GLASNIK MATEMATIČKI
Vol. 39(59)(2004), 303 – 314

# BOUNDED 2-LINEAR OPERATORS ON 2-NORMED SETS

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Abstract. In this paper properties of bounded 2-linear operators from a 2-normed set into a normed space are considered. The space of these operators is a Banach space and a symmetric 2-normed space. In the third part we will formulate Banach-Steinhaus Theorems for a family of bounded 2-linear operators from a 2-normed set into a Banach space.

## 1. Introduction

In [1] S. Gähler introduced the following definition of a 2-normed space:

Definition 1.1. [1] *Let $X$ be a real linear space of dimension greater than 1 and let $\| \cdot , \cdot \|$ be a real valued function on $X \times X$ satisfying the following four properties:*

(G1) $\|x, y\| = 0$ *if and only if the vectors $x$ and $y$ are linearly dependent;*
(G2) $\|x, y\| = \|y, x\|$;
(G3) $\|x, \alpha y\| = \mid \alpha \mid \cdot \|x, y\|$ *for every real number $\alpha$;*
(G4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ *for every $x, y, z \in X$.*

*The function $\| \cdot , \cdot \|$ will be called a 2-norm on $X$ and the pair $(X, \| \cdot , \cdot \|)$ a linear 2-normed space.*

In [4] and [5] we gave a generalization of the Gähler's 2-normed space. Namely a generalized 2-norm need not be symmetric and satisfy the first condition of the above definition.

Definition 1.2. [4] *Let $X$ and $Y$ be real linear spaces. Denote by $\mathcal{D}$ a non-empty subset of $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets*

2000 *Mathematics Subject Classification.* 46A32,47B99.
*Key words and phrases.* 2-normed set, 2-normed space, bounded 2-linear operator, Banach-Steinhaus theorems.

*$\mathcal{D}_x = \{y \in Y;\ (x,y) \in \mathcal{D}\}$ and $\mathcal{D}^{\,y} = \{x \in X;\ (x,y) \in \mathcal{D}\}$ are linear subspaces of the space $Y$ and $X$, respectively.*

*A function $\| \cdot , \cdot \| \colon \mathcal{D} \to [0,\infty)$ will be called a generalized 2-norm on $\mathcal{D}$ if it satisfies the following conditions:*

(N1) $\|x, \alpha y\| = |\alpha| \cdot \|x,y\| = \|\alpha x, y\|$ *for any real number* $\alpha$ *and all* $(x,y) \in \mathcal{D}$;

(N2) $\|x, y+z\| \le \|x,y\| + \|x,z\|$ *for* $x \in X,\ y,z \in Y$ *such that* $(x,y),\ (x,z) \in \mathcal{D}$;

(N3) $\|x+y, z\| \le \|x,z\| + \|y,z\|$ *for* $x,y \in X,\ z \in Y$ *such that* $(x,z),\ (y,z) \in \mathcal{D}$.

*The set $\mathcal{D}$ is called a 2-normed set.*

*In particular, if $\mathcal{D} = X \times Y$ , the function $\| \cdot , \cdot \|$ will be called a generalized 2-norm on $X \times Y$ and the pair $(X \times Y, \| \cdot , \cdot \|)$ a generalized 2-normed space.*

*Moreover, if $X = Y$, then the generalized 2-normed space will be denoted by $(X, \| \cdot , \cdot \|)$.*

Assume that the generalized 2-norm satisfies, in addition, the symmetry condition. Then we will define the 2-norm as follows:

Definition 1.3. [4] *Let $X$ be a real linear space. Denote by $\mathcal{X}$ a non-empty subset of $X \times X$ with the property $\mathcal{X} = \mathcal{X}^{-1}$ and such that the set $\mathcal{X}^y = \{x \in X; (x,y) \in \mathcal{X}\}$ is a linear subspace of $X$, for all $y \in X$.*

*A function $\| \cdot , \cdot \| \colon \mathcal{X} \to [0,\infty)$ satisfying the following conditions:*

(S1) $\|x,y\| = \|y,x\|$ *for all* $(x,y) \in \mathcal{X}$;

(S2) $\|x, \alpha y\| = |\alpha| \cdot \|x,y\|$ *for any real number* $\alpha$ *and all* $(x,y) \in \mathcal{X}$;

(S3) $\|x, y+z\| \le \|x,y\| + \|x,z\|$ *for* $x,y,z \in X$ *such that* $(x,y),\ (x,z) \in \mathcal{X}$;

*will be called a generalized symmetric 2-norm on $\mathcal{X}$. The set $\mathcal{X}$ is called a symmetric 2-normed set. In particular, if $\mathcal{X} = X \times X$, the function $\| \cdot , \cdot \|$ will be called a generalized symmetric 2-norm on $X$ and the pair $(X, \| \cdot , \cdot \|)$ a generalized symmetric 2-normed space.*

In [4], [5], [6], [7] we considered properties of generalized 2-normed spaces and 2-normed sets.

In what follows we shall use the following results:

Theorem 1.4. [4] *Let $(X \times Y, \| \cdot , \cdot \|)$ be a generalized 2-normed space. Then the family $\mathcal{B}$ of all sets defined by*

$$\bigcap_{i=1}^{n} \{x \in X;\ \|x, y_i\| < \varepsilon\},$$

*where $y_1, y_2, ..., y_n \in Y, n \in N$ and $\varepsilon > 0$, forms a complete system of neighborhoods of zero for a locally convex topology in $X$.*

We will denote it by the symbol $\mathcal{T}(X,Y)$. Similarly, we have the preceding theorem for a topology $\mathcal{T}(Y,X)$ in the space $Y$. In the case when $X=Y$ we will write: $\mathcal{T}_1(X)=\mathcal{T}(X,Y)$ and $\mathcal{T}_2(X)=\mathcal{T}(Y,X)$.

Let $(X\times Y,\|\cdot,\cdot\|)$ be a generalized 2-normed space and let $\Sigma$ be a directed set. A net $\{x_\sigma;\sigma\in\Sigma\}$ is convergent to $x_o\in X$ in $(X,\mathcal{T}(X,Y))$ if and only if for all $y\in Y$ and $\varepsilon>0$ there exists $\sigma_o\in\Sigma$ such that $\|x_\sigma-x_o,y\|<\varepsilon$ for all $\sigma\geq\sigma_o$. Similarly we have the notion of convergence in $(Y,\mathcal{T}(Y,X))$.

A sequence $\{x_n;n\in N\}\subset X$ is a Cauchy sequence in $(X,\mathcal{T}(X,Y))$ if and only if for every $y\in Y$ and $\varepsilon>0$ there exists a number $n_o\in N$ such that inequality $n,m>n_o$ implies $\|x_n-x_m,y\|<\varepsilon$. A space $(X,\mathcal{T}(X,Y))$ is called sequentially complete if every Cauchy sequence in $(X,\mathcal{T}(X,Y))$ is convergent in this space. Analogously we have the notion of sequential completeness for the space $(Y,\mathcal{T}(Y,X))$.

Example 1.5. [4] Let $X$ be a real linear space which have two norms (seminorms) $\|\cdot\|_1,\|\cdot\|_2$. Then $(X,\|\cdot,\cdot\|)$ is a generalized 2-normed space with the 2-norm defined by the formula

$$\|x,y\|=\|x\|_1\cdot\|y\|_2 \text{ for each } x,y\in X.$$

Let us remark that topologies generated by these norms $\|\cdot\|_1$ and $\|\cdot\|_2$ coincide with the topologies $\mathcal{T}_1(X)$ and $\mathcal{T}_2(X)$ given in Theorem 1.4.

Example 1.6. In Example 1.5 we can get $\|\cdot\|_1=\|\cdot\|_2$. Then $(X,\|\cdot,\cdot\|)$ is a generalized symmetric 2-normed space with the symmetric 2-norm defined by the formula

$$\|x,y\|=\|x\|\cdot\|y\| \text{ for each } x,y\in X. \tag{1.1}$$

Let us remark that a symmetric 2-normed space need not be a 2-normed space in the sense of Gähler. For instance given in Example 1.6 $x\neq\theta,y=kx,k\neq 0$ we obtain

$$\|x,y\|=\|x,kx\|=\mid k\mid\cdot\|x,x\|=\mid k\mid\cdot\|x\|^2>0,$$

but in spite of this $x$ and $y$ are linearly dependent. The 2-normed space from Example 1.6 is not a 2-normed space in the sense of Definition 1.1.

It is easy to see that if $(X,\|\cdot\|)$ is a normed space, $\mathcal{T}_1-$the topology generated by this norm and $\mathcal{T}_2-$the topology generated by the 2-norm defined by the formula (1.1), then $\mathcal{T}_1=\mathcal{T}_2$. Moreover a sequence $\{x_n;n\in N\}$ is a Cauchy sequence in $(X,\|\cdot\|)$ if and only if it is a Cauchy sequence in $(X,\|\cdot,\cdot\|)$ with the 2-norm defined in Example 1.6.

Thus the following theorem follows.

Theorem 1.7. *A normed space $(X,\|\cdot\|)$ is a Banach space if and only if the symmetric 2-normed space with the 2-norm defined by (1.1) is sequentially complete.*

## 2. The space of all bounded 2-linear operators

In [8] A. G. White defined and considered the properties of bounded 2-linear functionals from $B \times B$, where $B$ denotes a 2-normed space in the sense of Gähler. He proved that the set of all bounded 2-linear functionals is a Banach space.

S. S. Kim, Y. J. Cho and A. G. White in [3] and A. Khan in [2] gave the properties of bounded operators from $X \times X$ with values in a normed space $Y$, where $X$ denotes a 2-normed space in the sense of Gähler. They showed that the set $B(X \times X, Y)$ of all bounded operators from $X \times X$ into $Y$ is a seminormed space. Moreover, if $Y$ is a Banach space, then $B(X \times X, Y)$ is a complete space.

In this section we will consider bounded 2-linear operators defined on a 2-normed set into a normed space. We will show, like in the above mentioned papers, that the space of these operators is a Banach space. We will prove that under some additional conditions it is a symmetric 2-normed space.

Let us consider a real linear space $X$. Let $\mathcal{D} \subset X \times X$ be a 2-normed set, $Y$ a normed space.

Definition 2.1. *An operator $F\colon \mathcal{D} \to Y$ is said to be 2-linear if it satisfies the following conditions:*

1. $F(a+c, b+d) = F(a,b) + F(a,d) + F(c,b) + F(c,d)$ *for* $a,b,c,d \in X$ *such that* $a, c \in \mathcal{D}^{\,b} \cap \mathcal{D}^{\,d}$.
2. $F(\alpha a, \beta b) = \alpha \cdot \beta \cdot F(a,b)$ *for* $\alpha, \beta \in \mathbb{R}, (a,b) \in \mathcal{D}$.

Definition 2.2. *A 2-normed operator $F$ is said to be bounded if there is a positive number $K$ such that*

$$\|F(a,b)\| \le K \cdot \|a,b\| \textit{ for all } (a,b) \in \mathcal{D}.$$

Definition 2.3. *If $F$ is a bounded operator, then the following number*

$$\|F\| = \inf\{K > 0; \|F(a,b)\| \le K \cdot \|a,b\| \textit{ for } (a,b) \in \mathcal{D}\}$$

*will be called the norm of the 2-linear operator $F$.*

Example 2.4. Let $\Big(X, (\,\cdot \mid \cdot\,)\Big)$ be a real inner product space. Then $X$ is a generalized symmetric 2-normed space with the 2-norm defined as follows:

$$\|x,y\| = |\,(x \mid y)\,| \quad \text{for all } x,y \in X.$$

This 2-norm generates a weak topology in the Hilbert space (see Example 1.5 in [4]). An operator $F\colon X \times X \to \mathbb{R}$ defined by the formula

$$F(a,b) = (a \mid b) \text{ for } a,b \in X$$

is 2-linear and bounded. Moreover $\|F\| = 1$.

In the next theorem we will give properties of the above mentioned notions.

THEOREM 2.5. *Let $F$ be a bounded 2-linear operator. Then:*

(a) $\|F\| \leq K$ *for* $K \in \mathcal{P}^{(F)} = \{K' > 0; \|F(a,b)\| \leq K' \cdot \|a,b\|$ *for* $(a,b) \in \mathcal{D}\}$*;*

(b) $\|F(a,b)\| \leq \|F\| \cdot \|a,b\|$ *for each* $(a,b) \in \mathcal{D}$*;*

(c)

$$\begin{aligned}\|F\| &= \sup\{\|F(a,b)\|;\ (a,b) \in \mathcal{D}, \|a,b\| = 1\}\\ &= \sup\{\|F(a,b)\|;\ (a,b) \in \mathcal{D}, \|a,b\| \leq 1\}\\ &= \sup\left\{\frac{\|F(a,b)\|}{\|a,b\|};\ (a,b) \in \mathcal{D}, \|a,b\| \neq 0\right\}.\end{aligned}$$

PROOF. The condition (a) follows from the Definition 2.3.

(b) Because the operator $F$ is bounded, then there exists $K > 0$ such that

$$\|F(a,b)\| \leq K \cdot \|a,b\| \text{ for } (a,b) \in \mathcal{D}.$$

Thus $\|F(a,b)\| \leq \inf_{K' \in \mathcal{P}^{(F)}} K' \cdot \|a,b\|$, i.e.

$$\|F(a,b)\| \leq \|F\| \cdot \|a,b\|.$$

(c) By (b), $\sup\left\{\frac{\|F(a,b)\|}{\|a,b\|};\ (a,b) \in \mathcal{D}, \|a,b\| \neq 0\right\} \leq \|F\|$.

Let $A = \sup\{\|F(a,b)\|;\ (a,b) \in \mathcal{D}, \|a,b\| = 1\}$. Then

$$\begin{aligned}A &= \sup\left\{\frac{\|F(a,b)\|}{\|a,b\|};\ (a,b) \in \mathcal{D}, \|a,b\| = 1\right\}\\ &\leq \sup\left\{\frac{\|F(a,b)\|}{\|a,b\|};\ (a,b) \in \mathcal{D}, \|a,b\| \leq 1\right\}\\ &\leq \sup\left\{\frac{\|F(a,b)\|}{\|a,b\|};\ (a,b) \in \mathcal{D}, \|a,b\| \neq 0\right\}\\ &\leq \|F\|.\end{aligned} \tag{2.1}$$

Moreover

$$A \leq \sup\{\|F(a,b)\|;\ (a,b) \in \mathcal{D}, \|a,b\| \leq 1\}. \tag{2.2}$$

Let $(a,b) \in \mathcal{D}$ be such that $\|a,b\| \neq 0$. Because $\left\|\frac{a}{\|a,b\|}, b\right\| = 1$, then $\left\|F\left(\frac{a}{\|a,b\|}, b\right)\right\| \leq A$. And further by virtue of the equalities

$$\left\|F\left(\frac{a}{\|a,b\|}, b\right)\right\| = \left\|\frac{1}{\|a,b\|} \cdot F(a,b)\right\| = \frac{1}{\|a,b\|} \cdot \|F(a,b)\|$$

we obtain $\|F(a,b)\| \leq A \cdot \|a,b\|$. On the other hand, if $(a,b) \in \mathcal{D}$ and $\|a,b\| = 0$, then $0 \leq \|F(a,b)\| \leq \|F\| \cdot \|a,b\| = 0$, i.e. $\|F(a,b)\| = 0 = A \cdot \|a,b\|$.

Consequently $\|F(a,b)\| \leq A \cdot \|a,b\|$ for all $(a,b) \in \mathcal{D}$, which means that $A \in \mathcal{P}^{(F)}$. By virtue of (a) we obtain

$$\|F\| \leq A. \tag{2.3}$$

The conditions (2.1) and (2.3) imply

$$\begin{aligned}\|F\| &= \sup\{\|F(a,b)\|;\ (a,b) \in \mathcal{D}, \|a,b\| = 1\}\\ &= \sup\left\{\frac{\|F(a,b)\|}{\|a,b\|};\ (a,b) \in \mathcal{D}, \|a,b\| \neq 0\right\}.\end{aligned}$$

From (b) we have $\sup\{\|F(a,b)\|;\ (a,b) \in \mathcal{D}, \|a,b\| \leq 1\} \leq \|F\|$, which with (2.2) gives the equality $\|F\| = \sup\{\|F(a,b)\|;\ (a,b) \in \mathcal{D}, \|a,b\| \leq 1\}$, and the proof is completed. ∎

Definition 2.6. *Let $\mathcal{D} \subset X \times X$ be a 2-normed set and $Y$ a normed space. Denote by $L_2(\mathcal{D}, Y)$ the set of all bounded 2-linear operators from $\mathcal{D}$ into $Y$.*

*In particular, we will write $L_2(X,Y)$, if $X$ is a generalized 2-normed space and $\mathcal{D} = X \times X$.*

*Let $F, G \in L_2(\mathcal{D}, Y)$ and define*

1. $(F+G)(a,b) = F(a,b) + G(a,b)$ *for all* $(a,b) \in \mathcal{D}$;
2. $(\alpha \cdot F)(a,b) = \alpha \cdot F(a,b)$ *for* $\alpha \in \mathbb{R}, (a,b) \in \mathcal{D}$.

Theorem 2.7. *If $\mathcal{D}$ is a 2-normed set and $Y$ a normed space, then the set $L_2(\mathcal{D}, Y)$ is a normed space with the norm $\|\cdot\|$ defined in Definition 2.3.*

Proof. Let us take $F, G \in L_2(\mathcal{D}, Y), \alpha, \beta \in \mathcal{R}$ and $a, b, c, d \in X$ such that $a, c \in \mathcal{D}^{\,b} \cap \mathcal{D}^{\,d}$. For $F + G$ we obtain:

$$\begin{aligned}&(F+G)(a+c, b+d) =\\ &= (F+G)(a,b) + (F+G)(a,d) + (F+G)(c,b) + (F+G)(c,d);\end{aligned} \tag{2.4}$$

$$(F+G)(\alpha a, \beta b) = \alpha\beta \cdot (F+G)(a,b). \tag{2.5}$$

Moreover by virtue of the condition (b) of Theorem 2.5 we have

$$\begin{aligned}\|(F+G)(a,b)\| &= \|F(a,b) + G(a,b)\|\\ &\leq \|F(a,b)\| + \|G(a,b)\| \leq \|F\| \cdot \|a,b\| + \|G\| \cdot \|a,b\|\\ &= (\|F\| + \|G\|) \cdot \|a,b\|.\end{aligned} \tag{2.6}$$

Thus $F + G \in L_2(\mathcal{D}, Y)$.

Analogously we show that $\alpha \cdot F \in L_2(\mathcal{D}, Y)$ and

$$\|(\alpha \cdot F)(a,b)\| = \|\alpha \cdot F(a,b)\| \leq \mid \alpha \mid \cdot \|F\| \cdot \|a,b\|. \tag{2.7}$$

Moreover it is easy to prove that the set $L_2(\mathcal{D}, Y)$ is a real linear space.

Now we will show that the function $\|\cdot\|\colon L_2(\mathcal{D}, Y) \to [0, \infty)$ given in Definition 2.3 satisfies all conditions of a norm.

If $\|F\| = 0$, then $\|F(a,b)\| = 0$ for all $(a,b) \in \mathcal{D}$. Thus $F(a,b) = 0$ for every $(a,b) \in \mathcal{D}$. Conversely, if $F$ is a zero operator, then

$$\|F\| = \sup\{\|F(a,b)\|;\ (a,b) \in \mathcal{D}, \|a,b\| = 1\} = 0.$$

As a consequence we have the condition

$$\|F\| = 0 \text{ if and only if } F = 0.$$

From (2.7) we have $|\alpha| \cdot \|F\| \in \mathcal{P}^{(\alpha F)}$, which with Theorem 2.5 (a) implies the inequality $\|\alpha \cdot F\| \leq |\alpha| \cdot \|F\|$. Assume $\alpha \neq 0$. Then

$$\|F\| = \left\| \frac{1}{\alpha} \cdot \alpha \cdot F \right\| \leq \frac{1}{|\alpha|} \cdot \|\alpha F\|,$$

i.e. $|\alpha| \cdot \|F\| \leq \|\alpha \cdot F\|$; thus $|\alpha| \cdot \|F\| = \|\alpha \cdot F\|$.

For $\alpha = 0$ the equality $\|\alpha \cdot F\| = |\alpha| \cdot \|F\|$ is obvious. Therefore $\|\alpha \cdot F\| = |\alpha| \cdot \|F\|$ for $\alpha \in \mathbb{R}$.

The condition (2.6) implies $\|F\| + \|G\| \in \mathcal{P}^{(F+G)}$. Hence and from Theorem 2.5(a) we have $\|F + G\| \leq \|F\| + \|G\|$. This completes the proof.

□

THEOREM 2.8. *If $\mathcal{D}$ is a 2-normed set and $Y$ is a Banach space, then $L_2(\mathcal{D}, Y)$ is a Banach space.*

PROOF. According to Theorem 2.7, $L_2(\mathcal{D}, Y)$ is a normed space.

Let $\{F_n; n \in N\}$ be a Cauchy sequence in $L_2(\mathcal{D}, Y)$. Then

$$\lim_{n,m\to\infty} \|F_n - F_m\| = 0$$

and for every $(a,b) \in \mathcal{D}$ the following inequality

$$\|F_n(a,b) - F_m(a,b)\| = \|(F_n - F_m)(a,b)\| \leq \|F_n - F_m\| \cdot \|a,b\|$$

is true. Thus $\{F_n(a,b); n \in N\}$ is a Cauchy sequence in $Y$ for every $(a,b) \in \mathcal{D}$. Because $Y$ is complete, the sequence $\{F_n(a,b); n \in N\}$ is convergent for every $(a,b) \in \mathcal{D}$. Let us denote

$$F(a,b) = \lim_{n\to\infty} F_n(a,b).$$

We shall show that $F \in L_2(\mathcal{D}, Y)$.

For $a, b, c, d \in X$ such that $a, c \in \mathcal{D}^{\,b} \cap \mathcal{D}^{\,d}$ we have

$$\begin{aligned} F(a+c, b+d) &= \lim_{n\to\infty} F_n(a+c, b+d) \\ &= \lim_{n\to\infty} F_n(a,b) + \lim_{n\to\infty} F_n(a,d) + \lim_{n\to\infty} F_n(c,b) + \lim_{n\to\infty} F_n(c,d) \\ &= F(a,b) + F(a,d) + F(c,b) + F(c,d). \end{aligned}$$

Moreover for $\alpha, \beta \in \mathbb{R}$ and $(a,b) \in \mathcal{D}$ we have:

$$\begin{aligned} F(\alpha a, \beta b) &= \lim_{n\to\infty} F_n(\alpha a, \beta b) \\ &= \lim_{n\to\infty} \alpha\beta \cdot F_n(a,b) \\ &= \alpha\beta \cdot \lim_{n\to\infty} F_n(a,b) \\ &= \alpha\beta \cdot F(a,b). \end{aligned}$$

Thus $F$ is a 2-linear operator.

The inequality

$$|\ \|F_n\| - \|F_m\|\ | \leq \|F_n - F_m\|$$

implies that $\{\|F_n\|; n \in N\}$ is a Cauchy sequence in $\mathbb{R}$. As a consequence this sequence is bounded, that is, there exists $K > 0$ such that $\|F_n\| \leq K$ for all $n \in N$. Using this result we get

$$\begin{aligned} \|F(a,b)\| &\leq \|F_n(a,b)\| + \|F(a,b) - F_n(a,b)\| \\ &\leq \|F_n\| \cdot \|a,b\| + \|F(a,b) - F_n(a,b)\| \\ &\leq K \cdot \|a,b\| + \|F_n(a,b) - F(a,b)\|. \end{aligned}$$

Letting $n \to \infty$ we obtain $\|F(a,b)\| \leq K \cdot \|a,b\|$ for every $(a,b) \in \mathcal{D}$, which means that $F$ is bounded. So we have shown that $F \in L_2(\mathcal{D}, Y)$.

Now let us suppose that $(a,b) \in \mathcal{D}$ and $\|a,b\| \neq 0$. Let $\varepsilon > 0$. Because $\{F_n; n \in N\}$ is a Cauchy sequence, there exists $n_o \in N$ such that

$$\|F_n - F_m\| < \frac{\varepsilon}{4} \text{ for all } n, m \geq n_o.$$

Thus $\|F_n(a,b) - F_m(a,b)\| \leq \|F_n - F_m\| \cdot \|a,b\| < \frac{\varepsilon}{4} \cdot \|a,b\|$ for all $n, m \geq n_o$. The equality

$$F(a,b) = \lim_{n\to\infty} F_n(a,b)$$

implies that there exists $n_1 = n_1(a,b) \geq n_o$ such that

$$\|F_{n_1}(a,b) - F(a,b)\| < \frac{\varepsilon}{4} \cdot \|a,b\|.$$

As a consequence we obtain

$$\begin{aligned} \|F_n(a,b) - F(a,b)\| &\leq \|F_n(a,b) - F_{n_1}(a,b)\| + \|F_{n_1}(a,b) - F(a,b)\| \\ &< \frac{\varepsilon}{2} \cdot \|a,b\| \end{aligned}$$

for $n \geq n_o$, $(a,b) \in \mathcal{D}$ and $\|a,b\| \neq 0$. If $\|a,b\| = 0$, then $F_n(a,b) = 0 = F(a,b)$, so $\|F_n(a,b) - F(a,b)\| = \frac{\varepsilon}{2} \cdot \|a,b\|$. Thus $\|F_n(a,b) - F(a,b)\| \leq \frac{\varepsilon}{2} \cdot \|a,b\|$ for all $n \geq n_o, (a,b) \in \mathcal{D}$, i.e.

$$\frac{\varepsilon}{2} \in \mathcal{P}^{\,(F_n - F)} \quad \text{ for } n \geq n_o.$$

Therefore $\|F_n - F\| \leq \frac{\varepsilon}{2} < \varepsilon$ for $n \geq n_o$, which means that the sequence $\{F_n; n \in N\}$ is convergent to $F$ in $L_2(\mathcal{D}, Y)$. Hence we have shown that $L_2(\mathcal{D}, Y)$ is a Banach space, which finishes the proof. □

From Theorem 2.8 and Theorem 1.7 the following corollary follows.

Corollary 2.9. *If $\mathcal{X}$ is a symmetric 2-normed set and $Y$ is a Banach space, then $L_2(\mathcal{X}, Y)$ is a symmetric sequentially complete 2-normed space with the 2-norm defined as follows:*

$$\|F, G\| = \|F\| \cdot \|G\| \text{ for } F, G \in L_2(\mathcal{X}, Y).$$

## 3. Banach-Steinhaus Theorems for bounded 2-linear operators

In this section we will consider properties of sequences of operators from $L_2(\mathcal{D}, Y)$. We will formulate Banach-Steinhaus Theorems for a family of these operators.

Proposition 3.1. *Let $\mathcal{D}$ be a 2-normed set, $Y$ a normed space and $\{F_n; n \in N\} \subset L_2(\mathcal{D}, Y)$. If the sequence of norms $\{\|F_n\|; n \in N\}$ is bounded, then for each $(x, y) \in \mathcal{D}$ the sequence of norms $\{\|F_n(x, y)\|; n \in N\}$ is bounded.*

Proof. From the assumption it follows that there exists a positive number $M$ such that $\|F_n\| \leq M$ for each $n \in N$. Thus for $(x, y) \in \mathcal{D}$ we obtain

$$\|F_n(x, y)\| \leq \|F_n\| \cdot \|x, y\| \leq M \cdot \|x, y\| \text{ for each } n \in N.$$

□

Theorem 3.2. *Let $X$ be a generalized 2-normed space and $Y$ a normed space. If $\{F_n; n \in N\} \subset L_2(X, Y)$ is pointwise convergent to $F$ and the sequence of norms $\{\|F_n\|; n \in N\}$ is bounded, then $F \in L_2(X, Y)$.*

Proof. For all $x, y \in X$ we have

$$F(x, y) = \lim_{n \to \infty} F_n(x, y).$$

Thus the operator $F$ is a 2-linear operator.

Because the sequence of norms $\{\|F_n\|; n \in N\}$ is bounded, then there exists $M > 0$ such that $\|F_n\| \leq M$ for all $n \in N$. Thus $\|F_n(x, y)\| \leq \|F_n\| \cdot \|x, y\| \leq M \cdot \|x, y\|$. Let us take $x, y \in X$. Then

$$\begin{aligned} \|F(x, y)\| &\leq \|F_n(x, y) - F(x, y)\| + \|F_n(x, y)\| \leq \\ &\leq \|F_n(x, y) - F(x, y)\| + M \cdot \|x, y\|. \end{aligned} \tag{3.1}$$

By letting $n \to \infty$ we obtain $\|F(x, y)\| \leq M \cdot \|x, y\|$ for each $x, y \in X$. This gives that $F$ is bounded. As a consequence we have shown that $F \in L_2(X, Y)$. □

THEOREM 3.3. *Let $Y$ be a Banach space, $(X, \|\cdot,\cdot\|)$ a generalized 2-normed space and let $A$ be a linearly dense set in the spaces $(X, \mathcal{T}_1(X))$ and $(X, \mathcal{T}_2(X))$. If a sequence $\{F_n; n \in N\} \subset L_2(X, Y)$ is pointwise convergent on the set $A$ and the sequence of norms $\{\|F_n\|; n \in N\}$ is bounded, then the sequence $\{F_n(x, y); n \in N\}$ is convergent in $Y$ for each $x, y \in X$.*

PROOF. Let $X_o$ be the linear subspace of $X$ generated by $A$. We will consider $X_o$ as a 2-normed space with the same 2-norm induced by that of $X$. Let $x, y \in X_o$. Then $x = a_1x_1 + \cdots + a_kx_k$, $y = b_1y_1 + \cdots + b_ty_t$, where $a_i, b_j \in \mathbb{R}$, $x_i, y_j \in A$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, t$; $k, t \in N$, and

$$F_n(x, y) = \sum_{i=1}^{k}\sum_{j=1}^{t} a_ib_j \cdot F_n(x_i, y_j).$$

Because the sequence $\{F_n(x_i, y_j); n \in N\}$ is convergent for all $x_i, y_j \in A$, then $\{F_n(x, y); n \in N\}$ is convergent in $X_o$.

Let $\|F_n\| \le M$ for every $n \in N$. Let us take a number $\varepsilon > 0$ and $x, y \in X$. Since $X_o$ is a dense set in $(X, \mathcal{T}_1(X))$ we can choose $x_o \in X_o$ such that

$$\|x - x_o, y\| < \frac{\varepsilon}{6M}.$$

Moreover there exists $y_o \in X_o$ with the property

$$\|x_o, y - y_o\| < \frac{\varepsilon}{6M},$$

because $X_o$ is also a dense set in $(X, \mathcal{T}_2(X))$.

The sequence $\{F_n(x_o, y_o); n \in N\}$ is convergent, so it is a Cauchy sequence in $Y$. Therefore there exists a number $n_o \in N$ such that

$$\|F_n(x_o, y_o) - F_m(x_o, y_o)\| < \frac{\varepsilon}{3} \text{ for each } n, m \ge n_o.$$

As a consequence we obtain

$$\begin{aligned}
\|F_n(x, y) - F_m(x, y)\| &= \|F_n(x - x_o + x_o, y) - F_m(x - x_o + x_o, y)\| \\
&\le \|F_n(x - x_o, y)\| + \|F_m(x - x_o, y)\| \\
&\quad + \|F_n(x_o, y) - F_m(x_o, y)\| \\
&\le \|F_n(x - x_o, y)\| + \|F_m(x - x_o, y)\| \\
&\quad + \|F_n(x_o, y - y_o)\| + \|F_m(x_o, y - y_o)\| \\
&\quad + \|F_n(x_o, y_o) - F_m(x_o, y_o)\| \\
&\le \|F_n\| \cdot \|x - x_o, y\| + \|F_m\| \cdot \|x - x_o, y\| \\
&\quad + \|F_n\| \cdot \|x_o, y - y_o\| + \|F_m\| \cdot \|x_o, y - y_o\| + \frac{\varepsilon}{3} \\
&\le 2M \cdot \|x - x_o, y\| + 2M \cdot \|x_o, y - y_o\| + \frac{\varepsilon}{3} < \varepsilon
\end{aligned}$$

for $n, m \geq n_o$. Hence we have shown that $\{F_n(x,y); n \in N\}$ is a Cauchy sequence in $Y$ for each $x, y \in X$. Because $Y$ is complete, then the sequence $\{F_n(x,y); n \in N\}$ is convergent in $Y$, which finishes the proof. □

THEOREM 3.4. *Let $(X, \| \cdot , \cdot \|)$ be a generalized 2-normed space and $Y$ a Banach space. If a sequence $\{F_n; n \in N\} \subset L_2(X,Y)$ is pointwise convergent to $F \in L_2(X,Y)$ on a linearly dense set $A$ in the spaces $(X, \mathcal{T}_1(X))$ and $(X, \mathcal{T}_2(X))$ and the sequence of norms $\{\|F_n\|; n \in N\}$ is bounded, then $\{F_n; n \in N\}$ is pointwise convergent to $F$ and the inequality $\|F\| \leq \sup_n \|F_n\|$ holds.*

PROOF. It follows from Theorem 3.3 that the sequence $\{F_n(x,y); n \in N\}$ is convergent in $Y$ for each $x, y \in X$. Let us denote

$$H(x,y) = \lim_{n\to\infty} F_n(x,y) \text{ for every } x, y \in X.$$

We must show that $H(x,y) = F(x,y)$ for all $x, y \in X$. Using Theorem 3.2 we see that $H \in L_2(X,Y)$. From assumption it follows that $H(x,y) = F(x,y)$ for all $x, y \in A$, i.e. $(H-F)(x,y) = 0$ for $x, y \in A$. Because $L_2(X,Y)$ is a linear space, then $H - F \in L_2(X,Y)$. As a consequence $H - F$ is an 2-linear operator and $(H-F)(x,y) = 0$ for $x, y \in X_o$, where $X_o$ denote the set of all linear combinations of elements from $A$. Moreover $H - F$ is bounded, thus there exists $K > 0$ such that $\|(H-F)(x,y)\| \leq K \cdot \|x, y\|$ for every $x, y \in X$.

Let $\varepsilon > 0, x, y \in X$. Since the set $X_o$ is dense in $(X, \mathcal{T}_1(X))$ we can choose $x_o \in X_o$ such that

$$\|x - x_o, y\| < \frac{\varepsilon}{2K}.$$

There exists $y_o \in X_o$ with the property

$$\|x_o, y - y_o\| < \frac{\varepsilon}{2K},$$

because $X_o$ is also dense in $(X, \mathcal{T}_2(X))$. Then we have

$$\begin{aligned}
0 &\leq \|(H-F)(x,y)\| = \|(H-F)(x - x_o + x_o, y)\| \\
&= \|(H-F)(x - x_o, y) + (H-F)(x_o, y)\| \\
&= \|(H-F)(x - x_o, y) + (H-F)(x_o, y - y_o + y_o)\| \\
&= \|(H-F)(x - x_o, y) + (H-F)(x_o, y - y_o) + (H-F)(x_o, y_o)\| \\
&= \|(H-F)(x - x_o, y) + (H-F)(x_o, y - y_o)\| \\
&\leq \|(H-F)(x - x_o, y)\| + \|(H-F)(x_o, y - y_o)\| \\
&\leq K \cdot \|x - x_o, y\| + K \cdot \|x_o, y - y_o\| < \varepsilon.
\end{aligned}$$

This gives $\|(H-F)(x,y)\| = 0$ for each $x, y \in X$, i.e. $H(x,y) = F(x,y)$ for every $x, y \in X$.

Let us denote $M = \sup_n \|F_n\|$. Then for every $n \in N$ and $x, y \in X$ such that $\|x, y\| \leq 1$ we have

$$\|F_n(x, y)\| \leq \|F_n\| \cdot \|x, y\| \leq M.$$

Thus

$$\begin{aligned}\|F(x, y)\| &= \|F(x, y) - F_n(x, y) + F_n(x, y)\| \\ &\leq \|F(x, y) - F_n(x, y)\| + \|F_n(x, y)\| \\ &\leq \|F(x, y) - F_n(x, y)\| + M.\end{aligned}$$

By letting $n \to \infty$ we obtain $\|F(x, y)\| \leq M$ for $x, y \in X$ such that $\|x, y\| \leq 1$. This implies $\|F\| = \sup\{\|F(x, y)\|; x, y \in X, \|x, y\| \leq 1\} \leq M$, which finishes the proof. □

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*Received*: 1.10.2003.