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ON A LEMMA OF THOMPSON

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ABSTRACT. In Theorem 3 we improve [8, Lemma 5.41] (= Lemma 1, below) omitting one of its conditions. In Lemma 1 the structure of T, a Sylow 2-subgroup of G, is described only. In contrast to that lemma, we describe in detail the structure of the whole group G and embedding of T in G. In Theorem 4 we consider a similar, but more general, situation for groups of odd order.

In the first part [8] of his seminal N-paper Thompson considered, in particular, a number of special situations arising in the subsequent parts of that paper. He proved there the following

LEMMA 1 ([8, Lemma 5.41]). *Suppose that the following holds:*

(a) *G is a finite nonnilpotent solvable group.*
(b) *$O_{2'}(G) = \{1\}$.*
(c) *G has a proper noncyclic abelian subgroup of order* 8.
(d) *If K is any proper subgroup of G of index a power of 2, then K has no noncyclic abelian subgroup of order* 8.

Let T be a Sylow 2-subgroup of G. Then T is normal in G and one of the following holds:

(i) *T is abelian.*
(ii) *T is an extraspecial group.*
(iii) *T has a subgroup $T_0 \cong Q_8$ of index 2 and $T = T_0Z(T)$.*
(iv) *T is special and $Z(T) \cong E_4$.*

In Theorem 3 we eliminate condition (c) from the hypothesis of Lemma 1 and, as a result, we obtain three additional non 2-closed groups; we also describe the structure of G in some detail. Note also that our proof differs

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essentially from the original proof of Lemma 1. Theorem 4 is a stronger version of Theorem 3 for groups of odd order. In the proof of Theorem 4 we use [3, Theorem 4.1(i)] (= Lemma 2(e), below), a fairly deep result of finite p-group theory.

In what follows G is a finite group, p is a prime, π is a set of primes and π' is the set of primes not contained in π, $m, n \in \mathbb{N}$ and $\pi(m)$ is the set of all prime divisors of m. Next, C_m is the cyclic group of order m; E_{p^m} is the elementary abelian group of order p^m; SD_{2^n} $(n > 3)$, Q_{2^n} and D_{2^n} are the semidihedral, generalized quaternion group and dihedral groups of order 2^n, respectively (these groups exhaust the groups of maximal class and order 2^n); $\mathrm{A}_4(\mathrm{S}_4)$ are the alternating (symmetric) group of degree 4; $\mathrm{C}_G(M)$ $(\mathrm{N}_G(M))$ is the centralizer (normalizer) of the subset M in G; $\mathrm{Z}(G)$, G' and $\Phi(G)$ is the center, the derived subgroup and the Frattini subgroup of G, respectively; $\mathrm{O}_\pi(G)$ is the product of all normal π-subgroups of G. If G is a p-group, then $\Omega_1(G) = \langle x^p = 1 \mid x \in G\rangle$ and $\mho_1(G) = \langle x^p \mid x \in G\rangle$. By $A * B$ we denote a central product of A and B.

A p-group G is said to be *special* if $G' = \mathrm{Z}(G) = \Phi(G) > \{1\}$ (in that case, $\exp(G') \le \exp(G/G')$ so $\exp(G') = p$ and G' is elementary abelian). A p-group G is said to be *extraspecial* if it is special with $|G'| = p$.

Let G be a 2-group of maximal class. Then, if $G \not\cong \mathrm{Q}_8$, it contains a characteristic cyclic subgroup of index 2.

In Lemma 2 we gathered some known facts used in what follows.

Lemma 2. (a) [1, Proposition 19(a)] *Let B be a nonabelian subgroup of order p^3 in a p-group G. If G is not of maximal class, then $C_G(B) \not\le B$.*

(b) *Let G be a p-group generated by two elements. Then $\pi(|Aut(G)|) \subseteq \pi(p(p-1)(p+1))$. In particular, p is the maximal prime divisor of $|Aut(G)|$, unless $p = 2$. If, in addition, G has a characteristic subgroup of index p, then $\pi(|Aut(G)|) \subseteq \pi(p(p-1))$. In particular, if G is a 2-group of maximal class and $Aut(G)$ is not a 2-group, then $G \cong Q_8$.*

(b1) *$Aut(Q_8) \cong S_4$.*

(c) *Let α be a p'-automorphism of a p-group G acting trivially on $\Omega_1(G)$. If $p > 2$ or G is abelian, then $\alpha = id_G$.*

(d) *If a p-group G has no noncyclic abelian subgroup of order p^3, then one and only one of the following holds: (i) G is cyclic, (ii) $G \cong E_{p^2}$, (iii) G is a 2-group of maximal class, (iv) G is nonabelian of order p^3, $p > 2$.*

(e) [3, Theorem 4.1(i)] *Let G be a p-group, $p > 2$. Suppose that G has no normal elementary abelian subgroup of order p^3. Then one of the following holds: (i) G is metacyclic, (ii) G is an irregular 3-group of maximal class, (iii) $G = EC$, where $E = \Omega_1(G)$ is nonabelian of order p^3 and exponent p and C is cyclic.*

(f) *Let A be a π'-group acting on a π-group G. Let $\mathcal{C} : G = G_0 > G_1 > \cdots > G_n = \{1\}$ be a chain of A-invariant normal subgroups of G. If A centralizes all factors G_i/G_{i+1} of the chain $\mathcal{C}$ (i.e., A stabilizes $\mathcal{C}$), then A centralizes G.*

(g) *(Transfer Theorem) Suppose that a Sylow p-subgroup of a group G is abelian. If p divides $|Z(G)|$, then G has a normal subgroup of index p.*

According to Hall-Burnside, if α is a p'-automorphism of a p-group G inducing identity on $G/\Phi(G)$, then $\alpha = \mathrm{id}_G$. Indeed, assuming, without loss of generality, that $o(\alpha) = q$, a prime, we see that α fixes an element of every coset $x\Phi(G)$. Since these fixed elements generate G, our claim follows.

If d is a minimal number of generators of a p-group G, then (Hall) $|\mathrm{Aut}(G)|$ divides the number $(p^d-1)(p^d-p)\dots(p^d-p^{d-1})|\Phi(G)|^d$ (indeed, that number is the cardinality of the set $\mathfrak{B}$ of minimal bases of G, and G has no fixed points on the set $\mathfrak{B}$), and this justifies the main assertion of Lemma 2(b). If a two-generator p-group G has a characteristic subgroup H of index p and $\alpha \in \mathrm{Aut}(G)$ has prime order $q \not\in \pi(p(p-1))$, then α stabilizes the chain $\{1\} < H/\Phi(G) < G/\Phi(G)$ so $\alpha = \mathrm{id}_G$, by the previous paragraph and (f), a contradiction. In (c), the partial holomorph $\langle\alpha\rangle \cdot G$ has no minimal nonnilpotent subgroup so it is nilpotent, by Frobenius' Normal p-Complement Theorem [5, Theorem 9.18] (here we use the structure of minimal nonnilpotent groups; see [4, Satz 3.5.2]). Lemma 2(d) follows from Roquette's Lemma [4, Satz 3.7.6], in which the p-groups without normal abelian subgroups of type (p,p) are classified. Lemma 2(g) follows from Wielandt's Theorem [4, Satz 4.8.1] and Fitting's Lemma [2, Corollary 1.18]. As to Lemma 2(f), assume that A does not centralize G and $|AG|$ is as small as possible. Then AG is minimal nonnilpotent. Since all nilpotent images of AG must be π'-groups, we get a contradiction with hypothesis.

Recall that there are two representation groups of the symmetric group S_4, their orders are equal to 48, Sylow 2-subgroups of these groups are generalized quaternion and semidihedral, respectively; see [7, Theorem 3.2.21].

Now we are ready to prove our main results.

Theorem 3. *Suppose that the following holds:*

(a) *G is a nonnilpotent solvable group with a Sylow 2-subgroup T and $2'$-Hall subgroup H.*

(b) *$O_{2'}(G) = \{1\}$.*

(c) *Whenever K is a proper subgroup of G such that $|G : K|$ is a power of 2, then K has no noncyclic abelian subgroup of order 8 (or, what is the same, every maximal subgroup of G containing H, has no noncyclic abelian subgroup of order 8).*

Then one and only one of the following assertions is true:

A *If T is not normal in G, then either $G \cong S_4$ or G is one of two representation groups of S_4.*

B *If T is normal in G, then one of the following holds:*

(B1) *If T is abelian, then $T \in \{E_{2^m}, C_4 \times C_4\}$.*

(B1.1) *If $T \cong C_4 \times C_4$, then G is a Frobenius group with $|G:T| = 3$.*

(B1.2) *Let $T \cong E_{2^m}$ be not a minimal normal subgroup of G. Then either $G \in \{A_4 \times C_2, A_4 \times A_4\}$ or $m = 4$ and G is a Frobenius group with $|G:T| = 3$.*

(B2) *T is extraspecial of order 2^{2m+1}, $m \geq 1$. If $m = 1$, then $G \cong SL(2,3)$. Next assume that $m > 1$.*

(B2.1) *If $m > 2$, then $T/Z(T)$ is a minimal normal subgroup of $G/Z(T)$.*

(B2.2) *If $T/Z(T)$ is not a minimal normal subgroup of $G/Z(T)$, then $T = U*V$, where $U \cong V \cong Q_8$, $U, V \triangleleft G$; in that case, $G/T \cong H$ is isomorphic to a subgroup of E_{3^2}. Moreover, if $H \cong E_{3^2}$, then $G = A * B$, where $A \cong B \cong SL(2,3)$ and $A \cap B = Z(A) = Z(B)$. If $|H| = 3$, then $UH \cong SL(2,3) \cong VH$.*

(B3) *T has a G-invariant subgroup $T_0 \cong Q_8$ of index 2 and $T = T_0 Z(T)$. In that case, $G/T' \cong A_4 \times C_2$, $G' = T_0$ and, if $D/T_0 < G/T_0$ is of order 3, then $D \cong SL(2,3)$.*

(B4) *T is special with $Z(T) = Z(G) \cong E_4$ and $T/Z(T)$ is a minimal normal subgroup of $G/Z(T)$.*

PROOF. The solvable group G contains a $2'$-Hall subgroup H. Since $\mathrm{O}_{2'}(G) = \{1\}$, $T \in \mathrm{Syl}_2(G)$ is noncyclic (Lemma 2(b)), $\mathrm{C}_G(\mathrm{O}_2(G)) \leq \mathrm{O}_2(G)$ (Hall-Higman) so, if T is abelian, it is normal in G.

Suppose that T is abelian and $\exp(T) > 2$. Then $\Omega_1(T)$ is normal in G since $T \triangleleft G$. Next, $|G : \Omega_1(T)H| > 1$ is a power of 2 so $\Omega_1(T) \cong \mathrm{E}_4$ since T is noncyclic. The number $|G : H\Omega_2(T)|$ is a power of 2 and $H\Omega_2(T)$ contains a noncyclic abelian subgroup of order 8, so we get $\Omega_2(T)H = G$ and $\exp(T) = 4$. Since G has no normal 2-complement, T is abelian of type $(4,4)$ (Lemma 2(b)). Then $\Omega_1(T)H$ is a Frobenius group (otherwise, by Lemma 2(c), $\{1\} < H \triangleleft G)$ so $|H| = 3$; in that case, G is also a Frobenius group.

Now suppose that $T \cong \mathrm{E}_{2^m}$; then $m > 1$. If $m = 2$, then $G \cong \mathrm{A}_4$. Now we let $m > 2$ and suppose that T is not a minimal normal subgroup of G. Then $T = R \times R_1$, where $R, R_1 > \{1\}$ are normal in G (Maschke) and, since $|G : RH| > 1$ and $|G : R_1H| > 1$ are powers of 2, we conclude that $|R| \leq 4$, $|R_1| \leq 4$ so $m \in \{3, 4\}$. If $m = 3$, then $G \cong \mathrm{A}_4 \times \mathrm{C}_2$. If $m = 4$, then G is either a Frobenius group with kernel $T \cong \mathrm{E}_{2^4}$ of index 3 in G or $G \cong \mathrm{A}_4 \times \mathrm{A}_4$. Indeed, assume that G is not a Frobenius group; then $|H| > 3$. Setting $Z = \mathrm{C}_H(R)$, $Z_1 = \mathrm{C}_H(R_1)$, we have $|H : Z| = 3 = |H : Z_1|$ and $Z \cap Z_1 \leq \mathrm{O}_{2'}(G) = \{1\}$ so $H = Z_1 \times Z_2$, $RZ_1 \cong \mathrm{A}_4 \cong R_1 Z$ and $G = (RZ_1) \times (R_1 Z)$.

In what follows we assume that T is nonabelian.

A. Suppose that T is normal in G.

(i) Suppose that T has no noncyclic abelian subgroup of order 8. Then, by Lemma 2(d), T is of maximal class, and, by Lemma 2(b), $T \cong \mathrm{Q}_8$, which is extraspecial (in that case, $G \cong \mathrm{SL}(2,3)$).

In what follows we assume that T has a noncyclic abelian subgroup of order 8 so T is not of maximal class; then $|T| > 8$.

(ii) Suppose that $K < G$ and $|G : K| = 2$; then K has no noncyclic abelian subgroup of order 8, by hypothesis. We get $\mathrm{O}_{2'}(K) \leq \mathrm{O}_{2'}(G) = \{1\}$ so $T \cap K$ is noncyclic and is not of maximal class and order > 8, by Lemma 2(b). It follows from Lemma 2(d), that $T \cap K \cong \mathrm{Q}_8$ and, since T is not of maximal class, $T = (T \cap K)\mathrm{C}_T(K \cap T) = (T \cap K)\mathrm{Z}(T)$ since $|T| = 16$ (Lemma 2(a)). The subgroup $T \cap K \triangleleft G$. Then, in view of Lemma 2(b1) and (a) (see the theorem), we conclude that

$$|H| = |G : T| = 3,\ (T \cap K)H \cong \mathrm{SL}(2,3),\ G' = T \cap K,\ G/G' \cong \mathrm{C}_6$$

and so G is as in part (B3).

Next we assume that G has no subgroup of index 2; then $T \leq G'$.

(iii) Let R be a G-invariant subgroup of T such that T/R is a minimal normal subgroup of G/R; then $R > \{1\}$ since T is nonabelian. Since $|G : RH| > 1$ is a power of 2, R has no noncyclic abelian subgroup of order 8, by hypothesis (see (c)), so we have for R the following possibilities listed in Lemma 2(d): either $R \leq 4$ or $R \cong \mathrm{Q}_8$ (here we also use Lemma 2(b)).

(iv) Suppose that H centralizes R. Then $G/\mathrm{C}_G(R)$ is a 2-group, so $\mathrm{C}_G(R) = G$, by (ii). Thus, $R \leq \mathrm{Z}(G)$. By hypothesis (see (a)), $\mathrm{Z}(G)$ is a 2-subgroup and, in view of the maximal choice of R, we get $\mathrm{Z}(T) = R = \mathrm{Z}(G)$. Assume that $T' < R$; then $|R/T'| = 2$. In that case, by Lemma 2(g), applied to the pair $T/T' < G/T'$, the group G/T' has a normal subgroup of index 2, contrary to (ii). Thus, $T' = R = \Phi(T)$ so T is special since $\exp(T') \leq \exp(T/T') = 2$, and $R \in \{\mathrm{C}_2, \mathrm{E}_4\}$. Therefore, we are done if $|R| = 2$.

(v) Suppose that T is extraspecial of order 2^{2m+1}, $m > 1$, and $|R| > 2$; then, by (iv), H does not centralize R. If $|R| = 4$, then $|T : \mathrm{C}_T(R)| = 2$ and $\mathrm{C}_T(R)H$ has index 2 in G, contrary to (ii) (note that $\mathrm{C}_T(R)$ is normal in G since R and T are). Thus, $|R| > 4$ so $R \cong \mathrm{Q}_8$ (Lemma 2(d,b)). Let $R_1 = \mathrm{C}_T(R)$; then $R_1 \cong R \cong \mathrm{Q}_8$, by what has just been said. In that case, $T = R{*}R_1$ is extraspecial of order 2^5. Suppose that $|H|$ is not a prime. Setting $\mathrm{C}_H(R) = Z$ and $\mathrm{C}_H(R_1) = Z_1$, we get, by Lemma 2(b1), $|H/Z| = 3 = |H/Z_1|$, $Z \cap Z_1 = \{1\}$ and so $H = Z \times Z_1$, $RZ_1 \cong \mathrm{SL}(2,3) \cong R_1Z$, and we conclude that $G = (RZ_1) * (R_1Z)$ with $(RZ_1) \cap R_1Z = \mathrm{Z}(RZ_1)$. If $|H|$ is a prime, then $|G : T| = 3$ and, as above, $RH \cong \mathrm{SL}(2,3) \cong R_1H$. Thus, G as in part (B2).

In what follows we assume that T is not extraspecial.

(vi) Suppose that T has a maximal G-invariant cyclic subgroup Z of order ≥ 4. One may choose R so that it contains Z. Then H centralizes Z (Lemma

2(b)) so, by (iv), $Z \leq \mathrm{Z}(G)$. By Lemma 2(d), R must be cyclic, contrary to (iv).

Thus, T has no G-invariant cyclic subgroup of order 4 and so R is noncyclic. Therefore, by (iii), $R \in \{\mathrm{E}_4, \mathrm{Q}_8\}$.

(vii) Let $R \cong \mathrm{E}_4$. In that case, $\mathrm{C}_T(R)$ is normal in G and $|T : \mathrm{C}_T(R)| \leq 2$. Since $|T : \mathrm{C}_T(R)H| \leq 2$, we get $\mathrm{C}_T(R) = T$, by (ii). Since T is nonabelian, we get $R = \mathrm{Z}(T)$, by the maximal choice of R. As in (iv), we get $T' = R$ so $\Phi(T) = R$ and T is special since, by the above, $R = \mathrm{Z}(G)$.

(viii) Now let $R \cong \mathrm{Q}_8$. By (iv), $[R, H] > \{1\}$. By Lemma 2(b1), $G/\mathrm{C}_T(R)$ is a subgroup of S_4 containing a subgroup isomorphic to $R/\mathrm{Z}(R) \cong \mathrm{E}_4$ (Lemma 2(b1)). Since T is normal in G, we get $G/\mathrm{C}_T(R) \not\cong \mathrm{S}_4$. Thus, $|T : \mathrm{C}_T(R)| = 4 = |R : \mathrm{Z}(R)|$ so $|H| = 3$ and $T = R * \mathrm{C}_T(R)$, by the product formula. Thus, $T/\mathrm{C}_T(R) \cong \mathrm{E}_4$. By (ii), $|T : R| > 2$ so $\mathrm{C}_T(R)$ is noncyclic of order > 4. Then, by Lemma 2(d), $\mathrm{C}_T(R) \cong \mathrm{Q}_8$ so $T \cong \mathrm{Q}_8 * \mathrm{Q}_8$ is extraspecial of order 2^5.

The case where T is normal in G, is complete.

B. Now suppose that T is not normal in G. Then $T_0 = \mathrm{O}_2(G) > \{1\}$ since $\mathrm{O}_{2'}(G) = \{1\}$ and G is solvable. Since $|G : T_0H| > 1$ is a power of 2, T_0 is a group of Lemma 2(d). It follows from $\mathrm{C}_G(T_0) \leq T_0$ that T_0 is noncyclic and, if T_0 is of maximal class, then $T_0 \cong \mathrm{Q}_8$ (Lemma 2(b)). If $T_0 \cong \mathrm{E}_4$, then $G \cong \mathrm{S}_4$ since $\mathrm{Aut}(\mathrm{E}_4) \cong \mathrm{S}_3$. Now let $T_0 \cong \mathrm{Q}_8$. Since $\mathrm{Aut}(T_0) \cong \mathrm{S}_4$ (Lemma 2(b1)), we conclude that $G/\mathrm{Z}(T_0)$ is isomorphic to a nonnilpotent subgroup of S_4 containing the subgroup $T_0/\mathrm{Z}(T_0) \cong \mathrm{E}_4$ of even index (by assumption, $T_0 < T$). We conclude that $\mathrm{C}_T(T_0) < T_0$ so T is of maximal class, namely, T is generalized quaternion of semidihedral of order 16 (Lemma 2(a)). It follows that $G/\mathrm{Z}(T_0) \cong \mathrm{S}_4$ so G is a representation group of S_4.

Since all groups listed in the conclusion of the theorem, satisfy the hypothesis, the proof is complete. $\square$

Next we expand Theorem 3 to groups of odd order.

THEOREM 4. *Let G be a nonnilpotent group and let $p > 2$ be the least prime divisor of $|G|$. Suppose that the following holds:*

(a) *$O_{p'}(G) = \{1\}$.*
(b) *Whenever K is a proper subgroup of G such that $|G : K|$ is a power of p, then K has no elementary abelian subgroup of order p^3.*

Let T be a Sylow p-subgroup of G. Then T is normal in G and one and only one of the following assertions takes place:

A *T is a minimal normal subgroup of G, $d(T) > 2$.*
B *T is special of exponent p with $Z(T) = Z(G)$ is of order at most p^2, $T/Z(T)$ is a minimal normal subgroup of $G/Z(T)$.*

PROOF. Since G has odd order, it is solvable hence, in view of (a), $\mathrm{C}_G(\mathrm{O}_p(G)) \le \mathrm{O}_p(G)$ and so, if T is abelian, it is normal in G. By Lemma 2(b), $\mathrm{O}_p(G)$ is not two-generator. Let H be a p'-Hall subgroup of G.

($*$) Let $M < T$ be G-invariant. We contend that H centralizes M. Indeed, since $|G : MH| > 1$ is a power of p, M is a group of Lemma 2(e), by hypothesis (see (b)). Then, by Lemma 2(b), H centralizes M if $\mathrm{d}(M) \le 2$. Now let $\mathrm{d}(M) > 2$. Then, by Lemma 2(e), $M = \Omega_1(M)C$, where $\Omega_1(M)$ is nonabelian of order p^3 and exponent p and C is cyclic. Note, that $\Omega_1(M)$ is normal in G. By Lemma 2(b), H centralizes $\Omega_1(M)$ so H centralizes M, by Lemma 2(c).

1. Let T be normal in G.

(i) Assume that T is a group of Lemma 2(e). Then, as in ($*$), H centralizes T so H is normal in G, which is a contradiction. Thus, T possesses a subgroup $\cong \mathrm{E}_{p^3}$; then, by Lemma 2(e), T has a normal subgroup $\cong \mathrm{E}_{p^3}$.

(ii) Suppose that T is abelian. Since $|G : H\Omega_1(T)|$ is a power of p and, by (i), $\Omega_1(T)$ has a subgroup $\cong \mathrm{E}_{p^3}$, we get $T = \Omega_1(T)$ so T is elementary abelian. Assume that $T = V_1 \times V_2$, where $V_1 > \{1\}$ and $V_2 > \{1\}$ are normal in G. Then, by ($*$), H centralizes V_i, $i = 1, 2$ (Lemma 2(b)) so H centralizes T, which is not the case. Thus, T is a minimal normal subgroup of G (Maschke).

Next we assume that T is nonabelian; then $|T| \ge p^4$, by (i).

(iii) Assume that p divides $|G : G'|$. Then, by ($*$), H stabilizes the chain $\{1\} < T \cap G' < T$ so H is normal in G (Lemma 2(f)), a contradiction. Thus, p does not divide $|G : G'|$.

(iv) Let $A < T$ be a G-invariant subgroup. We claim that $A \le \mathrm{Z}(T)$. Assume that this is false. Since H centralizes A, by ($*$), $\mathrm{C}_G(A)$ is normal in G and $G/\mathrm{C}_G(A)$ is a p-group $> \{1\}$, contrary to (iii). Thus, $A \le \mathrm{Z}(T)$; moreover, $A \le \mathrm{Z}(G)$.

(v) Let $R < T$ be G-invariant and such that T/R is minimal normal in G/R. Then, by (iv), $R \le \mathrm{Z}(T)$; moreover, $R = \mathrm{Z}(T)$, by the maximal choice of R. It follows that the class of T equals 2 so, since $p > 2$, we get $\exp(\Omega_1(T)) = p$. By (i), T possesses a subgroup $E \cong \mathrm{E}_{p^3}$. Since $E \le \Omega_1(T)$ and $|G : H\Omega_1(T)|$ is a power of p, we get $G = H\Omega_1(T)$ so $T = \Omega_1(T)$ is of exponent p. It remains to show that T is special. Since $|G : RH| > 1$ is a power of p, R is elementary abelian of order at most p^2. If $T' < R$, then, by Lemma 2(g), applied to the pair $T/M < G/M$, the group G/M has a normal subgroup of index p, contrary to (iii). Thus, $T' = R$. Since T is of exponent p, we have $T' = \Phi(T)$. Thus, $\mathrm{Z}(G) = R = T' = \Phi(T)$ so T is special.

We see that if T is nonabelian, it is special of exponent p with $R = T' = \mathrm{Z}(T) = \Phi(T)$ of order $\le p^2$. By the maximal choice of R, T/R is a minimal normal subgroup of G/R so the case where T is normal in G, is complete.

It remains to show that T is normal in G always.

2. Now assume that T is not normal in G. Since $\mathrm{O}_{p'}(G) = \{1\}$ and G is solvable, we get $T > T_0 = \mathrm{O}_p(G) > \{1\}$. Therefore, we have $\mathrm{C}_G(T_0) \le T_0$ so H acts faithfully on T_0. Since $|G : T_0H| > 1$ is a power of p, T_0 has no

elementary abelian subgroup of order p^3. It follows that T_0 is a group of Lemma 2(e). However, as shows the argument in (i), H centralizes T_0, a final contradiction.

Since groups from parts A and B satisfy the hypothesis, the proof is complete. $\square$

Note that if G is a 2-group without normal elementary abelian subgroup of order 8, then it possesses a normal metacyclic subgroup M such that G/M is isomorphic to a subgroup of D_8 [6]. Therefore, it is natural to classify the nonnilpotent solvable groups G, satisfying (i) $\mathrm{O}_{2'}(G) = \{1\}$ and (ii) if $K < G$ is such that $|G : K|$ is a power of 2, then K has no elementary abelian subgroup of order 8. However, the proof of such result would be very long since the groups appearing in [6] are not so small as groups of Lemma 2(e).

Theorem 4 also holds for each odd prime divisor p of $|G|$ such that $|G|$ and $p^2 - 1$ are coprime (in that case, $|G|$ is odd so solvable). To prove this, we have to repeat, word for word, the proof of Theorem 4.

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