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ESTIMATES ON THE DIRICHLET HEAT KERNEL OF DOMAINS ABOVE THE GRAPHS OF BOUNDED $C^{1,1}$ FUNCTIONS

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Abstract. Suppose that D is the domain in $\mathbb{R}^d$, $d \geq 3$, above the graph of a bounded $C^{1,1}$ function $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ and that $p^D(t,x,y)$ is the Dirichlet heat kernel in D. In this paper we show that there exist positive constants C_1, C_2, C_3 and C_4 such that for all $t > 0$ and $x, y \in D$,

$$C_1(\frac{\rho(x)\rho(y)}{t} \wedge 1)t^{-d/2}e^{-\frac{C_2|x-y|^2}{t}} \leq p^D(t,x,y),$$

$$p^D(t,x,y) \leq C_3(\frac{\rho(x)\rho(y)}{t} \wedge 1)t^{-d/2}e^{-\frac{C_4|x-y|^2}{t}},$$

where $\rho(x)$ stands for the distance between x and ∂D.

1. Introduction

Suppose that D is a domain (i.e., a connected open set) in $\mathbb{R}^d$ and that $p^D(t,x,y)$ is the Dirichlet heat kernel for the Laplacian in D. Understanding the boundary behavior of p^D is of fundamental importance and a lot of progress has been made, see, for instance, [1, 6, 7, 8, 9, 13, 16] and [17]. It is known that when the Dirichlet heat semigroup on D is intrinsic ultracontractive, there is a $T > 0$ such that

$$\frac{1}{2}e^{-E_0t}\phi_0(x)\phi_0(y) \leq p^D(t,x,y) \leq \frac{3}{2}e^{-E_0t}\phi_0(x)\phi_0(y), \quad t \geq T, x, y \in D,$$

where E_0 is the smallest eigenvalue of the Dirichlet Laplacian $-\Delta|_D$ and ϕ_0 is the corresponding eigenfunction normalized by $\int_D \phi_0^2(x)dx = 1$. (For geometric conditions on D guaranteeing the intrinsic ultracontractivity, see

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[1, 6, 7] and [8].) However, the estimates above do not hold when t is small. In [16], Zhang proved that, when D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$, $d \geq 3$, there exist positive constants T, C_1, C_2, C_3 and C_4 such that for all $t \in (0,T]$ and $x, y \in D$,

$$
\begin{aligned}
&C_1\big(\frac{\rho(x)\rho(y)}{t} \wedge 1\big)t^{-d/2}e^{-\frac{C_2|x-y|^2}{t}} \leq p^D(t,x,y), \\
(1.1)\qquad &p^D(t,x,y) \leq C_3\big(\frac{\rho(x)\rho(y)}{t} \wedge 1\big)t^{-d/2}e^{-\frac{C_4|x-y|^2}{t}},
\end{aligned}
$$

where $\rho(x)$ stands for the distance between x and ∂D. In [13], it was shown that the result above remains valid in dimensions one and two. When D is the complement of a compact set in $\mathbb{R}^d$, $d \geq 3$, Grigor'yan and Saloff-Coste proved that $p^D(t,x,y)$ has upper and lower Gaussian estimates for x and y away from the boundary. In [17], Zhang proved that, when D is an exterior $C^{1,1}$ domain in $\mathbb{R}^d$, $d \geq 3$, there exist positive constants C_1, C_2, C_3 and C_4 such that for all $t \in (0,\infty)$ and $x, y \in D$,

$$
\begin{aligned}
&C_1\big(\frac{\rho(x)\rho(y)}{t \wedge 1} \wedge 1\big)t^{-d/2}e^{-\frac{C_2|x-y|^2}{t}} \leq p^D(t,x,y), \\
(1.2)\qquad &p^D(t,x,y) \leq C_3\big(\frac{\rho(x)\rho(y)}{t \wedge 1} \wedge 1\big)t^{-d/2}e^{-\frac{C_4|x-y|^2}{t}}.
\end{aligned}
$$

The estimates (1.1) and (1.2) were used in [13] to establish sharp bounds on the Green function, jump function and transition density of the subordinate killed Brownian motion in D, when D is either a bounded $C^{1,1}$ domain or an exterior $C^{1,1}$ domain. The main motivation for this paper is to extend the sharp estimates of [13] to other domains, such as domains above graphs of $C^{1,1}$ functions. To accomplish this, we need to establish explicit upper and lower estimates for $p^D(t,x,y)$ when D is the domain in $\mathbb{R}^d$ ($d \geq 3$) above the graph of a bounded $C^{1,1}$ function. More precisely we will, by adapting the arguments of [16] to the present case, show that (1.1) is valid for all $t \in (0,\infty)$ and $x, y \in D$. The main result of this paper is also valid when $d = 2$. To show this, one has to come up with an analogue in dimension two of Theorem 2.1 below and then follow the argument of Section 3. We are only going to deal with the case $d \geq 3$, we leave the case $d = 2$ to the interested reader.

The rest of the paper is organized as follows. In Section 2 we give the basic definitions and prove a simple geometric result which is essential for the argument of this paper. Section 3 contains the main result. In the last section, we apply our main result to get sharp estimates on the density, Green function and jumping function of subordinate killed Brownian motion in D.

After this paper was finished, the author came across the recent paper [15], in which Varopoulos proved, among other things, that the Dirichlet heat kernel in a domain D above the graph of a Lipschitz function satisfies for all

$(t,x,y) \in (0,\infty) \times D \times D$ the estimates:

$$
\begin{aligned}
&C_1 P(t,x)P(t,y)t^{-d/2}e^{-\frac{C_2|x-y|^2}{t}} \le p^D(t,x,y), \\
&p^D(t,x,y) \le C_3 P(t,x)P(t,y)t^{-d/2}e^{-\frac{C_4|x-y|^2}{t}},
\end{aligned}
\tag{1.3}
$$

where $P(t,x) = P^x(t < \tau_D)$ with τ_D being the first time that Brownian motion exits the domain D. If one knew that, when D is the domain above the graph of a bounded $C^{1,1}$ function,

$$
c\frac{\rho(x)}{\sqrt{t}} \wedge 1 \le P(t,x) \le C\frac{\rho(x)}{\sqrt{t}} \wedge 1, \quad (t,x) \in (0,\infty) \times D \tag{1.4}
$$

for some $0 < c \le C < \infty$, one could immediately get the main results of this paper from (1.3). However, (1.4) is not known. To get (1.4), one probably has to go through the main argument of the present paper. So in this sense, the explicit estimates of this paper are new. Also, the arguments of this paper are much more elementary.

2. Preliminaries

A bounded domain D in $\mathbb{R}^d$, $d \ge 2$, is said to be a bounded $C^{1,1}$ domain if there exist positive constants r_0 and M with the following property: for every $z \in \partial D$ and $r \in (0, r_0]$, there exist a function $\Gamma_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying the condition $|\nabla\Gamma_z(\xi) - \nabla\Gamma_z(\eta)| \le M|\xi - \eta|$ for all $\xi, \eta \in \mathbb{R}^{d-1}$ and an orthonormal coordinate system CS_z such that if $y = (y_1, \ldots, y_d)$ in the CS_z coordinates, then

$$
B(z,r) \cap D = B(z,r) \cap \{y : y_d > \Gamma_z(y_1, \ldots, y_{d-1})\}.
$$

The constant r_0 is called the localization constant of D and M is called the $C^{1,1}$ constant of D.

A domain D in $\mathbb{R}^d$, $d \ge 2$, is said to be an exterior domain if its complement is a compact set. An exterior domain is said to be an exterior $C^{1,1}$ domain if there exist positive constants r_0 and M with the following property: for every $z \in \partial D$ and $r \in (0, r_0]$, there exist a function $\Gamma_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying the condition $|\nabla\Gamma_z(\xi) - \nabla\Gamma_z(\eta)| \le M|\xi - \eta|$ for all $\xi, \eta \in \mathbb{R}^{d-1}$ and an orthonormal coordinate system CS_z such that if $y = (y_1, \ldots, y_d)$ in the CS_z coordinates, then

$$
B(z,r) \cap D = B(z,r) \cap \{y : y_d > \Gamma_z(y_1, \ldots, y_{d-1})\}.
$$

The following result played a very important role in [16] and [17].

Theorem 2.1. *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ ($d \ge 3$) with localization constant r_0 and $C^{1,1}$ constant M. Let* $\mathrm{diam}(D)$ *be the diameter of D. Then there exists a constant $C = C(r_0, M, \mathrm{diam}(D)) > 0$ such that the Green function G_D of D satisfies the following estimates:*

$$
\frac{1}{C}\left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1\right)\frac{1}{|x-y|^{d-2}} \le G_D(x,y) \le C\left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1\right)\frac{1}{|x-y|^{d-2}}
$$

for all $x, y \in D$. Here $\rho(x)$ denotes the Euclidean distance between x and ∂D. The dependence of $C = C(r_0, M, \mathrm{diam}(D))$ on r_0 and diam(D) *is only through the ratio* diam$(D)/r_0$.

The upper bound in the theorem above is due to [10], and the lower bound is due to [18]. The form of the theorem stated above (in particular, the last assertion) is taken from [3].

Suppose that $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ is a fixed bounded function such that

$$|\nabla(\xi)| \leq \lambda, \quad \forall \xi \in \mathbb{R}^{d-1}$$

and

$$|\nabla\Gamma(\xi) - \nabla\Gamma(\eta)| \leq \gamma|\xi - \eta|, \quad \forall \xi, \eta \in \mathbb{R}^{d-1}$$

for some constants λ and γ. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we write $\tilde{x} = (x_1, \dots, x_{d-1})$. In this paper we will concentrate on the domain

$$D = \{x \in \mathbb{R}^d : x_d > \Gamma(\tilde{x})\}.$$

As in [2], for any $x \in \mathbb{R}^d$ and constants $a, r > 0$, we define

$$\Delta(x, a, r) = \{y \in D : \Gamma(\tilde{y}) < y_d < \Gamma(\tilde{y}) + a, |\tilde{y} - \tilde{x}| < r\}.$$

For any $x \in D$, we use $\rho(x)$ to denote the Euclidean distance between x and ∂D and we use $\delta(x)$ to denote the vertical distance between x and ∂D:

$$\delta(x) = x_d - \Gamma(\tilde{x}).$$

It is easy to check that there exists $\kappa = \kappa(\lambda) \in (0, 1]$ such that

$$\kappa\delta(x) \leq \rho(x) \leq \delta(x), \quad x \in D. \tag{2.1}$$

The following geometric result will play a very important role in establishing our main result.

LEMMA 2.2. *For any $x \in \mathbb{R}^d$ ($d \geq 2$) and $r \geq 1$, there exists a bounded $C^{1,1}$ domain $D_{(x,r)}$ containing $\Delta(x, r, r)$ such that (1) $\{y \in \partial D : |\tilde{y} - \tilde{x}| \leq r\}$ and $\{y \in \mathbb{R}^d : |\tilde{y} - \tilde{x}| \leq r, y_d = \Gamma(\tilde{y}) + r\}$ are both contained in $\partial D_{(x,r)}$; (2) the $C^{1,1}$ constant of $D_{x,r}$ is bounded from above by a constant depending only on Γ; (3) the ratio* diam$(D_{(x,r)})/r_0$ *is bounded from above by a constant depending only on Γ, where* diam$(D_{(x,r)})$ *is the diameter of $D_{(x,r)}$ and r_0 is the localization constant of $D_{(x,r)}$.*

PROOF. Without loss of generality we may and do assume that $(\tilde{x}, \Gamma(\tilde{x}))$ is at the origin of $\mathbb{R}^d$. Let φ be a C^∞ function on $[0, \infty)$ such that (1) $\varphi(s) = 1$ for $s \in [0, 1]$; (2) $\varphi(s) = 0$ for $s \in [2, \infty)$; (3) $0 < \varphi(s) < 1$ for $s \in (1, 2)$. For any $r \geq 1$, define $\varphi_r(s) = \varphi(s/r)$. Then the function $\Gamma_r(\tilde{y}) = \Gamma(\tilde{y}) \cdot \varphi_r(|\tilde{y}|)$ coincides with $\Gamma(\tilde{y})$ when $|\tilde{y}| \leq r$ and equals 0 when $|\tilde{y}| \geq 2r$. Let's denote the following domain

$$\{y \in \mathbb{R}^d : |\tilde{y}| < 2r, \Gamma_r(\tilde{y}) < y_d < \Gamma_r(\tilde{y}) + r\}$$

by Ω_1. We are going to enlarge Ω_1 to get the desired domain.

Consider the following rectangle in the x_1x_d plane

$$Q = \{(y_1, y_d) : |y_1| < 2.5, 0 < y_d < 1\}.$$

By adding appropriate half-disk like regions at the right and left ends of Q, we can get a bounded $C^{1,1}$ domain V in the x_1x_d plane. Put $U = V \setminus \{(y_1, y_d) : |y_1| < 2, 0 < y_d < 1\}$, define $U_r = \{(ry_1, ry_d) : (y_1, y_d) \in U\}$ and let Ω_2 the subset of $\mathbb{R}^d$ obtained by revolving U_r around the x_d-axis. Then it is easy to check that the domain $\Omega_1 \cup \Omega_2$ satisfies all the requirements of the lemma.

□

3. The main results

In this section we are going to establish our main result. The argument of this section are adapted from that [16]. From now on we assume that $d \geq 3$. In the rest of this paper $p^D(t, x, y)$ stands for the Dirichlet heat kernel in D.

LEMMA 3.1. *Suppose that $\delta^2(x) \geq a_1 t$ and $\delta^2(y) \geq a_1 t$ for some $a_1 > 1$, then there exist positive constants C_1 and C_2 depending only on D and a_1 such that for all $t > 0$,*

$$\frac{1}{C_1 t^{d/2}} \exp(-\frac{|x-y|^2}{C_2 t}) \leq p^D(t, x, y) \leq \frac{C_1}{t^{d/2}} \exp(-\frac{C_2|x-y|^2}{t}).$$

PROOF. The proof of this lemma is similar to that of Lemma 2.1 of [16]. The upper bound is trivial, so we only need to prove the lower bound. The proof is divided into three steps.

Step 1: We prove the following claim: Suppose that $\delta^2(x) \geq a_1 t$ for some $a_1 > 1$, then there exists a positive constant c depending only on D such that

$$p^D(t, x, x) \geq \frac{c}{t^{d/2}}.$$

We pick a point $z \in D$ such that $\delta^2(z) \geq a_1 t$. Let $\phi \in C_0^\infty(B(z, \frac{\kappa\sqrt{t}}{2}))$ be such that $\phi(x) = 1$ when $x \in B(z, \frac{\kappa\sqrt{t}}{4})$ and $0 \leq \phi(x) \leq 1$ everywhere. Consider the function

$$u(x, s) = \int_D p^D(s, x, y)\phi(y)dy.$$

As in [12], we extend u by assigning $u(x, s) = 1$ when $s < 0$ and $x \in B(z, \frac{\kappa\sqrt{t}}{4})$, then u is a positive solution of $\frac{\partial u}{\partial s} = \Delta u$ in $B(z, \frac{\kappa\sqrt{t}}{4}) \times (-\infty, \infty) \subset D \times (-\infty, \infty)$. Using the parabolic Harnack inequality twice we get

$$\begin{aligned} u(z, 0) &\leq C_1 u(x, \frac{t}{4}) \\ p^D(\frac{t}{4}, z, y) &\leq C_1 p^D(t, z, z) \end{aligned}$$

for some constant $C_1 > 0$ independent of z. Hence

$$\begin{aligned} 1 &= u(z,0) \le C_1 u(x, \frac{t}{4}) = C_1 \int_{B(z, \frac{\kappa\sqrt{t}}{2})} p^D(\frac{t}{4}, z, y)\phi(y)dy \\ &\le C_1^2 p^D(t,z,z) \int_{B(z,\frac{\kappa\sqrt{t}}{2})} \phi(y)dy \le C_2 p^D(t,z,z) t^{-d/2}, \end{aligned}$$

for some constant $C_2 > 0$ depending only on D. Since z is arbitrary, the claim is proven.

Step 2: We prove the following claim: Suppose that $\delta^2(x) \ge a_1 t$ for some $a_1 > 1$ and $|x-y|^2 \le \kappa^2 t$, there exists a positive constant c depending only on D and a_1 such that

$$p^D(t,x,y) \ge \frac{c}{t^{d/2}}.$$

By Step 1, we know that there exists $c_1 > 0$ depending only on D such that

$$p^D(t,x,x) \ge \frac{c_1}{t^{d/2}}.$$

Consider the function $u(y,s) = p^D(s,x,y)$, which is a solution of $\frac{\partial u}{\partial s} = \Delta u$ in $B(x, \kappa(1+\epsilon)\sqrt{t}) \times (0,\infty) \subset D \times (0,\infty)$. Here $\epsilon > 0$ is sufficiently small. By the Harnack inequality we get

$$p^D(t,x,y) \ge c_2 p^D(\frac{t}{2}, x, x) > \frac{c_1 c_2}{t^{d/2}}$$

for some constant c_2 depending only on D and a_1. This completes Step 2.

Step 3: In this step, we treat the remaining case: $\delta^2(x) \ge a_1 t$, $\delta^2(y) \ge a_1 t$ and $|x-y|^2 \ge \kappa^2 t$.

By our assumption on D, we can easily see that there exists a length parameterized curve $l \subset D$ connecting x and y such that $|l| = \lambda_1 |x-y|$ for some $\lambda_1 \ge 1$. Here $\lambda_1 \le \lambda_0$ which is a constant depending on D only. Moreover l can be chosen so that $\rho(l(s), \partial D) \ge \lambda_2 \sqrt{t}$ for all $s \in [0, |l|]$. Here λ_2 is another constant depending on D only.

For $\lambda_3 > 0$ to be determined later, let m be the smallest positive integer satisfying

$$\frac{\lambda_3 |x-y|^2}{\kappa^2 t} \le m$$

and $x_j = l(\frac{j\lambda_1|x-y|}{m})$ for $j = 0, 1, \ldots, m$. Then

$$p^D(t,x,y) \ge \int p^D(\frac{t}{m}, x, y_1) p^D(\frac{t}{m}, y_1, y_2) \cdots p^D(\frac{t}{m}, y_{m-1}, y) dy_1 \cdots dy_{m-1}$$

where we integrate y_j over the set

$$\left\{ y_j : |y_j - x_j| \le \frac{\kappa}{4} (\frac{t}{m})^{1/2} \right\} \cap D.$$

Note that

$$\sqrt{\frac{\kappa^2 t}{m}} \le \sqrt{\frac{\kappa^4 t^2}{\lambda_3 |x-y|^2}} \le \sqrt{\frac{\kappa^2 t}{\lambda_3}} \le \frac{\kappa \rho(x_j)}{\lambda_2 \sqrt{\lambda_3}}.$$

Taking λ_3 sufficiently large, we have

$$\sqrt{\frac{t}{m}} \le \frac{\rho(x_j)}{4} \le \frac{|x_j - y_j|}{4} + \frac{\rho(y_j)}{4},$$

for $j = 0, 1, \ldots, m$. This shows that $\rho(y_j) > 2\sqrt{\frac{t}{m}}$ and hence

$$\left\{ y_j : |y_j - x_j| \le \frac{\kappa}{4} (\frac{t}{m})^{1/2} \right\} \cap D = \left\{ y_j : |y_j - x_j| \le \frac{\kappa}{4} (\frac{t}{m})^{1/2} \right\}.$$

Observe that

$$\begin{aligned}
|y_j - y_{j+1}| &\le |x_j - x_{j+1}| + |y_j - x_j| + |x_{j+1} - y_{j+1}| \\
&\le |x_j - x_{j+1}| + \frac{1}{2}\sqrt{\frac{\kappa^2 t}{m}} \le \frac{\lambda_1 |x-y|}{m} + \frac{1}{2}\sqrt{\frac{\kappa^2 t}{m}} \\
&= \frac{\lambda_1}{\sqrt{m}} \frac{|x-y|}{\sqrt{m}} + \frac{1}{2}\sqrt{\frac{\kappa^2 t}{m}} \le \frac{\lambda_1}{\sqrt{m}} \frac{|x-y|\sqrt{\kappa^2 t}}{\sqrt{\lambda_3}|x-y|} + \frac{1}{2}\sqrt{\frac{\kappa^2 t}{m}} \\
&\le \frac{2}{3}\sqrt{\frac{\kappa^2 t}{m}},
\end{aligned}$$

when λ_3 is sufficiently large. By step 2 there exists $c > 0$ depending only on D such that

$$p^D(\frac{t}{m}, y_k, y_{k+1}) \ge \frac{c}{(\frac{t}{m})^{d/2}}$$

for $y_1, \ldots, y_m$ in the region where the above integral takes place. Hence

$$p^D(t, x, y) \ge \prod_{j=0}^{m-1} \frac{c}{(\frac{t}{m})^{d/2}} \prod_{j=1}^{m-1} (\frac{\kappa^2 t}{16m})^{d/2} \ge c_1 \frac{c_2^m}{t^{d/2}},$$

for some positive constants c_1 and c_2 depending only on D and a_1. Since m is comparable to $\frac{|x-y|^2}{t}$, the above implies that, for some positive constants c_3 and c_4 depending only on D and a_1,

$$p^D(t, x, y) \ge \frac{1}{c_3 t^{d/2}} \exp(-\frac{|x-y|^2}{c_4 t}).$$

□

The following two lemmas are the analogs of Lemma 2.2 and Lemma 2.3 of [16] respectively. The difference is that the two lemmas below are valid for all t while Lemma 2.2 and Lemma 2.3 of [16] hold only for small t.

Lemma 3.2. *If $\delta^2(x) \le a_1 t$ and $\delta^2(y) \ge \frac{16a_1 t}{\kappa^2}$ for some $a_1 > 1$, then there exist positive constants C_1 and C_2 depending only on D and a_1 such that for all $t > 0$,*

$$\frac{\rho(x)}{C_1 t^{(d+1)/2}} \exp(-\frac{|x-y|^2}{C_2 t}) \le p^D(t,x,y) \le \frac{C_1\rho(x)}{t^{(d+1)/2}} \exp(-\frac{C_2|x-y|^2}{t}).$$

Proof. This proof is similar to the proof of Lemma 2.2 of [16]. The unspecified constants appearing in this proof are independent of t, x and y. We prove the upper bound first. Given $x \in D$ such that $\delta^2(x) \le a_1 t$, let $x = (\tilde{x}, \Gamma(\tilde{x}))$ and $x_t = (\tilde{x}, \Gamma(\tilde{x}) + \sqrt{2a_1 t})$. Then $|x - x_t| = \sqrt{2a_1 t}$. Put $x'_t = (\tilde{x}, \Gamma(\tilde{x}) + \frac{4\sqrt{a_1 t}}{\kappa})$. Clearly

$$|x_t - x'_t| \quad = \quad (\frac{4}{\kappa} - \sqrt{2})\sqrt{a_1 t} \tag{3.1}$$

$$|x - x'_t| \quad \ge \quad \frac{4\sqrt{a_1 t}}{\kappa} - \sqrt{a_1 t} \ge 3\sqrt{a_1 t}. \tag{3.2}$$

For any $y \in D$, write $u(z,s) = p^D(s,z,y)$ and $v(z) = G_\Omega(z, x'_t)$, where $\Omega = D_{x, 8\sqrt{a_1 t}/\kappa^2}$ is the bounded $C^{1,1}$ domain constructed in Lemma 2.2 with $r = 8\sqrt{a_1 t}/\kappa^2$. Both u and v are positive solutions of the equation $\partial_s u = \frac{1}{2}\Delta u$ in the region $\Delta(x, 3.5\sqrt{a_1 t}/\kappa, 3.5\sqrt{a_1 t}/\kappa) \times (0,\infty)$ and $u(z,s) = v(z) = 0$ when $z \in \partial D$. By the local comparison theorem in [11], there exists $c_1 > 0$ such that

$$\frac{u(x,t)}{v(x)} \le c_1 \frac{u(x_t, 2t)}{v(x_t)},$$

that is,

$$p^D(t,x,y) \le c_1 \frac{G_\Omega(x, x'_t)}{G_\Omega(x_t, x'_t)} p^D(2t, x_t, y). \tag{3.3}$$

By Lemma 2.2 we have that $\rho_\Omega(x) = \rho(x)$, $\rho_\Omega(x_t) = \rho(x_t)$ and $\rho_\Omega(x'_t) = \rho(x'_t)$, thus by Theorem 2.1 we have

$$\begin{aligned} G_\Omega(x, x'_t) &\le C\frac{\rho(x)}{|x - x'_t|^{d-1}} \le C\frac{\rho(x)}{t^{(d-1)/2}} \\ G_\Omega(x_t, x'_t) &\ge \frac{C}{|x_t - x'_t|^{d-2}} \ge \frac{C}{t^{(d-2)/2}}. \end{aligned}$$

Hence

$$p^D(t,x,y) \le C\frac{\rho(x)}{\sqrt{t}} p^D(2t, x_t, y). \tag{3.4}$$

When $y \in D$ satisfies $\delta(y)^2 \ge 16a_1 t/\kappa^2$,

$$\begin{aligned} |y - x| &\ge \rho(y) - \rho(x) \ge \kappa\delta(y) - \sqrt{a_1 t} \\ &\ge \sqrt{16 a_1 t} - \sqrt{a_1 t} \ge \frac{3}{\sqrt{2}}|x - x_t|. \end{aligned}$$

Hence

$$|y-x_t| \geq |y-x|-|x-x_t| \geq \frac{1}{2}|y-x| \tag{3.5}$$

$$|y-x_t| \leq |x-y|+|x-x_t| \leq 4|x-y|. \tag{3.6}$$

Thus by (3.4) and (3.5) we get

$$p^D(t,x,y) \leq C\frac{\rho(x)}{t^{(d+1)/2}}\exp(-\frac{c_2|x_t-y|^2}{t}) \leq C\frac{\rho(x)}{t^{(d+1)/2}}\exp(-\frac{c_3|x-y|^2}{t})$$

Now let us prove the lower bound. Keeping the same notation as above, by the local comparison theorem again, we have

$$\frac{u(x,t)}{v(x)} \geq c_1\frac{u(x_t,t/2)}{v(x_t)},$$

that is,

$$p^D(t,x,y) \geq c_1\frac{G_\Omega(x,x_t')}{G_\Omega(x_t,x_t')}p^D(\frac{t}{2},x_t,y). \tag{3.7}$$

By Theorem 2.1 we have

$$\begin{aligned} G_\Omega(x,x_t') &\geq C\left(\frac{\rho(x)\rho(x_t')}{|x-x_t'|^2}\wedge 1\right)\frac{1}{|x-x_t'|^{d-2}} \\ &\geq C\left(\frac{\rho(x)}{|x-x_t'|}\wedge 1\right)\frac{1}{|x-x_t'|^{d-2}} \\ &= C\frac{\rho(x)}{|x-x_t'|^{d-1}} \end{aligned}$$

and

$$G_\Omega(x_t,x_t') \leq \frac{1}{|x_t-x_t'|^{d-2}}.$$

Hence

$$p^D(t,x,y) \geq \frac{C\rho(x)|x_t-x_t'|^{d-2}}{|x-x_t'|^{d-1}}p^D(\frac{t}{2},x_t,y) \geq \frac{C\rho(x)}{t^{1/2}}p^D(\frac{t}{2},x_t,y), \tag{3.8}$$

where we used (3.1) and (3.2). Since $\delta^2(x_t) = 2a_1t$ and $\delta^2(y) \geq 16a_1t/\kappa^2$, Lemma 3.1 implies

$$p^D(\frac{t}{2},x_t,y) \geq \frac{c_1}{t^{d/2}}\exp(-\frac{c_2|x_t-y|^2}{t}) \geq \frac{c_1}{t^{d/2}}\exp(-\frac{c_3|x-y|^2}{t})$$

where the last inequality is due to (3.6). Therefore

$$p^D(t,x,y) \geq \frac{C\rho(x)}{t^{(d+1)/2}}\exp(-\frac{c_3|x-y|^2}{t}).$$

□

LEMMA 3.3. *If $\rho^2(x) \le a_2 t$ and $\rho^2(y) \le a_2 t$ for some $a_2 > 1$, then there exist positive constants C_1 and C_2 depending only on D and a_2 such that for all $t > 0$,*

$$\frac{\rho(x)\rho(y)}{C_1 t^{(d+2)/2}} \exp(-\frac{|x-y|^2}{C_2 t}) \le p^D(t,x,y) \le \frac{C_1\rho(x)\rho(y)}{t^{(d+2)/2}} \exp(-\frac{C_2|x-y|^2}{t}).$$

PROOF. The proof of this lemma is similar to that Lemma 2.3 of [16] and the unspecified constants appearing in this proof are independent of t, x and y. Let us keep the notations in the proof of Lemma 3.2 except replacing a_1 by a_2. Then following the proof of the upper bound in Lemma 3.2 we see that for all $y \in D$,

$$p^D(t,x,y) \le C\frac{\rho(x)}{\sqrt{t}} p^D(2t, x_t, y). \tag{3.9}$$

Now let $\tilde{u}(z,s) = p^D(s,z,x_t)$ and $\tilde{v}(z) = G_\Omega(z, y'_t)$, where y_t and y'_t are the counterparts of x_t and x'_t for y.

Both $\tilde{u}$ and $\tilde{v}$ are positive solutions of $\partial_s u = \frac{1}{2}\Delta u$ in the region $\Delta(x, 3.5\sqrt{a_2 t}/\kappa, 3.5\sqrt{a_2 t}/\kappa) \times (0,\infty)$ and $u(z,s) = v(z) = 0$ when $z \in \partial D$. By the local comparison theorem in [11], there exists $c_1 > 0$ such that

$$\frac{\tilde{u}(y,2t)}{\tilde{v}(y)} \le c_1 \frac{\tilde{u}(y_t, 4t)}{\tilde{v}(y_t)},$$

that is,

$$p^D(2t, y, x_t) \le c_1 \frac{G_\Omega(y, y'_t)}{G_\Omega(y_t, y'_t)} p^D(4t, y_t, x_t).$$

Since $|y - y'_t|$ and $|y_t - y'_t|$ are comparable with $\sqrt{a_2 t}$, we have as in the proof of the last lemma

$$p^D(2t, y, x_t) \le \frac{C\rho(y)}{t^{(d+1)/2}} \exp(-\frac{c_2|x_t - y_t|^2}{t}).$$

Since $|x_t - y_t| \ge |x-y| - |y-y_t| - |x-x_t| \ge |x-y| - C\sqrt{a_2 t}$, the above implies

$$p^D(2t, y, x_t) \le \frac{C\rho(y)}{t^{(d+1)/2}} \exp(-\frac{c_3|x-y|^2}{t}).$$

This and (3.9) yield the upper bound

$$p^D(t,x,y) \le \frac{C\rho(x)\rho(y)}{t^{(d+2)/2}} \exp(-\frac{c_3|x-y|^2}{t}).$$

Now we are going to prove the lower bound. Following the proof of the lower bound in Lemma 3.2, we get that for all $y \in D$,

$$p^D(t,x,y) \ge \frac{C\rho(x)}{t^{1/2}} p^D(\frac{t}{2}, x_t, y),$$

Switching the roles of x and y we get

$$p^D(\frac{t}{2}, x_t, y) = p^D(\frac{t}{2}, y, x_t) \geq \frac{C\rho(y)}{t^{1/2}} p^D(\frac{t}{4}, x_t, y_t).$$

Since $\delta(x_t)^2 = \delta(y_t)^2 = 2a_2 t$, Lemma 3.1 implies

$$p^D(\frac{t}{4}, x_t, y_t) \geq \frac{C}{t^{d/2}} \exp(-\frac{c_4|x_t - y_t|^2}{t}) \geq \frac{C}{t^{d/2}} \exp(-\frac{c_5|x-y|^2}{t}),$$

where we used the inequality $|x_t - y_t| \leq |x-y| + |x-x_t| + |y-y_t| \leq |x-y| + c\sqrt{t}$. The last three inequalities imply the desired lower bound. □

Now here is our main result.

THEOREM 3.4. *There exist positive constants C_1 and C_2 depending only on D such that for any $t > 0$ and any $x, y \in D$,*

$$\frac{1}{C_1}(\frac{\rho(x)\rho(y)}{t} \wedge 1) t^{-d/2} \exp(-\frac{|x-y|^2}{C_2 t}) \leq p^D(t, x, y),$$

$$p^D(t, x, y) \leq C_1(\frac{\rho(x)\rho(y)}{t} \wedge 1) t^{-d/2} \exp(-\frac{C_2|x-y|^2}{t}).$$

PROOF. For any $t > 0$ and $a_1 > 1$, put

$$\begin{aligned}
D_1 &= \{(x,y) \in D \times D : \rho^2(x) \geq a_1 t, \rho^2(y) \geq a_1 t\} \\
D_2 &= \{(x,y) \in D \times D : \rho^2(x) \leq a_1 t, \rho^2(y) \geq \frac{16 a_1 t}{\kappa^2}\} \\
D_3 &= \{(x,y) \in D \times D : \rho^2(x) \geq \frac{16 a_1 t}{\kappa^2}, \rho^2(y) \leq a_1 t\} \\
D_4 &= \{(x,y) \in D \times D : \rho^2(x) \leq \frac{16 a_1 t}{\kappa^2}, \rho^2(y) \leq \frac{16 a_1 t}{\kappa^2}\},
\end{aligned}$$

then $D \times D = D_1 \cup D_2 \cup D_3 \cup D_4$. The theorem follows by taking $a_2 = \frac{16a_1}{\kappa^2}$ in Lemma 3.3 and combining it with Lemmas 3.1 and 3.2. □

As a consequence of this result, we get the following sharp estimates on the Green function G_D.

THEOREM 3.5. *There exists a constant $C > 0$ such that for all $x, y \in D$,*

$$\frac{1}{C}\left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1\right) \frac{1}{|x-y|^{d-2}} \leq G_D(x,y) \leq C\left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1\right) \frac{1}{|x-y|^{d-2}}.$$

PROOF. The upper bound follows by integrating the upper bound in the theorem above with respect to t. The lower bounds can be obtained using an argument similar to the proof of Theorem 4.3 in [13]. □

As a consequence of Theorem 3.5, we immediately get the following sharp estimates on the Poisson kernel P_D.

THEOREM 3.6. *There exists a constants $C > 0$ such that for all $x \in D$ and $z \in \partial D$,*

$$\frac{1}{C}\frac{\rho(x)}{|x-z|^d} \le P_D(x,z) \le C\frac{\rho(x)}{|x-z|^d}.$$

Another consequence of Theorem 3.5 is the following 3G theorem which is very useful in analysis and probability (see [4] for one of the applications).

COROLLARY 3.7. *There exists a constant $C > 0$ such that*

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \le C\left(\frac{\rho(y)}{\rho(x)}G_D(x,y) + \frac{\rho(y)}{\rho(z)}G_D(y,z)\right), \quad x,y,z \in D.$$

4. Applications to subordinate killed Brownian motion

In this section we assume $\alpha \in (0,2)$. Let $\Delta|_D$ be the Dirichlet Laplacian in D. The fractional power $-(-\Delta|_D)^{\alpha/2}$ of the negative Dirichlet Laplacian is a very useful object in analysis and partial differential equations. There is a Markov process Z corresponding to $-(-\Delta|_D)^{\alpha/2}$ which can be obtained as follows: We first kill the Brownian motion X at τ_D, the first exit time of X from D, and then we subordinate the killed Brownian motion using the $\alpha/2$-stable subordinator T_t. Note that in comparison with the killed symmetric α-stable process on D the order of killing and subordination has been reversed. For the differences between Z and the killed symmetric α-stable process on D, please see [14].

It is well known that the Dirichlet form $\mathcal{E}$ of Z is given by

$$\mathcal{E}(u,u) = \int_D\int_D (u(x)-u(y))^2 J^D(x,y)dxdy + \int_D u^2(x)K^D(x)dx$$

where J^D and K^D are the jumping and killing functions of Z respectively given by

$$J^D(x,y) = \int_0^\infty p^D(t,x,y)\,\nu(dt) \tag{4.1}$$

$$K^D(x) = \int_0^\infty (1-P_t^D 1(x))\,\nu(dt), \tag{4.2}$$

where

$$\nu(dt) = \frac{\alpha/2}{\Gamma(1-\alpha/2)}t^{-\alpha/2-1}\,dt$$

is the Lévy measure of the $\alpha/2$-stable subordinator.

Let $r^D(t,x,y)$ be the density of Z and $\tilde{G}_D$ the Green function of Z. It follows from [14] and [13] that

$$\begin{aligned} r^D(t,x,y) &= \int_0^\infty p^D(s,x,y)\,\mu(t,s)ds \\ \tilde{G}_D(x,y) &= \int_0^\infty r^D(t,x,y)\,dt = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty p^D(t,x,y)t^{\alpha/2-1}dt \end{aligned}$$

where $\mu(t,s)$ is the density of the one-sided $\alpha/2$-stable convolution semigroup.

Sharp estimates on J^D, $\tilde{G}_D$ and r^D have been established when D is either a bounded $C^{1,1}$ domain or an exterior $C^{1,1}$ domain. In this section we deal with the case when D is the domain above the graph of a $C^{1,1}$ function, as specified in the previous section.

By using Theorem 3.4 and following the arguments in Section 4 of [13], we can get the following results.

Theorem 4.1. *Suppose that D is the domain specified in the previous section.*

(1) There exist positive constants C_1 and C_2 such that for all $x, y \in D$,

$$C_1 \left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{d+\alpha}} \le J^D(x,y) \le C_2 \left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{d+\alpha}}$$

(2) There exist positive constants C_3 and C_4 such that for all $x, y \in D$,

$$C_3 \left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{d-\alpha}} \le \tilde{G}_D(x,y) \le C_4 \left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{d-\alpha}}$$

Theorem 4.2. *Suppose that D is the domain specified in the previous section. There exist positive constants C_1 and C_2 such that*

$$C_1 \left(\frac{\rho(x)\rho(y)}{t^{2/\alpha} + |x-y|^2} \wedge 1 \right) t^{-\frac{d}{\alpha}} \left(1 + \frac{|x-y|^2}{t^{2/\alpha}} \right)^{-\frac{d+\alpha}{2}} \le r^D(t,x,y)$$

$$\le C_2 \left(\frac{\rho(x)\rho(y)}{t^{2/\alpha} + |x-y|^2} \wedge 1 \right) t^{-\frac{d}{\alpha}} \left(1 + \frac{|x-y|^2}{t^{2/\alpha}} \right)^{-\frac{d+\alpha}{2}} .$$

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References

[1] R. Bañuelos, *Intrinsic ultracontractivity and eigenfunction estimates for Schrödinger operators*, J. Funct. Anal. **100** (1991), 181–206.

[2] R. F. Bass and K. Burdzy, *A probabilistic proof of the boundary Harnack principle*, In: Seminar on Stochastic Processes, 1989, Birhäuser, Boston, 1990, 1–16.

[3] K. Bogdan, *Sharp estimates for the Green functions in Lipschitz domains*, J. Math. Anal. Appl. **243** (2000), 326-337.

[4] Z.-Q. Chen and R. Song, *General gauge and conditional gauge theorems*, Ann. Probab. **30** (2002), 1313-1339.

[5] K. L. Chung and Z. Zhao, From Brownian Motion to Schrödinger Equations, Springer, Berlin, 1995.

[6] E. B. Davies, Heat kernels and spectral theory, Cambridge University Press, Cambridge, 1989.

[7] E. B. Davies and B. Simon, *Ultracontractivity and heat kernels for Schrödinger operators and Dirichlet Laplacians*, J. Funct. Anal. **59** (1984), 335–395.
[8] B. Davis, *Intrinsic ultracontractivity and the Dirichlet Laplacian*, J. Funct. Anal. **100** (1991), 163–180.
[9] A. Grigor'yan and L. Saloff-Coste, *Dirichlet heat kernel in the exterior of a compact set*, Comm. Pure. Appl. Math. **55** (2002), 93–133.
[10] M. Grüter and K. O. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math. **37** (1982), 303–342.
[11] E. B. Fabes, N. Garofalo and S. Salsa, *A backward Harnack inequality and Fatou theorem for nonnegative solutions of parabolic equations*, Illinois J. Math. **30** (1986), 536–565.
[12] L. Saloff-Coste, *Uniformly elliptic operators on Riemannian manifolds*, J. Differential Geom. **36** (1992), 417–450.
[13] R. Song, *Sharp bounds on the density, Green function and jumping function of subordinate killed BM*, Probab. Th. rel Fields **128** (2004), 606–628.
[14] R. Song and Z. Vondraček, *Potential theory of subordinate killed Brownian motion in a domain*, Probab. Theory rel. Fields **125** (2003), 578–592.
[15] N. Th. Varopoulos, *Gaussian Estimates in Lipschitz domains*, Canad. J. Math. **55** (2003), 401–431.
[16] Q. S. Zhang, *The boundary behavior of heat kernels of Dirichlet Laplacians*, J. Differential Equations **182** (2002), 416–430.
[17] Q. S. Zhang, *The global behavior of heat kernels in exterior domains*, J. Functional Analysis **200** (2003) 160–176.
[18] Z. Zhao, *Green's function for Schrödinger operators and conditioned Feynman-Kac gauge*, J. Math. Anal. Appl. **116** (1986), 309–334.

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