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SMOOTH YIELD SURFACES OF ISOTROPIC MATERIALS

Summary

Two types of the tensor formulation of the general yield criteria are presented. The first one having an invariant form describes smooth yield surfaces such as the von Mises circular cylinder. The second type defines multiple yield surfaces and it is convenient for the description of a smooth prism, i.e. Tresca's hexagonal prism. Only the first invariant form has been investigated.

In the case of a general isotropic material having different tensile and compressive yield strengths, three independent material constants have to be measured in order to achieve a complete description of the yield surface. Usually, these constants are tensile yield strength $\sigma_{Y.t}$, compressive yield strength $\sigma_{Y.c}$, and shearing yield strength τ_Y. Furthermore, it has been shown that the first stress invariant σ_{kk} directly influences yielding. If an isotropic material exhibits equal tensile and compressive yield strengths, then σ_{kk} does not influence yielding. In that case only one material constant, usually $\sigma_{Y.t}$, suffices for the description of the yield surface.

Key words: *yield surface, smooth, yield strength, tensor formulation*

1. Introduction

The yield surface of an isotropic material is given by the equation [1]

$$f(\sigma_{ij}, \varepsilon_{ij}^{\mathrm{P}}, k) = 0 \,, \tag{1}$$

where σ_{ij} is the stress tensor, $\varepsilon_{ij}^{\mathrm{P}}$ is the plastic strain tensor, and k is the strain hardening parameter. In the six-dimensional stress space, $(\sigma_{ij} = \sigma_{ji})$ (1) represents a closed hypersurface. The state of stress inside a yield surface is elastic. If the state of stress is on the yield surface, plastic deformation or yielding commences. Since σ_{ij} is a second order tensor, the yield function has a tensor character; therefore, it can be represented as a power series of σ_{ij} such as

$$f' = C' + C'_{ij}\sigma_{ij} + C'_{ijkm}\sigma_{ij}\sigma_{km} + C'_{ijkmpq}\sigma_{ij}\sigma_{km}\sigma_{pq} + \ldots = 0 \,. \tag{2}$$

Dividing the above expression by C', the following expression is obtained:

$$f = 1 + C_{ij}\sigma_{ij} + C_{ijkm}\sigma_{ij}\sigma_{km} + C_{ijkmpq}\sigma_{ij}\sigma_{km}\sigma_{pq} + ... = 0 \, . \tag{3}$$

If materials are isotropic, the coordinate axes can be chosen to coincide with the principal stress directions without any loss of generality. In that case, the yield function $f(\sigma_{ij})$ represents a closed surface in the three-dimensional space, that is

$$f(\sigma_1, \sigma_2, \sigma_3) = 0 \tag{4}$$

where σ_1, σ_2 and σ_3 are the principal stresses.

A great number of proposed yield criteria [2-5] contain components of the stress tensor raised to the first and the second power so that the proposed tensor equations [6] will contain the stress tensor raised to the first and the second power. Thus, the tensor equation of yield surfaces will take the form

$$f = 1 + P_{ij}\sigma_{ij} + P_{ijkm}\sigma_{ij}\sigma_{km} = 0 \, , \tag{5}$$

$$f_{ij} = 1 + Q\sigma_{ij} + Q\sigma_{ki}\sigma_{kj} = 0 \, , \tag{6}$$

$$f_{ijkm} = \delta_{ij}\delta_{km} + R_{ij}\sigma_{km} + R_{ijkm}\sigma_{ij}\sigma_{km} = 0. \tag{7}$$

The equation (5) is convenient for the representation of smooth yield surfaces such as the von Mises cylindrical surface. On the other hand, the equations (6) and (7) are suitable for the representation of piece-wise smooth yield surfaces such as Tresca's hexagonal prism.

P_{ij} and P_{ijkm} are the plasticity tensors of the first type, while R_{ij} and R_{ijkm} are plasticity tensors of the second type. In the case of isotropic materials, they must also be isotropic. Therefore, they are of the even order, i.e. 2nd, 4th, 6th etc. since there are no isotropic tensors of the odd order. Quasi-isotropic tensors e_{ijk}, $\delta_{ij}e_{kmp}$, etc. are isotropic under rotation and they are not isotropic under reflection.

In this paper, only smooth surfaces represented by (5) will be investigated. Due to the symmetry of the stress tensor σ_{ij} and the structure of (5), the plasticity tensors P_{ij} and P_{ijkm} possess the following symmetry properties:

$$P_{ij} = P_{ji} \, , \quad P_{ijkm} = P_{kmij} \tag{8}$$

$$P_{ijkm} = P_{jikm} = P_{jimk} = P_{ijmk} \, . \tag{9}$$

The general isotropic tensors of the second and the fourth order are, respectively

$$I_{ij} = A\delta_{ij} \, , \tag{10}$$

$$I_{ijkm} = B\delta_{ij}\delta_{km} + C\delta_{ik}\delta_{jm} + D\delta_{im}\delta_{jk} \, . \tag{11}$$

Since the tensor I_{ijkm} does not possess symmetry properties (9) it will be symmetrized as follows

$$I_{ijkm} = L\delta_{ij}\delta_{km} + M(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}) + N(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}) \tag{12}$$

where

$$L = B, \; M + N = C, \; M - N = D \, .$$

The first two terms on the right side of (12) have the required symmetry properties and they will be retained, while the third term will be omitted. Thus, the fourth order isotropic tensor suitable for this purpose is

$$P_{ijkm} = L\delta_{ij}\delta_{km} + M(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})\,. \tag{13}$$

2. General isotropic materials

After substituting (13) into (5), it follows that

$$f(\sigma_{ij}) = 1 + A\sigma_{jj} + L\sigma_{jj}\sigma_{kk} + 2M\sigma_{jk}\sigma_{jk} = 0\,. \tag{14}$$

Three constants, A, L and M that appear in (14) have to be determined experimentally. For that, three experiments will be performed:

1. Simple tension test.
2. Simple compression test.
3. Pure shearing test.

In the first experiment, $\sigma_{11} > 0$, all other $\sigma_{ij} = 0$, yielding starts when σ_{11} reaches the yield point or the yield strength in tension $\sigma_{\mathrm{Y.t}}$. In that case (14) becomes

$$1 + A\sigma_{\mathrm{Y.t}} + L\sigma_{\mathrm{Y.t}}^2 + 2M\sigma_{\mathrm{Y.t}}^2 = 0\,. \tag{15}$$

In the second test, $\sigma_{11} < 0$, all other $\sigma_{ij} = 0$, yielding starts when $\sigma_{11} = -\sigma_{\mathrm{Y.c}}$, i.e. when σ_{11} reaches the yield strength in compression. If this is substituted into (14), one gets

$$1 - A\sigma_{\mathrm{Y.c}} + L\sigma_{\mathrm{Y.c}}^2 + 2M\sigma_{\mathrm{Y.c}}^2 = 0\,. \tag{16}$$

In the shearing test, $\sigma_{12} = \sigma_{21} \neq 0$, all other $\sigma_{ij} = 0$. Yielding starts when $\sigma_{12} = \sigma_{21} = \tau_{\mathrm{Y}}$, where τ_{Y} is the shearing yield strength. Then (14) becomes

$$1 + 4M\tau_{\mathrm{Y}}^2 = 0\,. \tag{17}$$

Solving simultaneously (15), (16) and (17), one gets

$$\begin{aligned} A &= \frac{\sigma_{\mathrm{Y.t}} - \sigma_{\mathrm{Y.c}}}{\sigma_{\mathrm{Y.t}}\sigma_{\mathrm{Y.c}}} \\ L &= \frac{\sigma_{\mathrm{Y.t}}\sigma_{\mathrm{Y.c}} - 2\tau_{\mathrm{Y}}^2}{2\tau_{\mathrm{Y}}^2\sigma_{\mathrm{Y.t}}\sigma_{\mathrm{Y.c}}} \\ M &= -\frac{1}{4\tau_{\mathrm{Y}}^2} \end{aligned} \tag{18}$$

After substituting (18) into (14), it follows that

$$\begin{aligned} f &= (\sigma_{\mathrm{Y.c}} - \sigma_{\mathrm{Y.t}})\tau_{\mathrm{Y}}^2\sigma_{jj} + (\sigma_{\mathrm{Y.c}}\sigma_{\mathrm{Y.t}} - 2\tau_{\mathrm{T}}^2)\sigma_{jj}\sigma_{kk} + \\ &\quad + \sigma_{\mathrm{Y.c}}\sigma_{\mathrm{Y.t}}\sigma_{kj}\sigma_{kj} - 2\tau_{\mathrm{Y}}^2\sigma_{\mathrm{Y.c}}\sigma_{\mathrm{Y.t}} = 0. \end{aligned} \tag{19}$$

Using coordinate axes to coincide with the principal stress directions, the expression (19), in the expanded form, reads

$$\begin{aligned} & 1+A(\sigma_1+\sigma_2+\sigma_3)+L\left[\sigma_1^2+\sigma_2^2+\sigma_3^2+2\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right)\right]+ \\ & +2M\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right)=0. \end{aligned} \tag{20}$$

It represents an ellipsoid of revolution whose axis coincides with the hydrostatic line $\sigma_1=\sigma_2=\sigma_3$. When the difference between the compressive and the tensile yield strength is negligible, the ellipsoid is prolate. However, if the difference is marked, the ellipsoid is oblate [7-9]. The ellipsoid is shown in Figure 1a, while its two parts cut by coordinate planes are depicted in Figure 1b. Intersections of the hydrostatic line and the ellipsoid are the vertices A and B, as shown in Figure 1a.

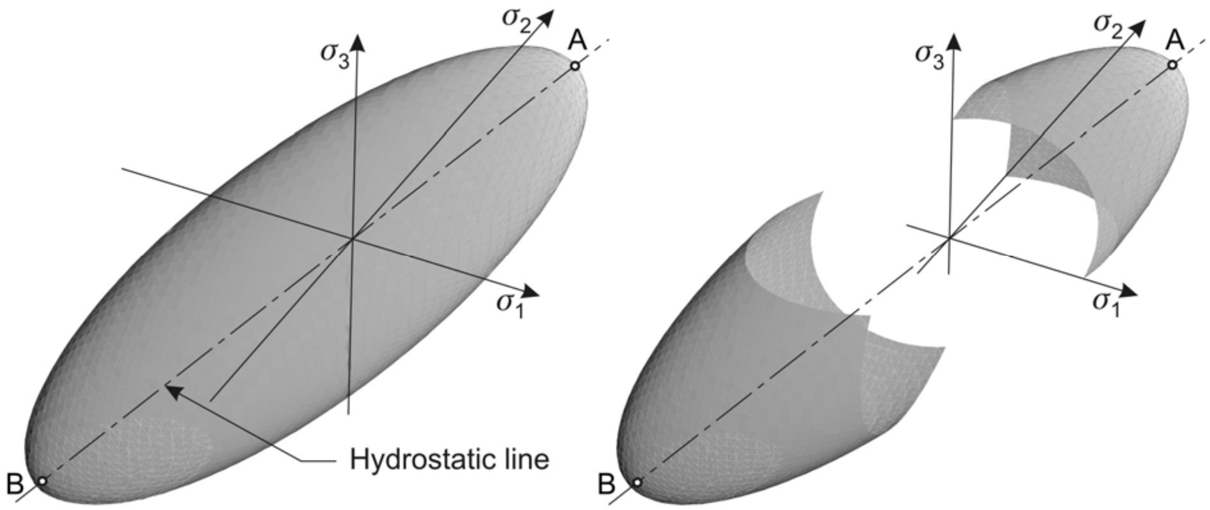

Fig. 1 Smooth yield surface of an isotropic material in the stress space

Intersection of the yield surface and the $O\sigma_1\sigma_2$ plane is an ellipse whose equation is

$$1+A(\sigma_1+\sigma_2)+L(\sigma_1+\sigma_2)^2+2M\left(\sigma_1^2+\sigma_2^2\right)=0. \tag{21}$$

The principal axes of the ellipse make an angle of 45° with the σ_1 and σ_2 axes. After the transformation

$$\begin{aligned} \bar{\sigma}_1 &= \frac{\sqrt{2}}{2}(\sigma_1+\sigma_2), \\ \bar{\sigma}_2 &= \frac{\sqrt{2}}{2}(-\sigma_1+\sigma_2), \end{aligned} \tag{22}$$

the expression (21) takes the following form

$$p\bar{\sigma}_1^2+q\bar{\sigma}_1+r\bar{\sigma}_2^2+1=0 \tag{23}$$

where

$$\begin{aligned} p &= 2(L+M), \\ q &= \sqrt{2}A, \\ r &= 2M. \end{aligned} \tag{24}$$

The equation (23) may be put in the form

$$\frac{(\bar{\sigma}_1+m)^2}{a^2}+\frac{\bar{\sigma}_2^2}{b^2}=1 \tag{25}$$

where

$$a=\frac{\sqrt{q^2-4p}}{2p},\quad b=\frac{1}{2}\sqrt{\frac{q^2-4p}{pr}} \tag{26}$$

are the major axes of the ellipse, as shown in Figure 2a, and $m=q/2p$. The intersection of the ellipsoid and the plane $\sigma_1=\sigma_2$ is the ellipse shown in Figure 2b. The plane $\sigma_1=\sigma_2$ contains the σ_3 axis as well as the hydrostatic line $\sigma_1=\sigma_2=\sigma_3$.

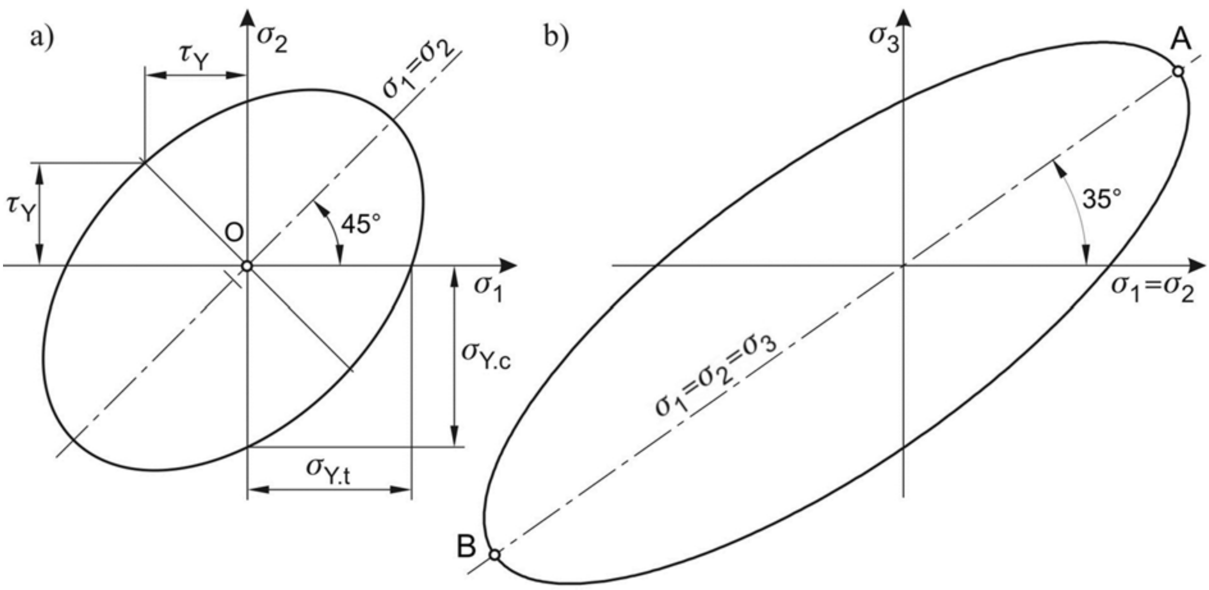

Fig. 2 Intersection of an ellipsoid and the planes a) $O\sigma_1\sigma_2$ and b) $\sigma_1=\sigma_2$

3. Yielding is independent of hydrostatic stress

When yielding of an isotropic material does not depend on the first stress invariant σ_{kk}, (14) is reduced to

$$f(\sigma_{ij})=1+2M\sigma_{jk}\sigma_{jk}=0. \tag{27}$$

In this case, it is convenient to use the deviator stress tensor s_{ij} instead of the stress tensor σ_{ij}

$$\sigma_{ij}=s_{ij}+\frac{1}{3}\sigma_{kk}\delta_{ij}.$$

In that case, (27) becomes

$$1 + 2M s_{ij} s_{ij} = 0 \tag{28}$$

or in the expanded form

$$2\left[s_{11}^2 + s_{22}^2 + s_{33}^2 + 2\left(s_{12}^2 + s_{23}^2 + s_{31}^2\right)\right] = -\frac{1}{M}.$$

The unknown constant M can be determined in the uniaxial tension test [10]. In that case, $\sigma_{11} > 0$, all other $\sigma_{ij} = 0$, i.e. $s_{11} = 2\sigma_{11}/3$, $s_{22} = s_{33} = -\sigma_{11}/3$, all other $s_{ij} = 0$. Yielding starts when σ_{11} reaches the yield strength σ_{Y}. Therefore (28) gives

$$M = -\frac{3}{4\sigma_{11}^2}. \tag{29}$$

After substituting (29) back into (28), one obtains

$$\frac{3}{2} s_{ij} s_{ij} = \sigma_{\mathrm{Y}}. \tag{30}$$

This is the well-known equation of the von Mises cylinder whose axis is the hydrostatic line.

4. Two examples of general isotropic materials

The intersections of the general yield surface, i.e. the yield ellipsoid, and the coordinate plane $\mathrm{O}\sigma_1\sigma_2$ are shown in Figures 3a and 3b. The first ellipse refers to the Titanium Ti6Al4V, annealed. The difference between its $\sigma_{\mathrm{Y.c}}$ and $\sigma_{\mathrm{Y.t}}$ is small and the major axis of the ellipse lies in the first and the third quadrant. The ellipse in Figure 3b refers to the Gray Cast Iron EN-GJL250. Since the difference $\sigma_{\mathrm{Y.c}}$-$\sigma_{\mathrm{Y.t}}$ is marked in this case [11], the major axis of the ellipse lies in the second and the forth quadrant as depicted in Figure 3b. The data about the yield strengths for both materials are given in Table 1.

Table 1 Yield strengths of the considered materials

Material	Compress. yield strength $\sigma_{\mathrm{Y.c}}$ /MPa	Tensile yield strength $\sigma_{\mathrm{Y.t}}$ /MPa	Shear yield strength τ_{Y} /MPa
Ti6Al4V, annealed	970	880	550
EN-GJL250	860	165	334

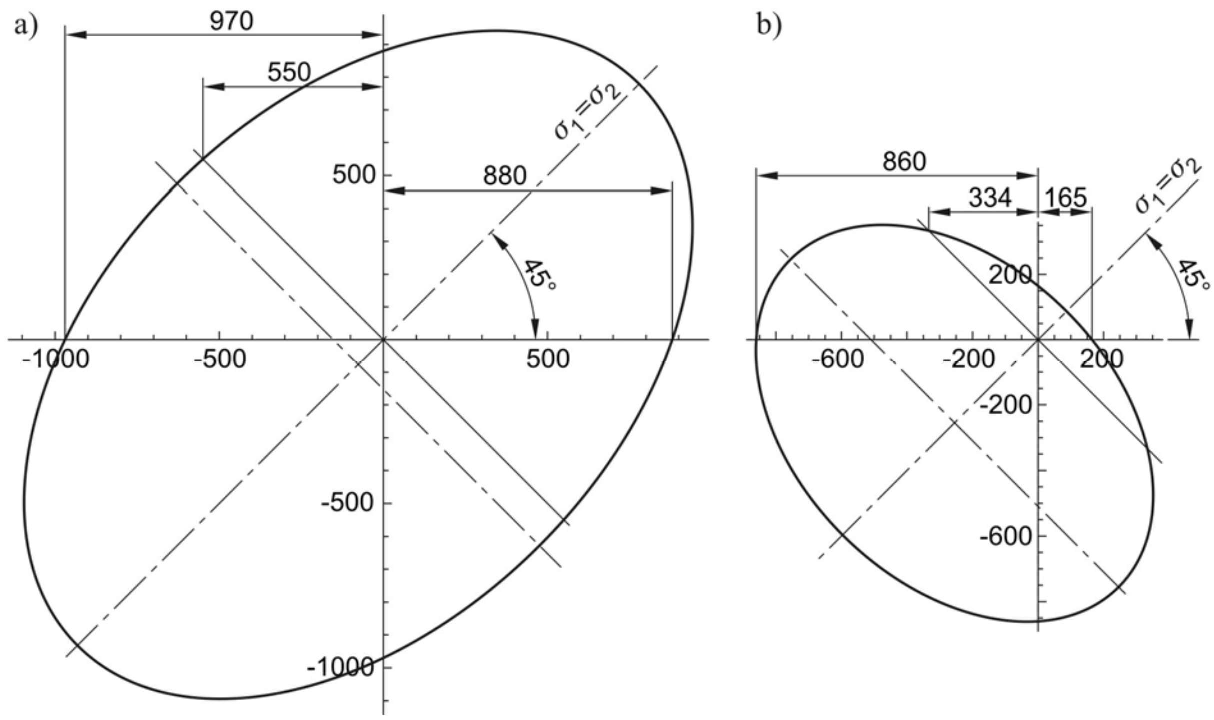

Fig. 3 Intersection of the ellipsoids and the plane $O\sigma_1\sigma_2$ for a) Titanium alloy and b) Gray Cast Iron

5. Conclusion

A general yield criterion containing the stress tensor raised to the first and the second power can be formulated in two ways, namely in an invariant tensor expression and using the tensor expression of the fourth order. The former gives smooth yield surfaces such as the von Mises cylinder. The latter gives piece-wise smooth yield surfaces such as Tresca's hexagonal prism. In the case of a general isotropic material having different tensile and compressive yield strengths, three independent material constants have to be measured in order to achieve a complete description of the yield surface. Usually these constants are the tensile yield strength $\sigma_{Y.t}$, the compressive yield strength $\sigma_{Y.c}$, and the shearing yield strength τ_Y. Furthermore, it has been shown that the first stress invariant σ_{kk} directly influences yielding. If the compressive and tensile yield strengths do not differ from each other very much, the ellipsoid is prolate. However, if the difference between them is marked, the ellipsoid is oblate.

When an isotropic material exhibits equal tensile and compressive yield strengths, then the first stress invariant σ_{kk} does not influence yielding. In that case only one material constant, usually $\sigma_{Y.t}$, suffices for the description of the yield surface which is the von Mises cylinder.

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