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SEIFFERT MEANS, ASYMPTOTIC EXPANSIONS AND INEQUALITIES

Lenka Vukšić

Abstract. In this paper we study inequalities of the form

$$(1-\mu)M_1(s,t)+\mu M_3(s,t) \leq M_2(s,t) \leq (1-\nu)M_1(s,t)+\nu M_3(s,t),$$

which cover some classical bivariate means and Seiffert means. Using techniques of asymptotic expansions detailed analysis was made and the method for obtaining optimal parameters μ and ν was described.

1. Introduction

Let $0 < s < t$. Seiffert means are defined by (see [4]):

$$P(s,t) = \frac{t-s}{2\arcsin\frac{t-s}{t+s}},$$

and

$$T(s,t) = \frac{t-s}{2\arctan\frac{t-s}{t+s}}.$$

There is a large number of papers studying inequalities between Seiffert means and convex combinations of other means. Some of the known results are the following. In [7] authors established that the double inequality

$$\mu A(s,t)+(1-\mu)H(s,t) < P(s,t) < \nu A(s,t)+(1-\nu)H(s,t) \tag{1.1}$$

holds for all $s,t > 0$ with $s \neq t$ if and only if $\mu \leq \frac{2}{\pi}$ and $\nu \geq \frac{5}{6}$. Authors in [9] gave the optimal parameters $\mu = \frac{3}{5}$, $\nu = \frac{\pi}{4}$ for the following double inequality:

$$\mu T(s,t)+(1-\mu)G(s,t) < A(s,t) < \nu T(s,t)+(1-\nu)G(s,t) \tag{1.2}$$

where $s,t > 0$, $s \neq t$. Furthermore, in [8] it was proved that the double inequality

$$\mu Q(s,t)+(1-\mu)A(s,t) < T(s,t) < \nu Q(s,t)+(1-\nu)A(s,t) \tag{1.3}$$

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holds for all $s,t>0$ with $s\neq t$, if and only if $\mu\leq\frac{4-\pi}{\pi(\sqrt{2}-1)}$ and $\nu\geq\frac{2}{3}$. In [13] it was proved that the double inequality

$$\mu C(s,t)+(1-\mu)H(s,t)<P(s,t)<\nu T(s,t)+(1-\nu)H(s,t) \tag{1.4}$$

holds for all $s,t>0$ with $s\neq t$, if and only if $\mu\leq\frac{3}{2\pi}$ and $\nu\geq\frac{5}{8}$. In [17] authors proved that the double inequality

$$\mu N(s,t)+(1-\mu)G(s,t)<P(s,t)<\nu N(s,t)+(1-\nu)G(s,t) \tag{1.5}$$

holds for all $s,t>0$ with $s\neq t$, if and only if $\mu\leq\frac{2}{9}$ and $\nu\geq\frac{1}{\pi}$.

The subject of this paper is to give a systematic study of inequalities of the form

$$(1-\mu)M_1+\mu M_3\leq M_2\leq(1-\nu)M_1+\nu M_3 \tag{1.6}$$

where M_i are chosen from the class of elementary means given below.

Here is the list of means which along with the Seiffert means take part in the inequalities of the type (1.6):

$$H(s,t)=\frac{2st}{s+t},\qquad G(s,t)=\sqrt{st},\qquad L(s,t)=\frac{t-s}{\log t-\log s},$$
$$A(s,t)=\frac{s+t}{2},\qquad C(s,t)=\frac{2}{3}\cdot\frac{s^2+st+t^2}{s+t},\qquad Q(s,t)=\sqrt{\frac{s^2+t^2}{2}},$$
$$N(s,t)=\frac{s^2+t^2}{s+t}.$$

These means are harmonic mean (H), geometric mean (G), logarithmic mean (L), arithmetic mean (A), centroidal mean (C), root mean square (Q), and contraharmonic mean (N). Since it holds

$$H\leq G\leq L\leq P\leq A\leq T\leq C\leq Q\leq N \tag{1.7}$$

we assume

$$M_1\leq M_2\leq M_3. \tag{1.8}$$

Hence, (1.6) is equivalent to

$$\mu\leq\frac{M_2-M_1}{M_3-M_1}\leq\nu \tag{1.9}$$

and we are dealing with the problem of finding minimum and maximum of the function in the middle.

It is explained in [10] how asymptotic expansions can be used in finding optimal constants a, b and c such that inequality

$$aM_1+bM_2+cM_3\geq 0$$

would be possible. For some general information concerning asymptotic series see [12]. Applying the same method here we find the smallest value of μ and the largest value of ν such that (1.6) is possible. This optimality will be ensured by conditions imposed on appropriate coefficients in asymptotic power series expansion of considered combinations of means.

2. Asymptotic expansions

Asymptotic expansion of a mean M has the following form

$$M(x+s,x+t) \sim x + c_1(s,t) + \frac{c_2(s,t)}{x} + \frac{c_3(s,t)}{x^2} + \dots \quad \text{as } x \to \infty,$$

where $c_n(t,s)$ is a polynomial of order n. The coefficients c_n have a simpler form when they are presented in terms of variables α and β where

$$\alpha = \frac{t+s}{2}, \quad \beta = \frac{t-s}{2}.$$

Asymptotic expansions of harmonic, geometric, logarithmic, arithmetic centroidal, quadratic mean and contraharmonic mean among other classical means are given in [11]:

(2.1)

$$\begin{aligned}
H(x+\alpha-\beta,x+\alpha+\beta) &\sim x+\alpha-\beta^2x^{-1}+\alpha\beta^2x^{-2}-\alpha^2\beta^2x^{-3}+\dots\\
G(x+\alpha-\beta,x+\alpha+\beta) &\sim x+\alpha-\frac{1}{2}\beta^2x^{-1}+\frac{1}{2}\alpha\beta^2x^{-2}\\
&\qquad -\frac{1}{8}\beta^2(4\alpha^2+\beta^2)x^{-3}+\dots\\
L(x+\alpha-\beta,x+\alpha+\beta) &\sim x+\alpha-\frac{1}{3}\beta^2x^{-1}+\frac{1}{3}\alpha\beta^2x^{-2}\\
&\qquad -\frac{1}{45}\beta^2(15\alpha^2+4\beta^2)x^{-3}+\dots\\
A(x+\alpha-\beta,x+\alpha+\beta) &\sim x+\alpha\\
C(x+\alpha-\beta,x+\alpha+\beta) &\sim x+\alpha+\frac{1}{3}\beta^2x^{-1}-\frac{1}{3}\alpha\beta^2x^{-2}+\frac{1}{3}\alpha^2\beta^2x^{-3}+\dots\\
Q(x+\alpha-\beta,x+\alpha+\beta) &\sim x+\alpha+\frac{1}{2}\beta^2x^{-1}-\frac{1}{2}\alpha\beta^2x^{-2}\\
&\qquad +\frac{1}{8}\beta^2(4\alpha^2-\beta^2)x^{-3}+\dots\\
N(x+\alpha-\beta,x+\alpha+\beta) &\sim x+\alpha+\beta^2x^{-1}-\alpha\beta^2x^{-2}+\alpha^2\beta^2x^{-3}+\dots
\end{aligned}$$

It remains to find asymptotic expansions of Seiffert means. Computation of coefficients in asymptotic expansion of these means is based on manipulation with series expansions of functions arcsin and arctan. The following lemma, known in the context of power series, will be used here. We give its version for asymptotic series (see [6]).

Lemma 2.1. *Let g be a function with asymptotic expansion (as $x \to \infty$):*

$$g(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \qquad (c_0 \neq 0).$$

Then for all real r it holds

$$[g(x)]^r \sim \sum_{n=0}^{\infty} P_n x^{-n}$$

where

$$\begin{aligned} P_0 &= c_0^r, \\ P_n &= \frac{1}{nc_0} \sum_{k=1}^{n} [k(1+r)-n] c_k P_{n-k}, \quad n \geq 1. \end{aligned}$$

In particular, for the choice $r = -1$, coefficients P_n of the reciprocal value of an asymptotic series are given by the recursive relation:

$$\begin{aligned} P_0 &= \frac{1}{c_0}, \\ P_n &= -\frac{1}{c_0} \sum_{k=1}^{n} c_k P_{n-k}, \quad n \geq 1. \end{aligned}$$

Let

$$f(x) = \sum_{n=1}^{\infty} a_n x^n$$

be Maclaurin expansion for any of the functions arcsin or arctan. Then for any of Seiffert means M:

$$\begin{aligned} \frac{1}{M(x+s,x+t)} &= \frac{1}{\beta} \sum_{n=1}^{\infty} a_n \beta^n (x+\alpha)^{-n} \\ &= \sum_{n=1}^{\infty} a_n \beta^{n-1} \sum_{k=0}^{\infty} \binom{-n}{k} x^{-n-k} \alpha^k \\ &= \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} a_k \beta^{k-1} \binom{-k}{n-k} \alpha^{n-k} \right] x^{-n} \\ &= \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} (-1)^{n-k} a_k \beta^{k-1} \binom{n-1}{k-1} \alpha^{n-k} \right] x^{-n} \\ &= x^{-1} \sum_{n=0}^{\infty} (-1)^n \left[\sum_{k=0}^{\lfloor n/2 \rfloor} a_{2k+1} \beta^{2k} \binom{n}{2k} \alpha^{n-2k} \right] x^{-n} \\ &= x^{-1} \sum_{n=0}^{\infty} r_n x^{-n}. \end{aligned}$$

Now we have

$$a_{2k+1} = \binom{k-\frac{1}{2}}{k} \frac{1}{2k+1}$$

for the first Seiffert mean, and

$$a_{2k+1} = (-1)^k \frac{1}{2k+1}$$

for the second one. In order to find asymptotic expansion of Seiffert means, it is sufficient to apply Lemma 2.1. Thus we have proved the following theorem.

THEOREM 2.2. *Seiffert means have the asymptotic expansion (M stands for either P or T):*

$$M(x+s, x+t) \sim x \sum_{n=0}^{\infty} c_n x^{-n}, \quad x \to \infty,$$

where coefficients c_k are given by recursive relation

$$\begin{aligned} c_0 &= 1, \\ c_n &= \sum_{k=1}^{n} (-1)^{k+1} \left[\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} a_{2j+1} \binom{k}{2j} \beta^{2j} \alpha^{k-2j} \right] c_{n-k}, \quad n \geq 1. \end{aligned}$$

In the case of the first Seiffert mean $a_{2j+1} = \binom{j-\frac{1}{2}}{j} \frac{1}{2j+1}$, and in the case of the second Seiffert mean $a_{2j+1} = (-1)^j \frac{1}{2j+1}$.

Here are the first few coefficients for both of these means.
The first Seiffert mean P:

$$\begin{aligned} c_0 &= 1, & c_4 &= -\frac{\beta^2(60\alpha^2 + 17\beta^2)}{360}, \\ c_1 &= \alpha, & c_5 &= \frac{\alpha\beta^2(20\alpha^2 + 17\beta^2)}{120}, \\ c_2 &= -\frac{\beta^2}{6}, & c_6 &= -\frac{\beta^2(2520\alpha^4 + 4284\alpha^2\beta^2 + 367\beta^4)}{15120}, \\ c_3 &= \frac{\alpha\beta^2}{6}, & &\vdots \end{aligned}$$

The second Seiffert mean T:

$$\begin{aligned} c_0 &= 1, & c_4 &= \frac{\beta^2(15\alpha^2 - 4\beta^2)}{45}, \\ c_1 &= \alpha, & c_5 &= -\frac{\alpha\beta^2(5\alpha^2 - 4\beta^2)}{15}, \\ c_2 &= \frac{\beta^2}{3}, & c_6 &= \frac{\beta^2(315\alpha^4 - 504\alpha^2\beta^2 + 44\beta^4)}{945}, \\ c_3 &= -\frac{\alpha\beta^2}{3}, & &\vdots \end{aligned}$$

3. Comparison of means. Asymptotic inequalities

In this section we present a method for obtaining the best parameters μ and ν such that (1.6) is possible for all combinations of means mentioned in the introduction that include one or both Seiffert means. To this end we introduce the notion of asymptotic inequality.

Definition 3.1. *Let $F(s,t)$ be any homogenous bivariate function such that*

$$F(x+s,x+t)=c_k(t,s)x^{-k+1}+\mathcal{O}(x^{-k}).$$

If $c_k(s,t)>0$ for all s and t, then we say that F is asymptotically greater than zero, and write

$$F\succ 0.$$

Theorem 3.2. *If $F\geq 0$, then $F\succ 0$.*

Proof. For x large enough, $F(x+s,x+t)$ has the same sign as the first term in its asymptotic expansion. ∎

Therefore, one may consider asymptotic inequalities as a necessary relation between comparable means. Furthermore, for the asymptotic inequalities it is sufficient to observe the case $\alpha=0$ as explained in [10].

In this paper we will compare convex combination of two means with the third mean. Let

$$F(s,t;\mu)=(1-\mu)M_1(s,t)+\mu M_3(s,t)-M_2(s,t).$$

Then function F has the following asymptotic expansion:

$$F(x+s,x+t;\mu)\sim F^{(0)}(\mu)x+F^{(1)}(\mu)+F^{(2)}(\mu)x^{-1}+F^{(3)}(\mu)x^{-2}+\cdots$$

where $F^{(j)}$ is expressed through j-th coefficients of means M_i, that is

$$F^{(j)}(\mu)=(1-\mu)M_1^{(j)}+\mu M_3^{(j)}-M_2^{(j)}.$$

Asymptotic expansions of all of the means under consideration start with $x+\alpha$. Hence, first two coefficients in expansion of the function F equal zero. Suppose that also $F^{(2)}(\mu)=0$ and that the following inequality holds

$$(1-\mu)M_1+\mu M_3-M_2\geq\ 0. \tag{3.1}$$

Thus the linear combination on the left is asymptotically greater than 0. As a consequence of theorem (3.2) and sequence of inequalities (1.8) we obtain the monotonicity of

$$(1-\mu)M_1^{(2)}+\mu M_3^{(2)}-M_2^{(2)}$$

in variable μ. This can also be seen from asymptotic expansions (2.1). Hence, in the case where $M_1^{(2)}$ and $M_3^{(2)}$ are different, taking smaller μ will result by decreasing $F^{(2)}(\mu)$. Then the linear combination will be asymptotically smaller than zero and thus inequality (3.1) cannot be true for some point

(s,t). If $M_1^{(2)}$ equals $M_3^{(2)}$ then obviously $M_2^{(2)}$ is the same as those two so we proceed similarly with the next coefficients. For example, that situation can happen if we involve Heronian mean in combination with first Seiffert and identric mean. Analogous conclusions are drawn if in the (3.1) stands ≤ 0.

Values of μ such that $F^{(2)}(\mu) = 0$ are given in the table below and the corresponding first nonzero term in asymptotic expansion of $F(x-\beta, x+\beta; \mu)$ is calculated. Combinations of means for which the numerical calculation showed that proper inequality couldn't hold are excluded from the table.

TABLE 1

H	G	L	P	A	C	Q	N	$\times\beta^4/x^3$
2/5	−1		3/5					29/300
1/5		−1	4/5					23/450
1/6			−1	5/6				17/360
3/8			−1		5/8			17/360
7/12			−1				5/12	17/360
	1/2	−1	1/2					1/360
	1/3		−1	2/3				1/180
	−3/5		1		−2/5			1/36
	−2/3		1			−1/3		7/90
	−7/9		1				−2/9	1/20
		−3/4	1		−1/4			7/360
		−4/5	1			−1/5		11/225
		−7/8	1				−1/8	11/225
			−2/3	1	−1/3			17/540
			−3/4	1		−1/4		1/15
			−6/7	1			−1/7	17/420
			−1/4		1	−3/4		19/180
			−4/7		1		−3/7	17/630
			3/7			−1	4/7	11/105

This table should be interpreted as follows:

$$\frac{2}{5}H + \frac{3}{5}P \succ G$$
$$\frac{1}{5}H + \frac{4}{5}P \succ L$$
$$\frac{1}{6}H + \frac{5}{6}A \succ P$$
$$\frac{3}{8}H + \frac{5}{8}C \succ P$$
$$\vdots$$

Other parameter can be obtained from the boundary condition in the most of the cases. Because of homogeneity of means it suffices to observe relations on some curve that intersects all lines passing through the origin. We chose the line segment $\{(s, 1-s) \colon s \in [0,1]\}$. Moreover, in our case means are symmetric and the problem reduces on finding infimum and supremum of the function in the middle of (1.9) in variables $(s, 1-s)$ for $s \in [0, \frac{1}{2}]$. For the most of the combinations of means mentioned in the introduction, this function appears to be monotonic and takes the minimum and maximum values at the edges. Therefore, in order to reach the optimality we require

$$(1-\nu)M_1(0,1) + \nu M_3(0,1) = M_2(0,1)$$

which makes sense if the value of M_1 differs from value of M_3 in $(0,1)$. Then we calculate corresponding value of $F^{(2)}(\nu)$ to determine the direction of the inequality. Such inequalities still have to be proved. Notice that μ and ν obtained by procedure described above, coincide with those from (1.1), (1.2), (1.3), (1.4) and (1.5). Among the other combinations of means there is a large number of those which can hold for all $s, t \geq 0$. Some of them are proved in the following subsections and some are stated in the form of conjectures.

3.1. *Comparison of the first Seiffert mean with other means.* Let us illustrate more precisely previously explained method for obtaining the best parameters by taking the example of the first Seiffert, arithmetic and contraharmonic mean. We want to find the largest μ and the smallest ν such that the inequality

$$(1-\mu)P(s,t) + \mu N(s,t) \leq A(s,t) \leq (1-\nu)P(s,t) + \nu N(s,t) \tag{3.2}$$

is possible. We read from the Table 1:

$$\tfrac{6}{7}P(x-\beta, x+\beta) + \tfrac{1}{7}N(x-\beta, x+\beta) - A(x-\beta, x+\beta) \sim -\tfrac{17}{420}\beta^4 x^{-3} + \dots$$

and conclude that $\mu = \frac{1}{7}$. On the other side, we have

$$\frac{A(0,1) - P(0,1)}{N(0,1) - P(0,1)} = \frac{\pi - 2}{2\pi - 2},$$

and

$$\tfrac{\pi}{2\pi-2}P(x-\beta, x+\beta) + \tfrac{\pi-2}{2\pi-2}N(x-\beta, x+\beta) - A(x-\beta, x+\beta) \sim \tfrac{5\pi-12}{12(\pi-1)}\beta^2 x^{-1} + \dots$$

Hence, $\nu = \frac{\pi-2}{2\pi-2}$. The double inequality (3.2) really holds for such μ and ν which will be proved in the next theorem.

Theorem 3.3. *The following double inequalities hold for all $s, t > 0$:*

$$\tfrac{6}{7}P(s,t) + \tfrac{1}{7}N(s,t) \leq A(s,t) \leq \tfrac{\pi}{2\pi-2}P(s,t) + \tfrac{\pi-2}{2\pi-2}N(s,t) \tag{3.3}$$

$$\tfrac{2}{3}P(s,t) + \tfrac{1}{3}C(s,t) \leq A(s,t) \leq \tfrac{\pi}{4\pi-6}P(s,t) + \tfrac{3\pi-6}{4\pi-6}C(s,t) \tag{3.4}$$

$$\tfrac{4}{7}P(s,t) + \tfrac{3}{7}N(s,t) \leq C(s,t) \leq \tfrac{\pi}{3\pi-3}P(s,t) + \tfrac{2\pi-3}{3\pi-3}N(s,t) \tag{3.5}$$

and the choice of parameters is the best possible.

Proof. First, we prove (3.3) which is equivalent to

$$\frac{1}{7} \leq \frac{A(s,t)-P(s,t)}{N(s,t)-P(s,t)} \leq \frac{\pi-2}{2\pi-2} \tag{3.6}$$

for all $s,t \geq 0$. Since these means are symmetric and homogeneous, it is sufficient to show (3.6) for all $t \geq 1, s=1$. Let $t = \frac{1+\sin\varphi}{1-\sin\varphi}$, $\varphi \in \left[0, \frac{\pi}{2}\right\rangle$. Then (3.6) becomes

$$\frac{1}{7} \leq M(\varphi) \leq \frac{\pi-2}{2\pi-2}$$

where

$$M(\varphi) = \frac{\varphi - \sin\varphi}{\varphi + \sin\varphi(\varphi\sin\varphi - 1)}.$$

As it was proved in [16], function

$$h(\varphi) = \frac{1}{\varphi\sin\varphi} - \frac{1}{\sin^2\varphi} + 1$$

is strictly decreasing on $[0,\pi]$. In particular, $h(\varphi)-1$ is strictly decreasing on $\left[0, \frac{\pi}{2}\right\rangle$ and negative so

$$M(\varphi) = \frac{1}{1 + \frac{1}{1-h(\varphi)}}$$

is strictly increasing. Values at the edges

$$\lim_{\varphi\to 0} M(\varphi) = \frac{1}{7}, \qquad M\left(\frac{\pi}{2}\right) = \frac{\pi-2}{2\pi-2}$$

complete the proof of (3.3). Analogously

$$M_{PAC}(\varphi) = \frac{A(1,t)-P(1,t)}{C(1,t)-P(1,t)} = \frac{1}{1 + \frac{1}{3}\frac{1}{1-h(\varphi)}}$$

is strictly increasing and

$$\lim_{\varphi\to 0} M_{PAC}(\varphi) = \frac{1}{3}, \qquad M_{PAC}\left(\frac{\pi}{2}\right) = \frac{3\pi-6}{4\pi-6}.$$

Finally,

$$M_{PCN}(\varphi) = \frac{C(1,t)-P(1,t)}{N(1,t)-P(1,t)} = \frac{1}{3} + \frac{2}{3}\frac{1}{1 + \frac{1}{1-h(\varphi)}}$$

is strictly increasing and

$$\lim_{\varphi\to 0} M_{PCN}(\varphi) = \frac{3}{7}, \qquad M_{PCN}\left(\frac{\pi}{2}\right) = \frac{2\pi-3}{3\pi-3}.$$

∎

Other than these inequalities there are unproved ones of this type. We mention some interesting examples verified through CAS. For the sake of simplicity, variables s,t are omitted.

Conjecture 3.4. *The following double inequalities hold true with the best possible parameters:*

$$\frac{\pi-2}{\pi}G + \frac{2}{\pi}A \le P \le \frac{1}{3}G + \frac{2}{3}A \tag{3.7}$$

$$\frac{2}{3}G + \frac{1}{3}Q \le P \le \frac{\pi-\sqrt{2}}{\pi}G + \frac{\sqrt{2}}{\pi}Q \tag{3.8}$$

$$\frac{4}{5}L + \frac{1}{5}Q \le P \le \frac{\pi-\sqrt{2}}{\pi}L + \frac{\sqrt{2}}{\pi}Q \tag{3.9}$$

$$\frac{7}{8}L + \frac{1}{8}N \le P \le \frac{\pi-1}{\pi}L + \frac{1}{\pi}N \tag{3.10}$$

$$\frac{3}{4}P + \frac{1}{4}Q \le A \le \frac{(\sqrt{2}-1)\pi}{\sqrt{2}\pi-2}P + \frac{\pi-2}{\sqrt{2}\pi-2}Q \tag{3.11}$$

3.2. *Comparison of the second Seiffert mean with other means.* We proceed equally with the second Seiffert mean. Here are given optimal parameters for the asymptotic side of the double inequality.

Table 2

H	G	L	A	T	C	Q	N	$\times\beta^4/x^3$
5/8	−1			3/8				11/120
1/2		−1		1/2				2/45
−1/4			1	−3/4				1/15
0				−1	1			4/45
−1/9				1		−8/9		1/45
1/3				−1			2/3	4/45
	−2/5		1	−3/5				31/300
	0			−1	1			4/45
	−1/6			1		−5/6		13/360
		−1/2	1	−1/2				4/45
		0		−1	1			4/45
		−1/5		1		−4/5		13/450
			0	−1	1			4/45
			1/3	−1		2/3		1/180
			2/3	−1			1/3	4/45
				−1	1	0		4/45
				−1	1		0	4/45
				3/4		−1	1/4	7/120

Consider the arithmetic, second Seiffert and contraharmonic mean. By equating $(1-\mu)A^{(2)} + \mu N^{(2)}$ with $T^{(2)}$ we obtain $\mu = \frac{1}{3}$ (as it is written in Table 2). In that case we have

$$\frac{2}{3}A(x-\beta, x+\beta) + \frac{1}{3}N(x-\beta, x+\beta) - T(x-\beta, x+\beta) \sim \frac{4}{45}\beta^4 x^{-3} + \cdots$$

Hence, by the definition

$$\frac{2}{3}A + \frac{1}{3}N \succ T.$$

On the other side, from the boundary condition

$$(1-\nu)A(0,1)+\nu N(0,1)=T(0,1)$$

we get $\nu=\frac{4-\pi}{\pi}$ and

$$\tfrac{2\pi-4}{\pi}A(x-\beta,x+\beta)+\tfrac{4-\pi}{\pi}N(x-\beta,x+\beta)-T(x-\beta,x+\beta)\sim-\tfrac{4(\pi-3)}{3\pi}\beta^4x^{-1}+\cdots$$

Therefore,

$$\tfrac{2\pi-4}{\pi}A+\tfrac{4-\pi}{\pi}N\prec T.$$

Inequality

$$\tfrac{2\pi-4}{\pi}A(s,t)+\tfrac{4-\pi}{\pi}N(s,t)\le T(s,t)\le\tfrac{2}{3}A(s,t)+\tfrac{1}{3}N(s,t) \tag{3.12}$$

was proved in [15]. Furthermore, we can prove some double inequalities that cover centroidal mean C.

Theorem 3.5. *The following double inequalities hold for all $s,t>0$*

$$\tfrac{1}{4}H(s,t)+\tfrac{3}{4}T(s,t)\le A(s,t)\le\tfrac{4-\pi}{4}H(s,t)+\tfrac{\pi}{4}T(s,t) \tag{3.13}$$

$$\tfrac{\pi-3}{\pi}H(s,t)+\tfrac{3}{\pi}C(s,t)\le T(s,t)\le C(s,t) \tag{3.14}$$

$$\tfrac{\pi-2}{\pi}H(s,t)+\tfrac{2}{\pi}N(s,t)\le T(s,t)\le\tfrac{1}{3}H(s,t)+\tfrac{2}{3}N(s,t) \tag{3.15}$$

$$\tfrac{4\pi-12}{\pi}A(s,t)+\tfrac{12-3\pi}{\pi}C(s,t)\le T(s,t)\le C(s,t) \tag{3.16}$$

and the chosen parameters are the best possible.

Proof. For the first double inequality consider the function

$$M_{HAT}(t)=\frac{A(1,t)-H(1,t)}{T(1,t)-H(1,t)},\quad t\ge1.$$

Because of the homogeneity and symmetry of means and since

$$\lim_{t\to1}M_{HAT}(t)=\tfrac{3}{4},\quad\lim_{t\to\infty}M_{HAT}(t)=\tfrac{\pi}{4},$$

it suffices to show M_{HAT} is increasing. The discussion from the beginning of this section will provide the optimality of given parameters.

Substituting $\frac{t-1}{t+1}$ with $\tan\varphi$, we obtain

$$M_{HAT}(t)=\frac{\varphi}{\varphi+\cot\varphi-\varphi\cot^2\varphi}=\frac{1}{M(\varphi)+1},\quad\varphi\in\left[0,\tfrac{\pi}{4}\right\rangle$$

where

$$M(\varphi)=\tfrac{1}{\varphi}\cot\varphi-\cot^2\varphi. \tag{3.17}$$

By the monotonicity of the function $M(\varphi)$ proved in [18], inequality (3.13) follows.

Other double inequalities follow similarly since

$$M_{HTC}(\varphi)=\frac{T(1,t)-H(1,t)}{C(1,t)-H(1,t)}=\frac{3}{4}(1+M(\varphi)),$$

$$M_{HTN}(\varphi) = \frac{T(1,t) - H(1,t)}{N(1,t) - H(1,t)} = \frac{1}{2}(1 + M(\varphi)),$$
$$M_{ATC}(\varphi) = \frac{T(1,t) - A(1,t)}{C(1,t) - A(1,t)} = 3M(\varphi).$$

□

As in the case of the first Seiffert mean, among other combinations of means we find inequalities that could be proved. Some of them are given in the following conjecture and the corresponding parameters are optimal.

Conjecture 3.6. *The following double inequalities hold true with the best possible parameters:*

$$\tfrac{1}{4}H + \tfrac{3}{4}T \le A \le \tfrac{4-\pi}{4}H + \tfrac{\pi}{4}T \tag{3.18}$$

$$\tfrac{1}{9}H + \tfrac{8}{9}Q \le T \le \tfrac{\pi-2\sqrt{2}}{\pi}H + \tfrac{2\sqrt{2}}{\pi}Q \tag{3.19}$$

$$\tfrac{\pi-2}{\pi}H + \tfrac{2}{\pi}N \le T \le \tfrac{1}{3}H + \tfrac{2}{3}N \tag{3.20}$$

$$\tfrac{1}{6}G + \tfrac{5}{6}Q \le T \le \tfrac{\pi-2\sqrt{2}}{\pi}G + \tfrac{2\sqrt{2}}{\pi}Q \tag{3.21}$$

$$\tfrac{1}{2}L + \tfrac{1}{2}T \le A \le \tfrac{4-\pi}{4}L + \tfrac{\pi}{4}T \tag{3.22}$$

$$\tfrac{1}{5}L + \tfrac{4}{5}Q \le T \le \tfrac{\pi-2\sqrt{2}}{\pi}L + \tfrac{2\sqrt{2}}{\pi}N \tag{3.23}$$

$$\tfrac{2\pi-4}{\pi}A + \tfrac{4-\pi}{\pi}N \le T \le \tfrac{1}{2}A + \tfrac{1}{3}N \tag{3.24}$$

$$\tfrac{(2-\sqrt{2})\pi}{2\pi-4}T + \tfrac{\sqrt{2}\pi-4}{2\pi-4}N \le Q \le \tfrac{3}{4}T + \tfrac{1}{4}N \tag{3.25}$$

3.3. *Comparison of the first and second Seiffert mean with other means.* We find the best parameters for the combinations of the first and second Seiffert mean with other means. The following table was obtained analogously to previous cases.

Table 3

G	L	P	A	T	C	Q	$\times\beta^4/x^3$
$-3/5$		1		$-2/5$			$19/300$
	$-3/4$	1		$-1/4$			$1/24$
		$-2/3$	1	$-1/3$			$11/180$
		0		-1	1		$4/45$
		$-1/4$		1		$-3/4$	$1/60$

Finally, we claim that the following inequalities are true.

Conjecture 3.7. *The following double inequalities hold true with the best possible parameters:*

$$\tfrac{3}{5}G + \tfrac{2}{5}T \le P \le \tfrac{1}{2}G + \tfrac{1}{2}T \tag{3.26}$$

$$\frac{3}{4}L + \frac{1}{4}T \le P \le \frac{1}{2}L + \frac{1}{2}T \tag{3.27}$$

$$\frac{2}{3}P + \frac{1}{3}T \le A \le \frac{4-\pi}{2}P + \frac{\pi-2}{2}T \tag{3.28}$$

$$\frac{1}{4}P + \frac{3}{4}Q \le T \le \frac{\pi-2\sqrt{2}}{\pi-\sqrt{2}}P + \frac{\sqrt{2}}{\pi-\sqrt{2}}Q \tag{3.29}$$

All of these conjectures are verified using CAS. Applying methods shown in this paper it is possible to draw the similar conclusions for the other combinations of means.

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Asimptotski razvoji i nejednakosti za Seiffertove sredine

Lenka Vukšić

Sažetak. U ovom radu proučavaju se nejednakosti među klasičnim sredinama oblika $(1-\mu)M_1(s,t)+\mu M_3(s,t) \leq M_2(s,t) \leq (1-\nu)M_1(s,t)+\nu M_3(s,t)$, koje obuhvaćaju posebno i Seiffertove sredine. Primjenom tehnika asimptotskih razvoja napravljena je detaljna analiza te je opisana metoda određivanja optimalnih parmetara μ i ν.

Lenka Vukšić
Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia
E-mail: lenka.vuksic@fer.hr

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