GLASNIK MATEMATIČKI
Vol. 39(59)(2004), 221 – 233

MINIMAL NONMODULAR FINITE p-GROUPS

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Abstract. We describe first the structure of finite minimal nonmodular 2-groups G. We show that in case $|G| > 2^5$, each proper subgroup of G is Q_8-free and $G/\mho_2(G)$ is minimal nonabelian of order 2^4 or 2^5. If $|G/\mho_2(G)| = 2^4$, then the structure of G is determined up to isomorphism (Propositions 2.4 and 2.5). If $|G/\mho_2(G)| = 2^5$, then $\Omega_1(G) \cong E_8$ and $G/\Omega_1(G)$ is metacyclic (Theorem 2.8).

Then we classify finite minimal nonmodular p-groups G with $p > 2$ and $|G| > p^4$ (Theorems 3.5 and 3.7). We show that $G/\mho_1(G)$ is nonabelian of order p^3 and exponent p and $\mho_1(G)$ is metacyclic. Also, $\Omega_1(G) \cong E_{p^3}$ and $G/\Omega_1(G)$ is metacyclic.

1. Introduction and Known Results

A group is called modular if its subgroup lattice is. It is known (Suzuki [3]) that a finite p-group G is modular if and only if any subgroups X and Y of G are permutable (i.e., $XY = YX$). According to Iwasawa's classification of modular groups, a finite 2-group is modular if and only if it is D_8-free (see Suzuki [3]). First we classify minimal nonmodular finite 2-groups G. Hence G is not D_8-free but each proper subgroup of G is D_8-free. The structure of such groups G is described in Propositions 2.1 to 2.7 and this is summarized in Theorem 2.8. Then we classify also minimal nonmodular finite p-groups for $p > 2$ (Theorems 3.5 and 3.7).

We consider here only finite groups and our notation is standard. In particular, we recall that a metacyclic 2-group H is called "ordinary metacyclic" (with respect to A) if H possesses a cyclic normal subgroup A such that H/A

2000 *Mathematics Subject Classification.* 20D15.
Key words and phrases. p-group, modular group, nonmodular group, quaternion group, minimal nonabelian group.

is cyclic and H centralizes $A/\mho_2(A)$. Further, a p-group G is called an A_2-group if each subgroup of index p^2 in G is abelian and at least one maximal subgroup of G is nonabelian. For convenience, we state here some known results which are used often in this paper.

Proposition 1.1. (Suzuki [3]) *Let G be a modular 2-group. If G is not Q_8-free, then G is Hamiltonian, i.e., $G = Q \times E$ where $Q \cong Q_8$ is quaternion and $exp(E) \leq 2$.*

Proposition 1.2. (Wilkens [4, Lemmas 1 and 2]) *Let G be a Q_8-free modular 2-group. Then G is powerful, i.e., $G/\mho_2(G)$ is abelian. Also, $\Omega_1(G)$ is elementary abelian, $\Omega_2(G)$ is abelian, and $d(G) = d(\Omega_1(G))$.*

Proposition 1.3. (Wilkens [4, Lemma 1]) *Let $G = \langle x, y\rangle$ be a two-generated Q_8-free modular 2-group. Then G is ordinary metacyclic and $[x, y] \in \langle x^4, y^4\rangle = \mho_2(G)$.*

Proposition 1.4. (Janko [2, Proposition 1.7]) *Suppose that a nonabelian p-group G possesses an abelian maximal subgroup. Then $|G| = p|G'||Z(G)|$.*

Proposition 1.5. (A. Mann, see Berkovich [1]) *Let A and B be two distinct maximal subgroups in a p-group G. Then $|G' : (A'B')| \leq p$.*

Proposition 1.6. (Janko [2, Proposition 1.10]) *Let G be a 2-group of order $> 2^4$ with $\Omega_2(G) = \langle a, b\rangle \times \langle u\rangle$, where $\langle a, b\rangle = Q \cong Q_8$ and u is an involution. Then G is a uniquely determined group of order 2^5. Set $\langle z\rangle = Z(Q)$, where $a^2 = b^2 = z$. There is an element y of order 8 in $G - \Omega_2(G)$ such that*

$$y^2 = ua,\ u^y = uz,\ a^y = a^{-1},\ b^y = bu.$$

Proposition 1.7. (N. Blackburn, see Berkovich [1]) *A 2-group G is metacyclic if and only if $G/\mho_2(G)$ is metacyclic.*

Proposition 1.8. (Suzuki [3]) *Let G be a modular p-group, $p > 2$. Then $G/\mho_1(G)$ and $\Omega_1(G)$ are elementary abelian and $d(G) = d(\Omega_1(G))$.*

Proposition 1.9. (N. Blackburn, see Berkovich [1]) *Let G be a minimal nonmetacyclic p-group, $p > 2$. Then G is one of the following groups:*

(a) *Any group of order p^3 and exponent p;*
(b) *The group G of order 3^4 and class 3 with $|\Omega_1(G)| = 3^2$.*

Proposition 1.10. (B. Huppert, see Berkovich [1]) *A p-group G with $p > 2$ is metacyclic if and only if $|G/\mho_1(G)| \leq p^2$.*

Please note that our proofs will be completely elementary and the computations are reduced to a minimum.

2. New Results for $p = 2$

Let G be a minimal non-modular 2-group. Then G has a normal subgroup N such that $G/N \cong D_8$. It is clear that $N \leq \Phi(G)$ and so $d(G) = 2$. We shall use this notation throughout this section.

Our first proposition is actually contained in Proposition 3.3 which is proved for all p-groups. But the proof below is typical for 2-groups.

Proposition 2.1. *We have* $d(N) \leq 2$.

Proof. Suppose $d(N) \geq 3$. Then N possesses a G-invariant subgroup R such that $N/R \cong E_8$. We want to determine the structure of G/R (which is also minimal non-modular) and so we may assume $R = \{1\}$ which implies that $N \cong E_8$. Let S/N be any subgroup of order 2 in G/N. Then $exp(S) \leq 4$ and S is Q_8-free and so S (being modular) is abelian (Proposition 1.2). On the other hand, G/N is generated by its subgroups of order 2 and so $N \leq Z(G)$.

Assume $Z(G) > N$ so that $Z(G)/N = Z(G/N) = \Phi(G/N) = (G/N)'$ and therefore we get $Z(G) = \Phi(G)$. Each maximal subgroup of G is abelian and so G is minimal nonabelian. In particular, $|G'| = 2$ and since G' covers $Z(G)/N$, we get $Z(G) = N \times G' \cong E_{16}$. This is a contradiction since minimal nonabelian 2-groups have the property $|\Omega_1(G)| \leq 8$. We have proved that $N = Z(G)$.

Let L/N be the unique cyclic subgroup of index 2 in G/N. Then L is abelian. If $L = N \times L_1$ with $L_1 \cong C_4$, then $\mho_1(L) = \mho_1(L_1)$ is of order 2 and so $\mho_1(L) \leq Z(G)$. But $\mho_1(L) \not\leq N = Z(G)$, a contradiction. Hence L does not split over N and so $L = NC$, where $C \cong C_8$ and $C \cap N = C_0$ is of order 2. We have $\Phi(L) = \Phi(C) = C_1 \cong C_4$, where $C_0 < C_1$ and $\Phi(G) = C_1 N$. For each $x \in G - L$, $x^2 \in N$ and so there is $b \in G - L$ with $b^2 \in N - C_0$ (otherwise, $\Phi(G) = \mho_1(G) = C_1$). Since $C_1 \not\leq Z(G)$ and C_1 is normal in G, it follows that b inverts C_1. We have $D = \langle C_1, b\rangle$ is of order 2^4 and $D/\langle b^2\rangle \cong D_8$, a contradiction. ∎

Proposition 2.2. *Suppose $d(N) = 1$ and some proper subgroup of G is not Q_8-free. Then G is isomorphic to Q_{2^4} or to the uniquely determined group X of order 2^5 with $\Omega_2(X) \cong Q_8 \times C_2$ (given in Proposition 1.6).*

Proof. Suppose that N is cyclic and G has a maximal subgroup M which is not Q_8-free. Since M is modular, it follows that M is Hamiltonian, i.e., $M = Q \times E$ with $Q \cong Q_8$ and $exp(E) \leq 2$ (Proposition 1.1). In particular, $exp(M) = 4$ and $\mho_1(M) = \Phi(M) = \mho_1(Q)$ which implies $|N| \leq 4$. If $|G'| = 2$, then $d(G) = 2$ implies that G is minimal nonabelian. This is a contradiction since M is nonabelian. Hence $|G'| \geq 4$ which implies that G has at most one abelian maximal subgroup (Proposition 1.5). Let $L/N \cong C_4$ be the unique cyclic subgroup of index 2 in G/N.

Suppose $|N| = 4$ so that $|\mho_1(L)| \geq 4$ and therefore L is not Hamiltonian. In that case L is ordinary metacyclic (Proposition 1.3). Suppose that L is abelian. We have $|\mho_1(L)| = 4$ and $\mho_1(L) > \mho_1(N) = \mho_1(M)$. Note that L cannot be cyclic since G has the Hamiltonian subgroup M of order 2^4. Let K be the maximal subgroup of G distinct from M and L and note that (since G is non-cyclic) $\mho_1(G) = \Phi(G) = \mho_1(M)\mho_1(L)\mho_1(K)$. We know that K must be nonabelian and $K/N \cong E_4$. Since $N \cong C_4$ does not lie in $Z(M)$, we have $N \not\leq Z(G)$ and so $C_G(N) = L$. This implies that $|K : C_K(N)| = 2$. Let $k \in K - C_K(N)$ so that $k^2 \in N$ and therefore $\langle N, k\rangle \cong Q_8$. It follows that K is Hamiltonian and so $\mho_1(K) = \mho_1(N) < \mho_1(L)$. Hence $\mho_1(G) = \Phi(G) = \mho_1(L)$ is of order 4. This is a contradiction since $|\Phi(G)| = 8$ in view of the fact that $\Phi(G) > N$.

We have proved that L is nonabelian. In particular, $N = \langle n\rangle \not\leq Z(L)$ and if we set $L = \langle N, l\rangle$, then $n^l = n^{-1}$ and $l^4 \in \langle n^2\rangle$. If $o(l) = 4$, then $L/\langle l^2\rangle \cong D_8$, a contradiction. Hence $o(l) = 8$ and so $L \cong M_{16}$ with $\langle l^4\rangle = \langle n^2\rangle = L'$ and $\Phi(L) = Z(L) = \langle l^2\rangle \cong C_4$. Note that $\Omega_2(L) = N\Phi(L)$ is abelian of type $(4, 2)$. Set $K = C_G(N)$ so that K is the maximal subgroup of G distinct from M and L and (noting that $\langle l^2\rangle > \langle n^2\rangle$) we get

$$\Phi(G) = \Phi(M)\Phi(L)\Phi(K) = \langle n^2\rangle\langle l^2\rangle\Phi(K) = \langle l^2\rangle\Phi(K).$$

Since $K/N \cong E_4$, we have $\Phi(K) \leq N$. But $\Phi(G) \geq N$ and so we must have $\Phi(K) = N$. It follows that K has a cyclic subgroup of index 2 and K is Q_8-free since K cannot be Hamiltonian (because $|\mho_1(K)| = 4$). Hence K is either abelian of type $(8, 2)$ or $K \cong M_{16}$. In any case, $\Omega_2(K) = \Phi(L)N = \Phi(G)$ is abelian of type $(4, 2)$. It follows that $\Omega_2(G) = M \cong Q_8 \times C_2$ and consequently G is the uniquely determined group of order 2^5 described in Proposition 1.6.

Now assume that $|N| = 2$ so that $M = Q \cong Q_8$, where $Z(M) = \mho_1(M) = N$. On the other hand, $Z(G/N) = \Phi(G)/N$, where $\Phi(G) < M$ and $\Phi(G) \not\leq Z(M)$. Hence $Z(G) = N \cong C_2$. But G' covers $\Phi(G)/N = (G/N)'$ and so $G' \not\leq Z(G)$. It follows that G is of class 3 and G is of maximal class. The only possibility is $G \cong Q_{2^4}$ and we are done. ∎

Proposition 2.3. *Suppose $d(N) = 1$, $N > \{1\}$, and each proper subgroup of G is Q_8-free. Then $\mho_2(G) = \Phi(N)$ and $G/\Phi(N)$ is minimal nonabelian of order 2^4 and exponent 4. Thus $G/\Phi(N)$ is isomorphic to one of the following groups:*

(a) $\langle x, y \mid x^4 = y^2 = 1,\ [x, y] = z,\ z^2 = [x, z] = [y, z] = 1\rangle$ *(non-metacyclic),*
(b) $\langle x, y \mid x^4 = y^4 = 1,\ x^y = x^{-1}\rangle$ *(metacyclic).*

Proof. By assumption, $N \neq \{1\}$ is cyclic and each maximal subgroup of G is Q_8-free. If $G/\Phi(N)$ is of exponent 8, then $G/\Phi(N)$ has a cyclic subgroup of index 2 which implies that $G/\Phi(N)$ is of maximal class (since $G/N \cong D_8$). But then $G/\Phi(N)$ has a proper subgroup which is isomorphic to D_8 or Q_8, a contradiction. Hence $G/\Phi(N)$ is of exponent 4 and each maximal subgroup of $G/\Phi(N)$ is D_8-free and Q_8-free and so is abelian (Proposition 1.2). Hence

$G/\Phi(N)$ is minimal nonabelian of order 2^4 and exponent 4 and so $G/\Phi(N)$ is isomorphic to the group (a) or (b) of our proposition.

Assume $\mho_2(G) < \Phi(N)$ so that $\mho_2(G) = \mho_2(N)$. Then $exp(G/\mho_2(N)) = 4$ and so each maximal subgroup of $G/\mho_2(N)$ is abelian (being D_8-free and Q_8-free). Thus $G/\mho_2(N)$ is minimal nonabelian of order 2^5 and exponent 4. In that case $G/\mho_2(N)$ is non-metacyclic and we know that $\Phi(G/\mho_2(N)) \cong E_{2^3}$. This is a contradiction since $\Phi(G)/\mho_2(N)$ contains a cyclic subgroup $N/\mho_2(N)$ of order 4. Our proposition is proved. □

In the next two propositions we shall determine completely the groups G of Proposition 2.3.

PROPOSITION 2.4. *Suppose $d(N) = 1$, $N > \{1\}$, and each proper subgroup of G is Q_8-free. If $G/\Phi(N)$ is not metacyclic, then G has a normal elementary abelian subgroup $E = \langle n, z, t\rangle$ of order 8 such that G/E is cyclic. We set $G = \langle E, x\rangle$, where $o(x) = 2^{s+1}$, $s \geq 1$, $E \cap \langle x\rangle = \langle n\rangle$, $[t, x] = z$, $[z, x] = n^{\epsilon}$, $\epsilon = 0, 1$, and $G = \langle x, t\rangle$. We have $G/\langle x^2\rangle \cong D_8$, $\Phi(G) = \langle x^2\rangle \times \langle z\rangle$, $\Omega_1(G) = E$, and G is Q_8-free. If $\epsilon = 0$, then G is minimal nonabelian non-metacyclic. If $\epsilon = 1$, then $s \geq 2$, $G' = \langle z, n\rangle \cong E_4$ and $Z(G) = \langle x^4\rangle$.*

PROOF. Suppose that we are in case (a) of Proposition 2.3 so that N is a non-trivial cyclic group and $G/\Phi(N)$ is isomorphic to the group (a) of Proposition 2.3. Let $M/\Phi(N)$ be a maximal subgroup of $G/\Phi(N)$ which is isomorphic to E_8. Set $|N| = 2^s$, $s \geq 1$, and note that N is a maximal cyclic subgroup of M. Let S/N be any subgroup of order 2 in M/N. Since M is D_8-free and Q_8-free, S cannot be of maximal class. It follows that S is either abelian of type $(2^s, 2)$ or $S \cong M_{2^{s+1}}$ $(s > 2)$. In any case, there exists an involution in $S - N$. Hence $\Omega_1(M)$ covers $M/N \cong E_4$ and (since M is modular) $\Omega_1(M)$ is elementary abelian (Proposition 1.2). It follows that $E = \Omega_1(M) \cong E_8$ is normal in G and $\Omega_1(M) \cap N = \Omega_1(N) = \langle n\rangle \leq Z(G)$. In particular, $\Phi(G)$ is abelian of type $(2^s, 2)$, $s \geq 1$. Indeed, $\Phi(G)$ cannot be isomorphic to $M_{2^{s+1}}$ $(s > 2)$ since in that case $Z(\Phi(G))$ is cyclic and so $\Phi(G)$ would be cyclic (Burnside). Take an involution $t \in E - \Phi(G)$.

Let K be another maximal subgroup of G such that $K/N \cong E_4$ so that $K \cap M = \Phi(G)$. Suppose that N is a maximal cyclic subgroup of K. Then, by the argument of the previous paragraph, $\Omega_1(K)$ covers K/N and so there is an involution $r \in K - M$. But then $\langle r, t\rangle = G$ and $\langle r, t\rangle$ is dihedral, a contradiction. We have proved that there is an element $x \in K - M$ such that $\langle x^2\rangle = N$ and so $o(x) = 2^{s+1}$, $s \geq 1$. Since $\langle x, t\rangle = G$ and $t \in E - \Phi(G)$, we get $G = E\langle x\rangle$ with $E \cap \langle x\rangle = \Omega_1(N) = \langle n\rangle \cong C_2$ and so G/E is cyclic of order 2^s and $|G| = 2^{s+3}$. In particular, $G' < E$ and so $|G'| = 2$ or 4.

We have $[x, t] \neq 1$ (since $\langle x, t\rangle = G$), $[x, t] \in E$ and so $z = [x, t]$ is an involution in $(E \cap \Phi(G)) - \langle n\rangle$ since $\langle x\rangle$ is not normal in G. Indeed, if $\langle x\rangle$ were normal in G, then the cyclic group $\langle x\rangle/\Phi(N) = \langle x\rangle/\langle x^4\rangle$ of order 4 is normal in $G/\Phi(N)$ which contradicts the structure of $G/\Phi(N)$. It follows that $E = \langle n, z, t\rangle$ and $\langle x\rangle$ is not normal in G. Thus $N_G(\langle x\rangle)$ is a maximal subgroup of G and so $z \in \Phi(G) \le N_G(\langle x\rangle)$ and therefore z normalizes $\langle x\rangle$. Since $\langle x, z\rangle$ cannot be of maximal class, we have either $[x, z] = 1$ (and then $G' = \langle z\rangle$ and G is minimal nonabelian) or $[x, z] = n$ (in which case $\langle x, z\rangle \cong M_{2^{s+1}},\ s > 1$). In the second case $\langle x\rangle$ induces an automorphism of order 4 on E. In any case, $G/\langle x^2\rangle \cong D_8$ and it is easy to see that G is Q_8-free.

Let u be an involution in $G - E$. Then $F = E\langle u\rangle \cong E_{16}$ since F is modular. But G/E is cyclic and so $F/E = (E\langle x^{2^{s-1}}\rangle)/E$ and so $x^{2^{s-1}}$ is an element of order 4 contained in $F - E$, a contradiction. We have proved that $\Omega_1(G) = E$. □

Proposition 2.5. *Suppose $d(N) = 1$, $N > \{1\}$, and each proper subgroup of G is Q_8-free. If $G/\Phi(N)$ is metacyclic, then G is also metacyclic and we have one of the following possibilities:*

(i) *$G = \langle x, y \mid x^4 = y^{2^{s+1}} = 1,\ s \ge 1,\ x^y = x^{-1}\rangle$, where G is minimal nonabelian and $N = \langle y^2\rangle$.*

(ii) *$G = \langle x, a \mid x^{2^{s+1}} = a^8 = 1,\ s \ge 2,\ x^{2^s} = a^4,\ a^x = a^{-1}\rangle$, where G is an A_2-group with $N = \langle x^2\rangle$, $Z(G) = N$, $G' = \langle a^2\rangle \cong C_4$ and G is of class 3.*

In both cases (i) and (ii), G is a minimal non-Q_8-free 2-group.

Proof. Suppose that we are in case (b) of Proposition 2.3 so that N is a non-trivial cyclic group and $G/\Phi(N)$ is isomorphic to the metacyclic group (b) of Proposition 2.3. We know that $\Phi(N) = \mho_2(G)$ and so G is also metacyclic (Proposition 1.7). Set $|N| = 2^s$, $s \ge 1$. If $\Phi(N) = \{1\}$, then G is isomorphic to the group (b) of Proposition 2.3 and we are done. Hence we may assume $s \ge 2$.

Let S/N be any subgroup of order 2 in G/N. Since S is D_8-free and Q_8-free, S is not of maximal class. Hence S is either cyclic of order 2^{s+1} or S is abelian of type $(2^s, 2)$ or $S \cong M_{2^{s+1}}$, $s > 2$. If $\Phi(G)$ is cyclic, then G has a cyclic subgroup of index 2, a contradiction. Also, $\Phi(G) \cong M_{2^{s+1}}$, $s > 2$, is not possible (Burnside). Hence $\Phi(G)$ is abelian of type $(2^s, 2)$, $s \ge 2$.

Let $\Omega_1(N) = \langle n\rangle$ so that $\langle n\rangle \le \Phi(N)$ and $\langle n\rangle \le Z(G)$. For any subgroup S/N of order 2 in G/N, we have seen (in the previous paragraph) that $S/\langle n\rangle$ is abelian. Since G/N is generated by its subgroups of order 2, we get $N/\langle n\rangle \le Z(G/\langle n\rangle)$.

Suppose for a moment that G is minimal abelian. Since $\Phi(G)$ is abelian of type $(2^s, 2)$, we get at once:

$$G = \langle x, y \mid x^4 = y^{2^{s+1}} = 1,\ s \ge 1,\ x^y = x^{-1}\rangle,$$

where $N = \langle y^2\rangle$. In what follows we assume that G is not minimal nonabelian. In particular, $|G'| \ge 4$.

We shall determine the structure of our three maximal subgroups of G. Let M be a maximal subgroup of G such that $M/N \cong E_4$. If N is a maximal cyclic subgroup of M, then for each subgroup S/N of order 2 of M/N, there is an involution in $S - N$. Hence $\Omega_1(M)$ covers M/N and (since M is D_8-free and Q_8-free), $\Omega_1(M)$ is elementary abelian and $\Omega_1(M) \cap N = \langle n \rangle$ so that $\Omega_1(M) \cong E_8$. This is a contradiction since G is metacyclic. It follows that N is not a maximal cyclic subgroup of M. Let M_0 be a maximal cyclic subgroup of M containing N so that $M_0 \cong C_{2^{s+1}}$ is a cyclic subgroup of index 2 in M. Since M is D_8-free and Q_8-free, M is not of maximal class and so M is either abelian of type $(2^{s+1}, 2)$ or $M \cong M_{2^{s+2}}$, $s \geq 2$. In any case, $N \leq Z(M)$ and $M/\langle n \rangle$ is abelian since $M' \leq \langle n \rangle$. Let K $(\neq M)$ be another maximal subgroup of G with $K/N \cong E_4$. Then K is either abelian of type $(2^{s+1}, 2)$ or $K \cong M_{2^{s+2}}$, $s \geq 2$, and again $N \leq Z(K)$ and $K/\langle n \rangle$ is abelian. We get $N \leq Z(G)$. If $\Phi(G) \leq Z(G)$, then each maximal subgroup of G would be abelian, contrary to our assumption that G is not minimal nonabelian. We have proved that $N = Z(G)$.

Let L be the unique maximal subgroup of G such that $L/N \cong C_4$. Then L is abelian and using Proposition 1.4 we get $|G'| = 4$. By a result of A. Mann (Proposition 1.5), L is the unique abelian maximal subgroup of G and so $M \cong K \cong M_{2^{s+2}}$ with $M' = K' = \langle n \rangle$. In particular, G is an A_2-group.

We have $G' > \langle n \rangle$ and G' covers $\Phi(G)/N$ and so $G' \cong C_4$ (since G is metacyclic), $\Phi(G) = NG'$, $N \cap G' = \langle n \rangle = \Omega_1(N)$. Since $G' \not\leq Z(G) = N$, G is of class 3.

Since G is metacyclic, there exists a cyclic normal subgroup Z of order 8 such that $Z > G'$. But $N \cap Z = N \cap G' = \langle n \rangle$ and so $NZ = L$ which determines the structure of the maximal subgroup L and shows that L does not split over N.

Set $Z = \langle a \rangle$. We know that there exists an element $x \in G - L$ such that $\langle x^2 \rangle = N$. Hence $G = Z\langle x \rangle$ with $Z \cap \langle x \rangle = \langle n \rangle$. Since $|G'| = 4$ and $G' < Z$, we get either $a^x = a^{-1}$ or $a^x = a^{-1}n$, where $n = a^4$. However, if $a^x = a^{-1}n$, then we replace $Z = \langle a \rangle$ with $Z^* = \langle as \rangle$, where $s \in N$ is such that $s^2 = n$. Then we compute

$$(as)^x = a^{-1}ns = a^{-1}s^{-1} = (as)^{-1}.$$

Since $\langle (as)^2 \rangle = \langle a^2 \rangle = G'$, we may assume from the start that $a^x = a^{-1}$ and so the structure of G is completely determined. □

In the rest of this section we consider the case $d(N) = 2$.

Proposition 2.6. *Suppose $d(N) = 2$. Then $G/\Phi(N)$ is the minimal nonabelian non-metacyclic group of order 2^5 and exponent 4. In particular, $G/\Phi(N)$ has the unique epimorphic image isomorphic to Q_8. Each maximal subgroup of G is Q_8-free and N is ordinary metacyclic.*

Proof. We want to determine the structure of $G/\Phi(N)$. Since $G/\Phi(N)$ is also minimal non-modular, we may assume for a moment $\Phi(N) = \{1\}$ so that $N \cong E_4$. Let S/N be any subgroup of order 2 in G/N. If S is nonabelian, then $S \cong D_8$, a contradiction. Hence S is abelian and so $N \leq Z(G)$.

Suppose that $Z(G) = N$. Let L/N be the unique cyclic subgroup of index 2 in G/N. Then N is abelian. If $L = N \times R$ with $R \cong C_4$, then $\mho_1(L) = \mho_1(R) \not\leq N$ and $\mho_1(L) \leq Z(G)$, contrary to our assumption. Hence $L = NL_1$ with $L_1 \cong C_8$ and $L_0 = L_1 \cap N \cong C_2$. We have $\Phi(L) = \Phi(L_1) \cong C_4$, where $\Phi(L) > L_0$. For each $x \in G - L$, $x^2 \in N$ and $\Phi(G) = \mho_1(G) = \Phi(L)N$. This implies that there exists $b \in G - L$ such that $b^2 \in N - L_0$. Since $\Phi(L) \not\leq Z(G)$, b inverts $\Phi(L)$. But then $D = \langle \Phi(L), b\rangle$ is of order 2^4 and $D/\langle b^2\rangle \cong D_8$, a contradiction.

We have proved that $Z(G) > N$ and so $Z(G) = \Phi(G)$. It follows that each maximal subgroup of G is abelian and so G is minimal nonabelian. In particular, $|G'| = 2$ and since G' covers $Z(G)/N = (G/N)'$, we have $Z(G) = N \times G'$ is elementary abelian of order 8. It follows that G is the uniquely determined minimal nonabelian non-metacyclic group of order 2^5 and exponent 4:

$$G = \langle a, b \mid a^4 = b^4 = 1,\ [a, b] = c,\ c^2 = [a, c] = [b, c] = 1\rangle,$$

where $Z(G) = \langle a^2, b^2, c\rangle$, $G' = \langle c\rangle$, and $G/\langle a^2c, b^2c\rangle$ is the unique factor-group of G which is isomorphic to Q_8. In particular, G is not Q_8-free.

We return now to the general case $d(N) = 2$, where $\Phi(N)$ is not necessarily trivial. Assume that N is not Q_8-free. Then N (being modular) is Hamiltonian. But $d(N) = 2$ and so $N \cong Q_8$. On the other hand, N is a G-invariant subgroup contained in $\Phi(G)$ and $Z(N)$ is cyclic. By a result of Burnside, N is cyclic, a contradiction. We have proved that N is Q_8-free and so $N/\mho_2(N)$ is abelian. Since $d(N/\mho_2(N)) = 2$, $N/\mho_2(N)$ is metacyclic. By a result of N. Blackburn (Proposition 1.7), N is metacyclic.

We want to show that N is ordinary metacyclic (although this is clear by Proposition 1.3). Let A be a cyclic normal subgroup of N such that N/A is cyclic. If N centralizes $A/\mho_2(A)$, we are done. Suppose that N does not centralize $A/\mho_2(A)$. Set $N = \langle A, g\rangle$ so that g inverts $A/\mho_2(A)$. If $|N : A| = 2$, then $N/\mho_2(A) \cong D_8$ or Q_8, a contradiction. Hence $|N : A| \geq 4$. Assume that $\langle g\rangle \cap A \leq \mho_2(A)$. Since g^2 centralizes $A/\mho_2(A)$, $Y = \mho_2(A)\langle g^2\rangle$ is normal in N and $N/Y \cong D_8$, a contradiction. It follows that $\langle g\rangle \cap A \not\leq \mho_2(A)$. Since g inverts $A/\mho_2(A)$, $\langle g\rangle \not\geq A$ and so $\langle g\rangle \cap A = \mho_1(A)$. Thus $\langle g\rangle$ is a cyclic subgroup of index 2 in N and g induces an involutory automorphism on A which centralizes a maximal subgroup of A. If $|A| \geq 8$, then g centralizes $A/\mho_2(A)$, a contradiction. Thus $A \cong C_4$ and $\langle g\rangle \cap A = \mho_1(A)$ is of order 2 so that $N' \leq \langle g\rangle \cap A$ and therefore $N' = \mho_1(A)$ and $o(g) \geq 8$. It follows that $N \cong M_{2^n}$, $n \geq 4$, and so N is ordinary metacyclic with respect to $\langle g\rangle$ since N centralizes $\langle g\rangle/\langle g\rangle \cap A$ which is of order ≥ 4.

Suppose that a maximal subgroup M of G is not Q_8-free. Then M (being modular) is Hamiltonian and so

$$M = Q \times E,\ Q \cong Q_8,\ \exp(E) \leq 2.$$

In particular, $\Phi(M)$ is of order 2 and $\exp(M)$=4. If $N \cong E_4$, then (by the above) G is minimal nonabelian. This is a contradiction since M is nonabelian. Since N is of exponent 4, $\mho_1(N)$ is of order 2 and N is abelian (being Q_8-free), we have $N \cong C_4 \times C_2$. We get $\Phi(M) = \mho_1(M) = \mho_1(N) = \Phi(N)$ and so $M/\Phi(N)$ is an elementary abelian subgroup of order 16 in the minimal nonabelian group $G/\Phi(N)$ of order 2^5 which was determined above. But that group $G/\Phi(N)$ has no such subgroup. We have proved that each maximal subgroup of G is Q_8-free. ☐

PROPOSITION 2.7. *Suppose $d(N) = 2$. Then for each maximal subgroup M of G we have $d(M) = 3$. Also, $\Phi(N) = \mho_2(G)$, $E = \Omega_1(G) = \Omega_1(\Phi(G)) \cong E_8$, $E \leq Z(\Phi(G))$, and either $G/E \cong Q_8$ (with $\Omega_2(G) = \Phi(G)$ being abelian of type $(4, 2, 2)$) or G/E is ordinary metacyclic (but not cyclic).*

PROOF. If G has two non-commuting involutions t, u, then $\langle t, u\rangle = G \cong D_{2^n}$, $n \geq 5$, since $|G| \geq 2^5$. But each proper subgroup of G must be D_8-free, a contradiction. We have proved that $\Omega_1(G)$ is elementary abelian.

Set $F = \Phi(G)$ so that we have

$$F/\Phi(N) = \Phi(G/\Phi(N)) = \Omega_1(G/\Phi(N)) \cong E_8.$$

Since $\Phi(N) \leq \Phi(F)$, we get $\Phi(N) = \Phi(F)$. Thus $d(F) = 3$ and so (since F is D_8-free and Q_8-free) $E = \Omega_1(F) \cong E_8$ is a normal elementary abelian subgroup of order 8 in G. Let M be any maximal subgroup of G so that $M/\Phi(N)$ is abelian of type $(4, 2, 2)$, $\Phi(N) \leq \Phi(M)$ and so $d(M) = 3$. But M is also D_8-free and Q_8-free and therefore $\Omega_1(M) \cong E_8$ which implies $\Omega_1(M) = \Omega_1(F) = \Omega_1(G)$.

We have $\Phi(N) \geq \mho_2(G)$. On the other hand, $\exp(G/\mho_2(G)) = 4$ and so each maximal subgroup of $G/\mho_2(G)$ (being D_8-free and Q_8-free) is abelian. Thus $G/\mho_2(G)$ is minimal nonabelian of exponent 4 and so $|G/\mho_2(G)| \leq 2^5$. It follows $\mho_2(G) = \Phi(N)$.

If G/E is not D_8-free, then there is a normal subgroup N^* of G such that $E \leq N^*$ and $G/N^* \cong D_8$. By Propositions 2.1 and 2.6, N^* must be metacyclic, a contradiction. Hence G/E is D_8-free.

Suppose that G/E is not Q_8-free. Then G/E is Hamiltonian. Since $d(G/E) = 2$, we get $G/E \cong Q_8$. On the other hand, G/E cannot act faithfully on E, and so $C_G(E) \geq \Phi(G)$. In particular, $\Phi(G)$ is abelian of type $(4, 2, 2)$ and so $E \leq Z(\Phi(G))$. For each $x \in G - \Phi(G)$, $x^2 \in \Phi(G) - E$ and so $o(x^2) = 4$. It follows that $\Omega_2(G) = \Phi(G)$.

We assume that G/E is Q_8-free. In that case G/E is ordinary metacyclic (but not cyclic since $\Phi(G) \geq E$). There is a cyclic normal subgroup $S/E \neq \{1\}$

of G/E with the cyclic factor-group $G/S \neq \{1\}$. Let $s \in S$ be such that $S = \langle E, s\rangle$ and let $r \in G - S$ be such that $G = \langle S, r\rangle$. Since $E \leq \Phi(G)$, we have $G = \langle r, s\rangle$.

Since $S = \langle E, s\rangle$ is a proper subgroup of G, it follows that S is D_8-free and Q_8-free. This implies that $\langle s\rangle$ is normal in S. Indeed, if $\langle s\rangle$ were not normal in S, then $S/\langle s^2\rangle \cong D_8$ since $|E \cap \langle s\rangle| = 2$ and s does not centralize the four-group $E/(E \cap \langle s\rangle)$. This is a contradiction and so $\langle s\rangle$ is normal in S. In particular, $S' \leq E \cap \langle s\rangle$ and so s induces an automorphism of order ≤ 2 on E which implies that s^2 centralizes E.

Since $\langle E, r\rangle < G$, we get (as in the previous paragraph) that r^2 centralizes E. On the other hand, $\Phi(G) = \langle E, r^2, s^2\rangle$ and so we get again $E \leq Z(\Phi(G))$. ∎

We summarize our results in a somewhat different form.

THEOREM 2.8. *Let G be a minimal non-modular 2-group of order ¿ 2^5. Then each proper subgroup of G is Q_8-free and $G/\mho_2(G)$ is minimal nonabelian of order 2^4 or 2^5.*

(a) *Suppose that $|G/\mho_2(G)| = 2^4$. If N is any normal subgroup of G such that $G/N \cong D_8$, then N is cyclic. If $G/\mho_2(G)$ is non-metacyclic, then G is Q_8-free and $\Omega_1(G) \cong E_8$ with $G/\Omega_1(G)$ cyclic . If $G/\mho_2(G)$ is metacyclic, then G is also metacyclic and G is not Q_8-free and G is either minimal nonabelian or an A_2-group.*

(b) *Suppose that $|G/\mho_2(G)| = 2^5$. Then $G/\mho_2(G)$ is non-metacyclic, G is not Q_8-free and $\Omega_1(G) \cong E_8$ with $G/\Omega_1(G) \cong Q_8$ or $G/\Omega_1(G)$ is ordinary metacyclic (but not cyclic). Moreover, if N is any normal subgroup of G such that $G/N \cong D_8$, then N is ordinary metacyclic but non-cyclic.*

3. NEW RESULTS FOR $p > 2$

We recall that a p-group G is modular if and only if any subgroups X and Y of G are permutable, i.e., $XY = YX$. We turn now to the case $p > 2$.

PROPOSITION 3.1. *Let G be a modular p-group with $p > 2$ and $d(G) = 2$. Then G is metacyclic.*

PROOF. Since G is modular, $G/\mho_1(G)$ is elementary abelian (Proposition 1.8). But $d(G/\mho_1(G)) \leq 2$ and so so $|G/\mho_1(G)| \leq p^2$. Then Proposition 1.10 implies that G is metacyclic. ∎

PROPOSITION 3.2. *Let G be a minimal nonmodular p-group, $p > 2$, which is generated by two subgroups A and B of order p. Then $G \cong S(p^3)$ (the nonabelian group of order p^3 and exponent p).*

PROOF. Since G is a p-group, $G_1 = \langle A^G \rangle$ and $G_2 = \langle B^G \rangle$ are proper normal subgroups of G and so G_1 and G_2 are modular. It follows that G_1 and G_2 are elementary abelian (Proposition 1.8). But

$$\langle G_1, B\rangle = \langle A, B\rangle = \langle G_2, A\rangle = G$$

and so G_1 and G_2 are two distinct maximal subgroups of G. By Proposition 1.5, we have $|G'| = p$ and $G' \leq G_1 \cap G_2$. Thus G/G' is abelian and G/G' is generated by elementary abelian subgroups G_1/G' and G_2/G'. Hence G/G' is elementary abelian and $d(G) = 2$ implies that $G/G' \cong E_{p^2}$. Hence $G \cong S(p^3)$ since the metacyclic nonabelian group of order p^3 is modular. □

PROPOSITION 3.3. *Let G be a minimal nonmodular p-group. Then G possesses a normal subgroup N such that $d(N) \leq 2$, $N \leq \mho_1(G)$, and G/N is a nonmodular group of order p^3. If $p = 2$, then $G/N \cong D_8$ and if $p > 2$, then $G/N \cong S(p^3)$, $N = \mho_1(G)$, and N is metacyclic.*

PROOF. Let A and B be subgroups of G such that $AB \neq BA$. Then $\langle A, B\rangle = G$ and there are cyclic subgroups $\langle a\rangle \leq A$ and $\langle b\rangle \leq B$ such that $\langle a\rangle\langle b\rangle \neq \langle b\rangle\langle a\rangle$. It follows $G = \langle a, b\rangle$ and so $d(G) = 2$. Since $\langle a^p, b^p\rangle \leq \Phi(G)$, the subgroups $E = \langle a^p, b\rangle$ and $F = \langle a, b^p\rangle$ are proper subgroups of G. Hence E and F are modular and so $E = \langle a^p\rangle\langle b\rangle$, $F = \langle a\rangle\langle b^p\rangle$, and $G = \langle E, F\rangle$. Set $N = \langle a^p\rangle\langle b^p\rangle$ so that $|E : N| = |F : N| = p$. It follows that N is normal in G, $N \leq \mho_1(G)$, and $d(N) \leq 2$. It remains to determine the structure of $\bar{G} = G/N = \langle \bar{a}, \bar{b}\rangle$, where $\bar{G}$ is a minimal nonmodular p-group generated by elements $\bar{a}$ and $\bar{b}$ of order p. If $p = 2$, then $\bar{G}$ is dihedral and (because of minimality) $\bar{G} \cong D_8$. If $p > 2$, then Proposition 3.2 implies that $\bar{G} \cong S(p^3)$. In that case we have $N = \mho_1(G)$ and Proposition 3.1 implies that N is metacyclic. □

PROPOSITION 3.4. *Let G be a minimal nonmodular p-group, $p > 2$, with $|G| > p^4$. Then $\Omega_1(G)$ is elementary abelian of order $\geq p^3$.*

PROOF. Let A and B be subgroups of order p in G such that $AB \neq BA$. Then $G = \langle A, B\rangle$ and so Proposition 3.2 implies that $G \cong S(p^3)$, a contradiction. We have proved that $AB = BA$ and so $\langle A, B\rangle$ is abelian. Hence $\Omega_1(G)$ is elementary abelian.

Assume that each proper subgroup of G is metacyclic. By Proposition 3.3, G is nonmetacyclic and so $|G| \leq p^4$ (Proposition 1.9), a contradiction.

Let M be a nonmetacyclic maximal subgroup of G. Since M is modular, Proposition 3.1 implies that $d(M) \geq 3$. On the other hand, $d(M) = d(\Omega_1(M))$ (see Suzuki [3]) and so $\Omega_1(M)$ is elementary abelian of order $\geq p^3$ and we are done. □

THEOREM 3.5. *Let G be a minimal nonmodular p-group, $p > 2$, with $|G| > p^4$. If $\mho_1(G)$ is cyclic, then $\Omega_1(G) \cong E_{p^3}$ and $G/\Omega_1(G)$ is cyclic of order $\geq p^2$ (i.e. G is an L_3-group).*

Proof. By assumption, $N = \mho_1(G)$ is cyclic. By Proposition 3.4, we have $E = \Omega_1(G)$ is elementary abelian of order $\geq p^3$. But $|E \cap N| = p$ and E does not cover G/N, and so $E \cong E_{p^3}$. On the other hand, there is $a \in G - N$ with $\langle a^p \rangle = N$. Since $|G : \langle a \rangle| = p^2$ and $|\langle a \rangle \cap E| = p$, we get $G = \langle E, a \rangle$ and we are done. $\square$

Proposition 3.6. *Let G be a minimal nonmodular p-group, $p > 2$, with $|G| > p^4$. Suppose $d(\mho_1(G)) = 2$ and let M be any maximal subgroup of G. Then $d(M) \leq 3$.*

Proof. Suppose false. Let M be a maximal subgroup of G with $d(M) \geq 4$. Set $N = \mho_1(G)$ so that $M/N \cong E_{p^2}$ and $N/\Phi(N) \cong E_{p^2}$. Since $\Phi(N) \leq \Phi(M)$, we must have $\Phi(M) = \Phi(N)$ so that $M/\Phi(N) \cong E_{p^4}$. We shall study the structure of $G/\Phi(N)$ (which is also minimal nonmodular of order p^4) and so we may assume $\Phi(N) = \{1\}$ which implies $M \cong E_{p^4}$. Since $\Omega_1(G)$ is elementary abelian, we have $M = \Omega_1(G)$. Let $x \in G - M$ so that $1 \neq x^p \in N$ and $x^p \in Z(G)$. There is $y \in G - M$ such that $y^p \in N - \langle x^p \rangle$ and $y^p \in Z(G)$. It follows that $N \leq Z(G)$. If $Z(G) > N$, then $Z(G)/N = Z(G/N) = \Phi(G/N)$ and so $Z(G) = \Phi(G)$. But then G is minimal nonabelian. By the structure of such groups, $|\Omega_1(G)| \leq p^3$, a contradiction.

We have proved that $Z(G) = N$. By Proposition 1.4, $|G| = p^5 = p|Z(G)||G'|$ and so $|G'| = p^2$. Since $G/N \cong S(p^3)$, $G' \not\leq N$ and so $G' \cong E_{p^2}$ and $G' \cap Z(G) \cong C_p$ so that $C_G(G') = M$. Since $\mho_1(G) = N$, there is $v \in G - M$ such that $v^p \in N - G'$. The subgroup $H = \langle G', v \rangle$ is of order p^4 and $H/\langle v^p \rangle \cong S(p^3)$ so that H is nonmodular, a contradiction. $\square$

Theorem 3.7. *Let G be a minimal nonmodular p-group, $p > 2$, with $|G| > p^4$. Then $\mho_1(G)$ is metacyclic and $G/\mho_1(G) \cong S(p^3)$ (nonabelian group of order p^3 and exponent p). If $\mho_1(G)$ is noncyclic, then $\Phi(G) = \mho_1(G) \times C_p$, $\Omega_1(\Phi(G)) = \Omega_1(G) \cong E_{p^3}$, $G/\Omega_1(G)$ is metacyclic and for each maximal subgroup M of G we have $d(M) = 3$.*

Proof. Set $N = \mho_1(G)$ and suppose $d(N) = 2$. By Proposition 3.4, $\Omega_1(G)$ is elementary abelian of order $\geq p^3$ and $\Omega_1(G) \cap N \cong E_{p^2}$. Since $\Omega_1(G)$ does not cover $G/N \cong S(p^3)$, $N\Omega_1(G)$ is contained in a maximal subgroup M of G. By Proposition 3.6, $d(M) \leq 3$ and the modularity of M implies $d(M) = d(\Omega_1(M))$. This implies $\Omega_1(G) \cong E_{p^3}$ and so $(N\Omega_1(G))/N = \Phi(G/N)$. Thus $\Phi(G) = N\Omega_1(G)$ and so for each maximal subgroup X of G, we have $d(X) = 3$ since $X \geq \Phi(G)$ and $d(X) = d(\Omega_1(X)) = d(\Omega_1(G))$.

We know that $Aut(\Omega_1(G))$ does not possess an automorphism of order p^2. There are elements $a, b \in G$ such that $N = \langle a^p \rangle \langle b^p \rangle$ and a^p and b^p centralize $\Omega_1(G)$. Hence $\Phi(G) = N \times Z$ with $|Z| = p$.

If $G/\Omega_1(G)$ is nonmodular, then (Proposition 3.3) there is a normal subgroup K of G with $K \geq \Omega_1(G)$, $G/K \cong S(p^3)$, and $d(K) \leq 2$. This is a contradiction since $d(K) = d(\Omega_1(K)) = 3$. Hence $G/\Omega_1(G)$ is modular and since $d(G/\Omega_1(G)) \leq 2$, $G/\Omega_1(G)$ is metacyclic and our theorem is proved. □

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Received: 17.12.2003.