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CONSTRAINED ABSTRACT REPRESENTATION PROBLEMS IN SEMIGROUPS AND PARTIAL GROUPOIDS

A. Mani
Calcutta Mathematical Society, India

Abstract. In this paper different constrained abstract representation theorems for partial groupoids and semigroups are proved. Methods for improving the retract properties of the structures are also developed. These have strong class-theoretical implications for many types of generalized periodic semigroups and related partial semigroups. The results are significant in a model-theoretical setting.

1. Introduction

In this paper different new methods for improving the retract properties of semigroups and partial semigroups are developed. The results are of much significance in model-theoretical settings too. Given an ideal or a generalised ideal in a semigroup and two or more subsemigroups subject to certain restrictions, new semigroups are derived over the same base set via multiple derived partial semigroups. The method is significant from the classification and embeddability viewpoints too. The connections between the retract properties of the structures is then shown to be significant. The results are extended to CSM-partial semigroups -these being defined by the constraints on their process of generation [12].

Within the classes of semigroups/partial semigroups for which the method is definable a classification theory appears possible. This directly relates to the automorphism group of the structures. New class-closure operators are introduced in the light of the above. From the model-theoretic viewpoint

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the theory is interpretable over different logics including FOPL, 3-valued incomplete predicate logic and FOPL_+(FOPL enhanced with weak equalities). The binary partial/total operation $\cdot$ is interpretable as a ternary predicate R with a weakened functionality $(R(a,b,c), R(a,b,e) \rightarrow c = e)$ in the 3-valued logic. A reason for not using presentations is the relative breadth of possible applications. But key issues in using presentations in the context for finitely generated commutative monoids are considered.

The reader is expected to be familiar with generalized periodic semigroups, partial semigroups and retracts in semigroup theory or algebra. Some references are [2, 10, 16, 8, 19, 15] and [9]. Some of the essential notions and terminology are repeated for convenience.

A *Partial Semigroup* is a tuple of the form $S = \langle \underline{S}, \cdot, 2\rangle$ satisfying

$$(xy)z \stackrel{w^*}{=} x(yz),$$

the strong weak equality meaning 'if either side is defined then the other is and the two are equal'. The usual omission of $\cdot$ will be followed. If S_1, S_2 are two partial semigroups then a map $\phi : S_1 \longrightarrow S_2$ is called a *Morphism* if

$$\phi(xy) \rightsquigarrow \phi(x)\phi(y),$$

i.e. if the LHS is defined then so is the RHS and the two are equal. A *Closed Subsemigroup* K is a tuple of a subset $\underline{K}$ of $\underline{S}$ together with the $\cdot$ symbol and an interpretation of it on $\underline{K}$, satisfying

$$\forall x, y \in K (xy = z \in S \longrightarrow z \in K).$$

If K is a closed subsemigroup of a partial semigroup S s.t. there exists a morphism $\phi : S \longrightarrow K$, under

$$\phi_{|K} \equiv I_K \quad \text{and} \quad \phi^2 = \phi,$$

then it will be called a *Closed Retract*. If S is total then it is a retract. A *Strong Retract* K of a semigroup S is a retract which satisfies,

$$\forall x, y \in S\, xy = \phi(x)\phi(y),$$

ϕ being the associated retraction. S is then said to be a *Strong Retract Extension* of K. It may be noted that the notion of *f-Clone Extension* in [19] also reduces to this in case of semigroups.

A *Tabular Partial Algebra* will be a finite partial algebra with operations of at most arity two with an explicit representation of the partial algebra as a set of ordered elements. Equivalently it can be required that the interpretation of the signature be decidable from the defining conditions. It is a constrained version of categoricity.

An *Abstract CSM-Type* is a tabular partial algebra $T = \langle \underline{T}, \Sigma, \gamma, 0, 1, U\rangle$ with $\underline{T}$ being a set of deemed types except for 0,1,U being distinguished elements, Σ being some set of partial function symbols and γ an interpretation of it on $\underline{T}$ subject to

1. $\forall f^{\underline{T}} \in \Sigma^{\underline{T}}\, \forall \overline{x}\ (0 \in \overline{x} \longrightarrow f^{\underline{T}}(\overline{x}) = 0)$
2. $\forall f^{\underline{T}} \in \Sigma^{\underline{T}}\, \forall \overline{x}\ (U \in \overline{x}, \neg(0 \in \overline{x}) \longrightarrow f^{\underline{T}}(\overline{x}) = U)$
3. $\forall f^{\underline{T}} \in \Sigma^{\underline{T}}\, \forall \overline{x}\ (1 \in \overline{x} \longrightarrow f^{\underline{T}}(\overline{x}) \in \{0, 1, U\})$.

A *UCSM-Type* is a CSM-type which is idempotent in all of its operations except atmost for the unary ones.

2. Constrained Abstract Representation

The theorems in this section can also be seen from the viewpoint of actions as in the classification theory of groups. First it is proved that given a semigroup which contains a pair of subsemigroups satisfying a coherence condition and a retract ideal then there is a process of defining a new semigroup with improved retract properties. This result is then extended in different directions and to partial semigroups.

Theorem 2.1. *Let $S = \langle \underline{S}, \cdot, \tau_1, \tau_2, \tau_3, 2, 1, 1, 1\rangle$ be a model of a semigroup endowed with three unary predicates subject to $\Sigma_0 \cup \Sigma_1 \cup \Sigma_2$.*

$$\begin{aligned}
\Sigma_0 :&\quad \forall x, y\ \exists a, b xy = a, yx = b\\
&\quad \forall x, y, z (xy)z = x(yz);\\
\Sigma_1 :&\quad (x)(y)(a)(b)(\tau_3 y, xy = a, yx = b \longrightarrow \tau_3 a, \tau_3 b)\\
&\quad (x)(y)(z)(\tau_1 x, \neg\tau_1 y, xy = z \longrightarrow \neg\tau_1 z)\\
&\quad (x)(y)(z)(\tau_1 x, \neg\tau_2 y, xy = z \longrightarrow \neg\tau_2 z)\\
&\quad (x)(y)(z)(\neg\tau_1 x, \tau_2 y, xy = z \longrightarrow \neg\tau_1 z)\\
&\quad (x)(y)(z)(\neg\tau_2 x, \tau_2 y, xy = z \longrightarrow \neg\tau_2 z)]\\
\Sigma_2 :&\quad [(x)(\tau_3 x \longrightarrow \exists y \phi(y) = x, \tau_3 y)\\
&\quad (x)(\tau_3 x \longrightarrow \tau_3 \phi(x))
\end{aligned}$$

$\phi \in Mor(S, \tau_3)$, then the models of the partial semigroups $P_1 = \langle \underline{S}, \odot, \tau_1, \tau_2, \tau_3, (2, 1, 1, 1)\rangle$ defined via

$$(x)(y)(z)(\tau_1 x \vee \tau_2 y, xy = z \leftrightarrow x \odot y = z),$$

and $P_2 = \langle \underline{S}, *, \tau_1, \tau_2, \tau_3, (2, 1, 1, 1)\rangle$ defined via

$$(x)(y)(z)(\tau_3 x, \tau_3 y, xy = z \longrightarrow x * y = z)$$

and the functorial compatibility

$$(x)(y)(a)(b)(x \odot y = a, x * y = b \longrightarrow a = b),$$

admit of embedding into a semigroup on the same base set $\underline{S}$ with three extra predicates of the same form.

Proof. The proof consists in defining the required semigroup by an universalization of restricted quantification. The initial universal quantifiers are omitted wherever they are clear.

Let $S^* = \langle \underline{S}, \oplus, \tau_1, \tau_2, \tau_3, (2,1,1,1)\rangle$ be a semigroup under the functorial correspondences

$$(\tau_1 x \vee \tau_2 y, xy = z \longrightarrow x \oplus y = z)$$

and

$$(\neg\tau_1 x, \neg\tau_2 y, \phi(x) \odot \phi(y) = z \longrightarrow x \oplus y = z).$$

The consequences include

$$(\phi(x) \oplus \phi(y) = a \leftrightarrow \phi(x) \oplus y = a),$$

$$(\tau_1(x), \tau_1(y), \tau_1(z), x \oplus y = a, a \oplus z = b \longrightarrow y \oplus z = c, x \oplus c = b)$$

and

$$(\phi(x) \oplus y = a \leftrightarrow x \oplus \phi(y) = a).$$

The coherence condition on S_1, S_2 also results in

$$(\phi(x) \oplus \phi(y) = a, a \oplus \phi(z) = b \leftrightarrow \phi(y) \oplus \phi(z) = c, \phi(x) \oplus c = b.$$

$$(\tau_1(x), \tau_1(y), \tau_1(z), y \oplus z = c, x \oplus c = b \longrightarrow x \oplus y = a, a \oplus z = b)$$

is a consequence of

$$(\tau_1(x), \tau_1(y), x \oplus y = z \longrightarrow \tau_1(z), x \cdot y = z)$$

and the other sentences.

$$(\tau_1(x), \neg\tau_1(y), \neg\tau_2(z), x \oplus y = a, a \oplus z = b \longrightarrow y \oplus z = c, x \oplus c = b)$$

and

$$(\tau_1(x), \neg\tau_1(y), \neg\tau_2(z), y \oplus z = c, x \oplus c = b \longrightarrow x \oplus y = a, a \oplus z = b)$$

are a consequence of

$$(\tau_1(x), \neg\tau_1(y), x \oplus y = a \longrightarrow \neg\tau_1(a))$$

and the other sentences. The restriction of the quantifications to $\neg\tau_1(x)$, $\tau_1(y)$, $\neg\tau_2(y)$ follows as a consequence of

$$(\tau_1(y), \neg\tau_2(y), y \oplus a = b \longrightarrow \neg\tau_2 b).$$

The other restrictions corresponding to

$$[\tau_1(x), \neg\tau_1(y), \tau_2(z)], [\neg\tau_1(x), \tau_2(y), \neg\tau_2(z)], [\neg\tau_1(x), \neg\tau_1(y), \neg\tau_2(z)],$$

$$[\tau_1(x), \neg\tau_1(y), \neg\tau_2(z)], [\neg\tau_1(x), \tau_2(y), \tau_2(z)], [\neg\tau_1(x), \neg\tau_1(y), \neg\tau_2(y), \neg\tau_2(z)]$$

$$\text{and } [\neg\tau_1(x), \neg\tau_1(y), \neg\tau_2(y), \tau_2(z)]$$

allow the universal. ∎

REMARK 2.2. In the above theorem three unary predicates respectively correspond to two subsemigroups satisfying a strong condition and the third to an ideal admitting a retraction from the semigroup.

REMARK 2.3. If a subsemigroup is involved in two instances of the embedding with the ideal remaining fixed, then it is not necessary that the other two subsemigroups allow a similar embedding.

THEOREM 2.4. *In the context of the above theorem if H is a retract ideal of the semigroup S which is also a subsemigroup of $\{x : x \in S, \tau_3(x)\}$ then the unary predicate can be replaced with τ_0 defined via*

$$\tau_0(x) \leftrightarrow \tau_3(x) \wedge x \in H.$$

THEOREM 2.5. *In the context of the first theorem, if Σ_1 is replaced by $\Sigma_!$ then also the result holds.*

$$\begin{aligned}
\Sigma_! : \qquad & [(\tau_1(x), \neg\tau_1(y), \tau_2(z), xy = a, az = b \longrightarrow \neg\tau_1(b)) \\
& (\tau_1(x), \neg\tau_2(y), \tau_2(z), xy = a, az = b \longrightarrow \neg\tau_2(b)) \\
& (\tau_1(x), \tau_2(y) \longrightarrow xy = yx) \\
& (\tau_3(y), xy = a, yx = b \longrightarrow \tau_3(a), \tau_3(b)) \\
& \forall x\, \exists y, a, b(\bigvee_\alpha \bigwedge_{i \in \alpha} \tau_i(y), xy = a, yx = b)].
\end{aligned}$$

PROOF. The proof is similar to that of Thm 2.1. ∎

THEOREM 2.6. *In the context of the first theorem if Σ_1 is replaced by Σ_4 then also the result holds provided † holds.*

$$\begin{aligned}
\Sigma_4 : \qquad & [(\neg\tau_1(x), \neg\tau_2(x), \tau_1(y), xy = z \longrightarrow \neg\tau_1(z)) \\
& (\tau_2(x), \neg\tau_1(y), \neg\tau_2(y), xy = z \longrightarrow \neg\tau_2(z)) \\
& (\neg\tau_1(x), \neg\tau_2(x), \tau_2(y), xy = z \longrightarrow \neg\tau_2(z)) \\
& \forall x\, \exists y, a, b(\bigvee_\alpha \bigwedge_{i \in \alpha} \tau_i(y), xy = a, yx = b)]
\end{aligned}$$

$$\dagger \;\equiv\; \{x : \tau_i(x) \cap \tau_j(x)\} \models \Sigma_0 \; i, j = 1, 2, 3.$$

PROOF. Clearly if $i \neq j, i, j = 1, 2, 3$, then

$$(\tau_i(x), \tau_i(y), \tau_j(y), xy = a, yx = b \longrightarrow \tau_i(a), \tau_i(b), \tau_j(a), \tau_j(b)).$$

The functorial definitions

$$(\tau_1(x) \vee \tau_2(y), xy = z \longrightarrow x \oplus y = z)$$

and

$$(\neg\tau_1(x), \neg\tau_2(y), \phi(x)\phi(y) = z \longrightarrow x \oplus y = z)$$

suffice for the model definition. ∎

The following are some of the types of semigroups in which the results are of direct interest. The results have strong implications in semigroups where the predicates are representable [11]. They also 'improve' presentations and normal identities in particular, but identifying semigroups from such perspectives is less convenient.

1. Weakly Periodic Semigroups for which there exist a generating set partible into a set of finite semigroup generators.
2. Periodic Semigroups, i.e.semigroups in which $\forall x \langle x \rangle$ is finite holds.
3. Cyclic Group-Bound Semigroups, i.e. semigroups in which $\forall x\, \exists n \in N\, x^n \in G$-a cyclic subgroup of S is true.
4. Weakly Periodic Semigroups satisfying
$$\forall x, a\, \exists! y \forall k \in N\, \exists z_k a^n x = a^n y, y = a^k z_k$$
and
$$\forall a, x\, \exists! y\, \exists z_1, z_2, \ldots, z_r, \ldots a^n x = a^n y \wedge y = a z_1 = a^2 z_2 = \ldots.$$
In such semigroups $\{x : \forall k \in N\, \exists x_k x = a^k z_k\}$ is an r-ideal. The context of the first theorem is attainable with commutativity or with weakened forms thereof.
5. Commutative and submedial semigroups with Tol(S) atomic, Tol(S) being the set of compatible tolerances on S.
6. Quasiregular Semigroups endowed with l/r- or 2-sided bases [6]. $S_1 \subset S$ is an l-base if $S_1 \cup S_1 S = S$ and S_1 is minimal among such sets. Similarly $S_1 \cup SS_1 \cup S_1 S \cup SS_1 S = S$ correspond to r- and two-sided bases. If S admits of an r-base, then S contains maximal left ideals. If the intersection of all nontrivial maximal left ideals is empty or a left covered ideal, then S contains at least one r-base.
7. Semigroups with compatible natural partial order [17] with maximal left ideals and
$$\forall x, e(e^2 = e \longrightarrow (ex)^n = (ex)^{n+l}); l \in N$$

2.1. *Presentations.* Many of the structures in which the theorems are relevant are f.g. commutative monoids. These have nice representation theories associated [20] and all principal ideals are determinable upto isomorphism by specifying a tuple of integers and a finite subset of $N^p \times N^p$. More specifically, let $S = \langle \underline{S}, + \rangle$ be a commutative monoid with a finite generating set $\{s_1, s_2 \ldots, s_p\}$, then there exists a morphism $\varphi : N^p \longmapsto S$ defined via
$$\varphi(x_1, x_2, \ldots, x_p) = x_1 s_1 + \ldots + x_p s_p,$$
N^p being the monoid over the p-th power of the set of natural numbers. If ρ_φ is the kernel congruence then
$$S \cong N^p \mid \rho_\varphi.$$
For a pair (m, γ) with $m \in N^p$ and
$$\gamma \in N^p \times N^p,\ \gamma = \{(a_1, b_1), \ldots, (a_t, b_t)\},$$

let $I = m + N^p$ and

$$\Gamma = \{(a_1 + m, b_1 + m), \ldots, (a_t + m, b_t + m)\} \subset I \times I.$$

If $\sigma_I = \sigma \cap (I \times I)$, where σ is the least congruence containing Γ then $I \mid \sigma_I$ is a principal ideal of $N^p \mid \sigma$.

Theorem 2.7. *If S is a completely regular semigroup with all its ideals being prime, and $x : \tau_2(x)$ is a SA-ideal too, then the context of the first theorem is generable, other aspects remaining the same.*

In the following section the extension and modifications of these to partial semigroups and groupoids is considered. The global significance is considered in the last section.

3. Partial Semigroups with CSM-Types

The results of the previous section are extended to partial semigroups with CSM-types, though in not necessarily 'proper' directions. Partial algebras with CSM-types have been recently introduced in [13] by the present author. These are basically partial algebras which admit explicit representation to a level of their process of generation.

Definition 3.1. *A* Partial Groupoid with CSM-Type *S will be a partial groupoid with a generating set K and a surjective morphism $\sigma : S \longmapsto C$ with C being a CSM-type of the same type and s.t. $Im(\sigma_{|K}) = C$.*

Definition 3.2. *In the above definition if the CSM-type is replaced by a UCSM-type then S will be said to be with a* UCSM-Type.

Note that related notions have been considered in semigroups, as in [1], [18] for example. In the following theorem, Thm 2.1 is extended in a modified way to a partial semigroup with UCSM-type endowed with a weak retract ideal and two closed subalgebras satisfying a coherence condition. The theorem allows a method of improving the retract properties of given partial semigroups and also constructing embeddable semigroups under suitable restrictions. Importantly the use of UCSM-types allows the required flexibility in contexts involving search for models.

Theorem 3.3. *In a partial groupoid S with UCSM-type (K, σ) (K being weak equationally axiomatizable), endowed with three extra unary predicates of the form $\langle \underline{S}, \cdot, \tau_1, \tau_2, \tau_3 \rangle$ and satisfying $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, let τ_i satisfy Γ_0 and τ_3 satisfy Γ with $\phi : S \longmapsto \tau_1$ being a closed retraction. Then there exist partial groupoid P_1, P_2 on the same base set for which an embedding analogous to* *that in Thm 2.1 is possible.*

$$\begin{aligned}
\Gamma_0 : & \quad \{x(yz) \stackrel{w}{=} (xy)z\} \\
\Gamma_1 : & \quad \{(x)(y)(a)(b)(\tau_3 y, xy = a, yx = b \longrightarrow \tau_3 a, \tau_3 b) \\
& \quad (x)(y)(z)(\tau_1 x, \neg\tau_1 y, xy = z \longrightarrow \neg\tau_1 z) \\
& \quad (x)(y)(z)(\tau_1 x, \neg\tau_2 y, xy = z \longrightarrow \neg\tau_2 z) \\
& \quad (x)(y)(z)(\neg\tau_1 x, \tau_2 y, xy = z \longrightarrow \neg\tau_1 z) \\
& \quad (x)(y)(z)(\neg\tau_2 x, \tau_2 y, xy = z \longrightarrow \neg\tau_2 z)\} \\
\Gamma : & \quad \{(x)(\tau_3 x \longrightarrow \exists y \phi(y) = x, \tau_3 y) \\
& \quad (x)(\tau_3 x \longrightarrow \tau_3 \phi(x))\}.
\end{aligned}$$

PROOF. The retraction classifies the defined instances of the partial operation, and the resulting conditional implications allow the result. ∎

REMARK 3.4. Results in the same spirit but under quite different conditions are proved in [14] by the present author. The related class operators are also investigated in it.

REMARK 3.5. Interestingly if the UCSM-type is assumed to satisfy stronger conditions like being 'axiomatizable by a set of strongly regular equations' ([23]), then the conditions in Γ_1 can be weakened. This is considered in [13].

4. Universal Aspects

An important aspect which arises from the first theorem in particular is the necessity of developing 'measures of existence of retracts' within classes of partial/total algebras. Categorical properties relating to retracts do not help in substantially comparing the distribution of retracts in two different structures with/without additional conditions. One natural way is via the properties of the associated endomorphism monoids. This is however not sufficiently developed for the purpose. In [19], [15] stronger forms of retract extensions are considered in varieties, but the issue in question is not addressed.

DEFINITION 4.1. *By the expression $\Upsilon(S, T_1, T_2, T_3, \phi)$ or $\Omega(S, \tau_1, \tau_2, \tau_3, \phi)$ will be meant a tuple for which the context of Thm 2.1 is valid, with T_i respectively corresponding to τ_i for all 'i'. $\Upsilon(S)$ will denote the set of all semigroups obtainable from S by the construction while $\Pi(S)$ will denote the particular semigroup.*

THEOREM 4.2. *If H is a retract ideal of S and $\Upsilon(S, T_1, T_2, T_3, \phi)$ then there exists a $T_0 = \{x : x \in H, \tau_3(x)\}$ such that $\Upsilon(S, T_1, T_2, T_0, \phi^0)$ for some ϕ^0.*

REMARK 4.3. Clearly this a restatement of Thm 2.2.

Definition 4.4. *θ will be a binary relation defined via $S\theta S^*$ if and only if [S^* is obtainable by a $\Upsilon(S, T_1, T_2, T_3, \phi)$ as a $\Pi(S)$].*

Theorem 4.5. *θ is a reflexive, antisymmetric and nontransitive relation in general over any subclass of semigroups.*

Proof. The verification is direct. □

Theorem 4.6. *S is not necessarily a retract extension of every element of $\Upsilon(S)$, but every retract of S is a retract of every other element of $\Upsilon(S)$ by using the induced reinterpretation.*

Note that in the context of Thm 2.1, T_3 is a retract of both S and the derived semigroup, but the range of values of the latter are more within it than in case of S. This can be because T_3 is a better retract of the latter than of the former in a structure theoretic sense too. Measuring this is apparently a qualitative aspect to supplement classification. But strong connections with different types of relations are clear in particular algebras, while a lot more is known in case of varieties.

Theorem 4.7. *In the context of Thm 2.1 in particular if $H \in \Upsilon(S)$, then $Aut(H)$ is a subgroup of $Aut(S)$, $Aut(*)$ being the automorphism group associated with the semigroup.*

Proof. If $\eta \in Aut(H)$ then it is also an automorphism on reinterpretation over the semigroup S. Let ι denote the operation of reinterpretation (strictly speaking a functor) of the automorphisms, then

$$\forall \eta_1, \eta_2 \in Aut(H)\, \iota(\eta_1 \circ \eta_2) = \iota(\eta_1) \circ \iota(\eta_2).$$

Further the restriction of the composition from $Aut(S)$ to $Aut(H)$ results in the composition of $Aut(H)$. So the result follows. □

Definition 4.8. *If $X \subseteq \underline{S}$ then ρ_X will be a relation on S defined via*

$$(x, y) \in \rho_X \longleftrightarrow \forall a, b \in S\, axb \in X \text{ if and only if } ayb \in S.$$

It is also called the principal congruence generated by X. If $\varphi : S \longmapsto S$ is a morphism then ρ_φ will denote $\rho_{Im(\varphi)}$.

Theorem 4.9. *The least congruence $\rho*$ containing $\rho_\varphi^S \backslash \rho_\varphi^{S^*}$ over the semigroup S is well defined.*

Proof. The proof follows from basic properties. □

Construction: Let S, S^* be two semigroups on identical base sets as in the context of Thm 2.1 (with S^* being the derived semigroup), then let $Mor(S, S \mid \rho*)$ and $Mor(S^*, S \mid \rho*)$ be the set of all morphisms into $S \mid \rho*$. $Mor(S^*, S \mid \rho*)$ is a subgroup of $Mor(S, S \mid \rho*)$. Denoting these respectively by $B(S)$ and $A(S)$, we will call $Mor(A(S), B(S))$ the *Characteristic Group* of the original construction. This is useful in comparing semigroups within equational classes at least.

Definition 4.10. *If $\mathcal{S}$ is a collection of semigroups then $\beth\mathcal{S}$ will be the smallest collection of semigroups containing all the semigroups obtainable as elements of $\Upsilon(S)$ for $S \in \mathcal{S}$.*

Definition 4.11. $\Re, \Re_s, \Xi, \Xi_s$ *will denote the class-closure operators corresponding to closure under retracts, strong retracts, retract extensions and strong retract extensions, respectively.*

Theorem 4.12. *For any collection $\mathcal{S}$ of semigroups the following holds.*

1. $\beth S(\mathcal{S}) \subset S\beth(\mathcal{S})$
2. $\beth(\mathcal{S}) \subset \Xi S(\mathcal{S})$
3. $\beth(\mathcal{S}) \subset \Xi H(\mathcal{S})$.

Proof. The proof is easy. □

Theorem 4.13. *For any collection of semigroups $\mathcal{S}$, the following holds.*

1. $\mathcal{S} \subset \beth(\mathcal{S})$
2. $HSP(\mathcal{S}) = HSP\beth(\mathcal{S})$
3. $\Re\beth(\mathcal{S}) \subset \beth\Re(\mathcal{S})$
4. $\Re_s\beth(\mathcal{S}) \subset \beth\Re_s(\mathcal{S})$
5. $\Re\beth(\mathcal{S}) \subset \Xi\Re(\mathcal{S})$

Proof. The first statement follows because of Thm 2.1, we can take the semigroup, the two subsemigroups and the retract ideal to be S.

Note that if S (in the context of Thm 2.1) is equational then the resulting semigroup is also equational. This ensures the second statement. □

Theorem 4.14. *For any collection of semigroups $\mathcal{S}$, the following holds.*

1. $\beth\Xi_s(\mathcal{S}) \subset \Xi\Re(\mathcal{S})$
2. $\Xi_s\beth(\mathcal{S}) \subset \Xi\Re(\mathcal{S})$
3. $\beth(\mathcal{S}) \subset \beth\beth(\mathcal{S})$

Proof. The proof is left to the reader. □

The above motivates the generalization of the constrained abstract representation or transformation context in particular. This and related aspects are also considered in [13].

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A. Mani
9b, Jatin Bagchi Road
Kolkata-700 029
India
E-mail: `a_mani_sc_gs@vsnl.net`

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