alpha^4 + 280\alpha^2\beta^2 + 55\beta^4). \\
d_8 &= -\frac{3\beta^2}{16384}(4096\alpha^6 + 8960\alpha^4\beta^2 + 3520\alpha^2\beta^4 + 125\beta^6).
\end{aligned}
$$

Remark 2.5. Asymptotic expansion of the geometric-harmonic mean can be also obtained through its relation to the arithmetic-geometric mean. It holds

$$
GH(s,t) = \frac{1}{AG(\frac{1}{s}, \frac{1}{t})}. \tag{2.16}
$$

Therefore, we can apply Lemma 1.1 ($p = -1$) with asymptotic expansion of AG mean derived in paper [4], but it was also interesting to examine stationary properties of GH mean through its iterative process.

3. Asymptotic comparison of Pythagorean means

In [5, 7], authors developed techniques for comparison of means through their asymptotic expansions.

Definition 3.1. *Let M_1 and M_2 be any two means and*

$$M_1(x+s,x+t)-M_2(x+s,x+t)=c_k(s,t)x^{-k+1}+O(x^{-k}).$$

If $c_k(s,t)>0$ for all s and t then we say that mean M_1 is asymptotically greater than mean M_2 and write

$$M_1 \succ M_2.$$

Of course, this is equivalent to

$$M_1 \prec M_2.$$

Theorem 3.2. *If $M_1 \geq M_2$, then $M_1 \succ M_2$.*

In other words, asymptotic inequalities can be considered as a necessary relation between comparable means, see cited papers.

We will now present asymptotic relation between Pythagorean means and their iterative combinations. It is shown in [5] that for the comparison of means, it is sufficient to consider the case $\alpha=0$. In this case $c_{2n+1}=0$, so in the next table we will show only even coefficients in the asymptotic expansions obtained in the previous section.

Table 1. Expansions of the iterative Pythagorean means

M	x	t^2/x	t^4/x^3	t^6/x^5	t^8/x^7
A	1	0	0	0	0
AG	1	$-\frac{1}{4}$	$-\frac{5}{64}$	$-\frac{11}{256}$	$-\frac{469}{16384}$
$G=AH$	1	$-\frac{1}{2}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$-\frac{5}{128}$
GH	1	$-\frac{3}{4}$	$-\frac{7}{64}$	$-\frac{11}{256}$	$-\frac{375}{16384}$
H	1	-1	0	0	0

As we can see, coefficients coincide with the known inequality between Pythagorean means:

$$H \leq GH \leq G=AH \leq AG \leq A. \tag{3.1}$$

Iterative means AG and GH obviously lie in the middle of their starting means, but they are both closer to the geometric mean. According to coefficients next to x^{-3}, it is also interesting to see that GH is a little bit closer to the geometric mean than AG. Using this method, one can easily compare GH with other classical means as well, see cited papers for details about this concept.

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Asimptotsko ponašanje iterativnih pitagorejskih sredina

Tomislav Burić

SAŽETAK. Dobiveni su i analizirani asimptotski razvoji i ponašanje iterativnih kombinacija pitagorejskih sredina (aritmetičke, geometrijske i harmonijske sredine). Rezultati se koriste za asimptotsku usporedbu sredina.

Tomislav Burić
Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia
E-mail: tomislav.buric@fer.hr

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